\[
\begin{align*}
\min \quad & f(x) \\
\text{s.t.} \quad & h(x) = 0 \\
& g(x) \leq 0
\end{align*}
\Rightarrow \text{Feasible set } S = \{ x \mid h(x) = 0, g(x) \leq 0 \}
\]

**Example:**
\[
\begin{align*}
\min \quad & c^T x_1 - c^T x_2 \\
\text{s.t.} \quad & x_1^2 + x_2^2 = 1
\end{align*}
\Rightarrow \text{Feasible set }
\]

**Example:**
\[
\begin{align*}
\min \quad & c^T x_1 - c^T x_2 \\
\text{s.t.} \quad & x_1^2 + x_2^2 \leq 1 \\
& x_1 \geq 0 \\
& x_2 \geq 0
\end{align*}
\Rightarrow \Rightarrow \Rightarrow
\]

- The satisfaction of all constraints simultaneously means the intersection of the above sets.

\[
\Rightarrow S:
\]
Convex set: A set $S$ is called a convex set if for every two points $x$ and $y$, the segment connecting the two points is entirely in the set.

Formal definition: A set $S \subseteq \mathbb{R}^n$ is convex if $\forall x, y \in \mathbb{R}^n$ and $\alpha \in [0, 1]$ then $\alpha x + (1 - \alpha)y \in S$.

Convex function versus convex set:

$n = 2$: $f(x) = x_1^2 + x_2^2$

$S = \{ x \mid x_1^2 + x_2^2 \leq 1 \}$

Note $\{ x \mid x_1^2 + x_2^2 = 1 \}$ is nonconvex but $\{ x \mid x_1^2 + x_2^2 \leq 1 \}$ is convex.
Consider a convex optimization problem:

\[
\begin{align*}
\min_{x \in \mathbb{R}^n} & \quad f(x) \quad \text{convex} \\
\text{s.t.} & \quad h(x) = 0 \quad \text{linear} \\
& \quad g(x) \leq 0 \quad \text{convex}
\end{align*}
\]

**Theorem:** The feasible set of a convex problem is a convex set.

**Proof:** 
\[ S = \{ x \mid h(x) = 0, \quad g(x) \leq 0 \} \]

- Consider two points \( y, z \in S \) and a scalar \( \alpha \in (0, 1) \).
- We need to show that \( \alpha y + (1 - \alpha) z \in S \)
- Note that:
  \[ y \in S \implies h(y) = 0, \quad g(y) \leq 0 \]
  \[ z \in S \implies h(z) = 0, \quad g(z) \leq 0 \]

One can write:
\[
\begin{align*}
 h(\alpha y + (1 - \alpha) z) &= \alpha h(y) + (1 - \alpha) h(z) \\
(\text{linearity of } h(\cdot)) \\
 g(\alpha y + (1 - \alpha) z) &\leq \alpha g(y) + (1 - \alpha) g(z) \\
(\text{convexity of } g(\cdot))
\end{align*}
\]

\[ \alpha y + (1 - \alpha) z \in S \]
- Convex optimization:

\[
\begin{align*}
\text{min} \quad & f(x) \quad \text{in convex } x \in \mathbb{R}^n \\
\text{s.t.} \quad & h(x) = 0 \quad \text{in linear } \quad g(x) \geq 0 \quad \text{in convex set}
\end{align*}
\]

\[
\Rightarrow \quad \text{Convex optimization is about minimizing a convex function over a convex set.}
\]

- Example:

\[
\begin{align*}
\text{min} \quad & x_1^2 + x_2^2 \\
\text{s.t.} \quad & x_1^4 + x_2^4 \leq 1 \\
& x_1 \geq 0 \\
& x_2 \geq 0
\end{align*}
\]

- Consider a convex optimization problem.
- Recall that local = global in this case.
- Note that second-order necessary conditions are automatically satisfied:
\[
\Delta x^T \left( \nabla^2 f(x_*) + \sum_{i=1}^{m} \lambda_i^* \nabla^2 h_i(x_*) + \sum_{i=1}^{k} \mu_i^* \nabla^2 g_i(x_*) \right) \Delta x \geq 0
\]

- **Theorem:** If \( x_* \) is a regular point, then KKT conditions are necessary and sufficient for optimality.

- **In other words:**
  \( x_* \) is a global minimum \( \iff x_* \) satisfies KKT.

- **We don't know** \( x_* \) so it's hard to check its regularity at the beginning.

- **The regularity condition in the above theorem can be replaced by Slater's condition.**

**Slater's Condition:** There is a point \( x \) such that \( h(x) = 0 \) and \( g(x) < 0 \) (i.e., the feasible set \( \mathcal{S} \) has a non-empty interior).
Example:

\[
\begin{align*}
\min_{x \in \mathbb{R}} & \quad x_1^2 + x_2^2 \\
x_1 + x_2 & \leq 1 \\
x_1^2 + x_2^2 & \leq \frac{1}{2} \\
x_1 - x_2 & \leq \frac{1}{2}
\end{align*}
\]

\[\Rightarrow x = [0, 0] \text{ satisfies the inequalities in a strict way.}\]
\[\Rightarrow \text{Slater's condition holds.}\]
\[\Rightarrow KKT \iff x^*: \text{global min}\]

Consider an arbitrary optimization problem:

\[
\begin{align*}
\min_{x \in \mathbb{R}^n} & \quad f(x) \\
\text{s.t.} & \quad h(x) = 0 \\
& \quad g(x) \leq 0
\end{align*}
\]

\[\Rightarrow \text{pick arbitrary numbers } \lambda_1, \ldots, \lambda_m \geq 0 \text{ and } \mu_1, \ldots, \mu_{k_c}.
\]
\[\Rightarrow \text{remove each constraint and add it to the objective after multiplying with its coefficient.}\]
\[\Rightarrow \min_{x \in \mathbb{R}^n} f(x) + \sum_{i=1}^{m} \lambda_i h_i(x) + \sum_{i=1}^{k_c} \mu_i h_i(x)\]

- What's the connection with the constraint opt in the left and the unconstrained opt in the right?
- Define:

\[ l(x, \lambda, \mu) = f(x) + \sum_{i=1}^{m} \lambda_i h_i(x) + \sum_{i=1}^{k} \mu_i g_i(x) \]

Lagrangian function

- Theorem:

\[
\begin{align*}
\min_{x \in \mathbb{R}^n} f(x) & \quad \text{s.t. } h(x) = 0 \\
& \quad g(x) \leq 0
\end{align*}
\]

\[ \Rightarrow \quad \min_{x \in \mathbb{R}^n} l(x, \lambda, \mu) \]

- Proof: Let \( x^* \) and \( \bar{x} \) denote solutions of left and right problems, respectively.

\[ \Rightarrow \quad l(\bar{x}, \lambda, \mu) = \min_{x \in \mathbb{R}^n} l(x, \lambda, \mu) \leq l(x^*, \lambda, \mu) \]

(minimum over all values of \( x \) is better than the objective at a sample point \( x^* \))

\[ = f(x^*) + \sum_{i=1}^{m} \lambda_i h_i(x^*) + \sum_{i=1}^{k} \mu_i g_i(x^*) \leq f(x^*) \]

\[ \Rightarrow \quad l(\bar{x}, \lambda, \mu) \leq f(x^*). \]