Theoretical Guarantees for the Design of Near Globally Optimal Static Distributed Controllers

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Abstract— **This paper is concerned with the optimal distributed control problem for linear discrete-time deterministic and stochastic systems. The objective is to design a stabilizing static distributed controller whose performance is close to that of the optimal centralized controller (if such controller exists). In our previous work, we have developed a computational framework to transform centralized controllers into distributed ones for deterministic systems. By building on this result, we derive strong theoretical lower bounds on the optimality guarantee of the designed distributed controllers and show that the proposed mathematical framework indirectly maximizes the derived lower bound while striving to achieve a closed-loop stability. Furthermore, we extend the proposed design method to stochastic systems that are subject to input disturbance and measurement noise. The developed optimization problem has a closed-form solution (explicit formula) and can be easily deployed for large-scale systems that require low computational efforts. The proposed approach is tested on a power network and random systems to demonstrate its efficacy.**

I. INTRODUCTION

The area of optimal distributed control has been created to address computation and communication challenges in the control of large-scale real-world systems. The main objective is to design a high-performance controller with a prescribed structure, as opposed to a traditional centralized controller, for an interconnected system consisting of an arbitrary number of interacting local subsystems. This structurally constrained controller is composed of a set of local controllers associated with different subsystems, which are allowed to interact with one another according to the given control structure [1], [2]. Since it is not possible to design an efficient algorithm to solve this complex problem in its general form unless $P = NP$, several methods have been devoted to solving the optimal distributed control problem for special structures, such as spatially distributed systems, strongly connected systems, and optimal static distributed systems [3]–[7].

Due to the evolving role of convex optimization in solving complex problems, more recent approaches for the optimal distributed control problem have shifted toward a convex reformulation of the problem [8], [9]. This has been carried out in the seminal work [10] by deriving a sufficient condition named quadratic invariance. These conditions have been further investigated in several other papers [11], [12]. Using the graph-theoretic analysis developed in [13] and [14], we have shown in [15] and [16] that a semidefinite programming (SDP) relaxation of the distributed control problem has a low-rank solution for finite- and infinite-time cost functions in both deterministic and stochastic settings. The low-rank SDP solution may be used to find a near-globally optimal distributed controller. Moreover, we have proved in [17] that either a large input weighting matrix or a large noise covariance can convexify the optimal distributed control problem for stable systems. Unfortunately, SDPs and iterative algorithms are often computationally expensive for large-scale problems and it is desirable to develop a computationally-cheap method for designing suboptimal distributed controllers.

Consider the gap between the optimal costs of the optimal centralized and distributed control problems. This gap could be arbitrarily large in practice (as there may not exist a stabilizing controller with the prescribed structure). This paper is focused on systems for which this gap is relatively small. Our methodology is built upon the recent work [18], where we have developed a low-complexity optimization problem to find a distributed controller for a given deterministic system whose corresponding states and inputs are similar to those of the optimal centralized controller.

In this paper, we obtain strong theoretical lower bounds on the optimality guarantee of the controller designed in [18]. This lower bound determines the maximum distance between the performances of the obtained distributed controller and the optimal centralized one. We show that the proposed convex program indirectly maximizes the derived lower bound while striving to achieve a closed-loop stability. We then generalize the results to stochastic systems that are subject to disturbance and measurement noises. We show that these systems benefit from similar lower bounds on optimality guarantee. As another contribution, the subspace of high-performance distributed controllers is explicitly characterized, and an optimization problem is designed to seek a stabilizing controller in the subspace of high-performance and structurally constrained controllers. The efficacy of the developed mathematical framework is demonstrated on a power network and random systems.

Notations: The space of real numbers is denoted by $\mathbb{R}$. The symbols $\mathrm{trace}\{W\}$ and $\mathrm{null}\{W\}$ denote the trace and the null space of a matrix W, respectively. I_m denotes the

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This work was supported by the ONR YIP Award, DARPA Young Faculty Award, NSF CAREER Award, and NSF EPCN Award.

identity matrix of dimension m. The symbols $(.)^T$ and $(.)^*$ are used for transpose and Hermitian transpose, respectively. The symbols $\|W\|_2$ and $\|W\|_F$ denote the 2-norm and Frobenius norm of W, respectively. The $(i,j)^{\text{th}}$ entry of a matrix W is shown as $W(i,j)$ or W_{ij}, whereas the i^{th} entry of a vector w is shown as $w(i)$ or w_i. The Frobenius inner product of matrices W_1 and W_2 is denoted as $\langle W_1, W_2\rangle$. The notations M^k and M_k are used to show k^{th} row and k^{th} column of M, respectively. The expected value of a random variable x is shown as $\mathcal{E}\{x\}$. The symbol $\lambda_{\max}(W)$ is used to show the maximum eigenvalue of a symmetric matrix W. The spectral radius of W is the maximum absolute value of the eigenvalues of W and is denoted by $\rho(W)$.

II. Problem Formulation

In this paper, the optimal distributed control (ODC) problem is studied. The objective is to develop a cheap, fast and scalable algorithm for the design of distributed controllers for large-scale systems. It is aimed to obtain a static distributed controller with a pre-determined structure that achieves a high performance compared to the optimal centralized controller.

Definition 1: Define $\mathcal{K} \subseteq \mathbb{R}^{m\times n}$ as a linear subspace with some pre-specified sparsity pattern (enforced zeros in certain entries). A feedback gain belonging to $\mathcal{K}$ is called a **distributed (decentralized) controller** with its sparsity pattern captured by $\mathcal{K}$. In the case of $\mathcal{K} = \mathbb{R}^{m\times n}$, there is no structural constraint imposed on the controller, which is referred to as a **centralized controller**. Throughout this paper, we use the notations K_c, K_d, and K to show an optimal centralized controller gain, a designed (near-globally optimal) distributed controller gain, and a variable controller gain (serving as a variable of an optimization problem), respectively.

In this work, we will study two versions of the ODC problem, which are stated below.

Infinite-horizon deterministic ODC problem: Consider the discrete-time system

$$x[\tau+1] = Ax[\tau] + Bu[\tau], \qquad \tau = 0, 1, ..., \infty \tag{1}$$

with the known matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $x[0] \in \mathbb{R}^n$. The objective is to design a stabilizing static controller $u[\tau] = Kx[\tau]$ to satisfy certain optimality and structural constraints. Associated with the system (1) under an arbitrary controller $u[\tau] = Kx[\tau]$, we define the following cost function for the closed-loop system:

$$J(K) = \sum_{\tau=0}^{\infty} \left(x[\tau]^T Q x[\tau] + u[\tau]^T R u[\tau] \right) \tag{2}$$

where Q and R are constant positive-definite matrices of appropriate dimensions. Assume that the pair (A, B) is stabilizable. The minimization problem of

$$\min_{K\in\mathbb{R}^{m\times n}} J(K) \tag{3}$$

subject to (1) and the closed-loop stability condition is an optimal centralized control problem and the optimal controller gain can be obtained from the Riccati equation. However, if there is an enforced sparsity pattern on the controller via the linear subspace $\mathcal{K}$, the additional constraint $K \in \mathcal{K}$ should be added to the optimal centralized control problem, and it is well-known that Riccati equations cannot be used to find an optimal distributed controller in general. We refer to this problem as the infinite-horizon deterministic ODC problem.

Infinite-horizon stochastic ODC problem: Consider the discrete-time system

$$\begin{cases} x[\tau+1] = Ax[\tau] + Bu[\tau] + Ed[\tau] \\ y[\tau] = x[\tau] + Fv[\tau] \end{cases} \qquad \tau = 0, 1, 2, ... \tag{4}$$

where A, B, E, F are constant matrices, and $d[\tau]$ and $v[\tau]$ denote the input disturbance and measurement noise, respectively. Furthermore, $y[\tau]$ is the noisy state measured at time τ. Associated with the system (4) under an arbitrary controller $u[\tau] = Ky[\tau]$, consider the cost functional

$$J(K) = \lim_{\tau\to+\infty} \mathcal{E}\left\{ x[\tau]^T Q x[\tau] + u[\tau]^T R u[\tau] \right\} \tag{5}$$

The infinite-horizon stochastic ODC aims to minimize the above objective function for the system (4) with respect to a stabilizing distributed controller K belonging to $\mathcal{K}$.

Finding an optimal distributed controller with a pre-defined structure is NP-hard and intractable in its worst case. Therefore, we seek to find a near-globally optimal distributed controller.

Definition 2: Consider the deterministic system (1) with the performance index (2) or the stochastic system (4) with the performance index (5). Given a matrix $K_d \in \mathcal{K}$ and a percentage number $\mu \in [0, 100]$, it is said that the distributed controller $u[\tau] = K_d x[\tau]$ has the global optimality guarantee of μ if

$$\frac{J(K_c)}{J(K_d)} \times 100 \geq \mu \tag{6}$$

To understand Definition 2, if μ is equal to 90% for instance, it means that the distributed controller $u[\tau] = K_d x[\tau]$ is at most 10% worse than the best (static) centralized controller with respect to the cost function (2) or (5). It also implies that if there exists a better static distributed controller, it outperforms K_d by at most a factor of 0.1. This paper aims to address two problems.

Objective 1) Distributed Controller Design: Given the deterministic system (1) or the stochastic system (4), find a distributed controller $u[\tau] = K_d x[\tau]$ such that

i) The design procedure for obtaining K_d is based on a simple formula with respect to K_c, rather than solving an optimization problem.
ii) The controller $u[\tau] = K_d x[\tau]$ has a high global optimality guarantee.
iii) The system (1) is stable under the controller $u[\tau] = K_d x[\tau]$.

Objective 2) Minimum number of required communications: Given the optimal centralized controller K_c, find the minimum number of free (nonzero) elements in the sparsity patterns imposed by $\mathcal{K}$ that is required to ensure the existence of a stabilizing controller K_d with a high global optimality guarantee.

III. Distributed Controller Design: Deterministic Systems

In this section, we study the design of static distributed controllers for deterministic systems. We consider two criteria in order to design a distributed controller.

Consider the optimal centralized controller $u[\tau] = K_c x[\tau]$ and an arbitrary distributed controller $u[\tau] = K_d x[\tau]$. Let $x_c[\tau]$ and $u_c[\tau]$ denote the state and input of the system (1) under the centralized controller. Likewise, define $x_d[\tau]$ and $u_d[\tau]$ as the state and input of the system (1) under the distributed controller. Consider the following optimization problem.

Optimization A. This problem is defined as

$$\min_{K} \ \text{trace}\left\{(K_c - K)P(K_c - K)^T\right\} \qquad (7a)$$

$$\text{s.t.} \ \ K \in \mathcal{K} \qquad (7b)$$

where the symmetric positive-semidefinite matrix $P \in \mathbb{R}^{n\times n}$ is the unique solution of the Lyapunov equation

$$(A + BK_c)P(A + BK_c)^T - P + x[0]x[0]^T = 0 \qquad (8)$$

The next theorem explains how this optimization problem can be used to study the analogy of the centralized and distributed control systems in terms of their state and input trajectories.

Theorem 1: Given the optimal centralized gain K_c, an arbitrary gain $K_d \in \mathcal{K}$ and the initial state $x[0]$, the relations

$$u_c[\tau] = u_d[\tau], \qquad \tau = 0, 1, 2, ... \qquad (9a)$$

$$x_c[\tau] = x_d[\tau], \qquad \tau = 0, 1, 2, ... \qquad (9b)$$

hold if and only if the optimal objective value of Optimization A is zero and K_d is a minimizer of this problem.

Proof: Please refer to [18]. ■

Theorem 1 states that if the optimal objective value of Optimization A is equal to 0, then there exists a distributed controller $u_d[\tau] = K_d x_d[\tau]$ with the structure induced by $\mathcal{K}$ whose global optimality guarantee is 100%. Roughly speaking, a small optimal value for Optimization A implies that the centralized and distributed control systems can become close to each other. This statement will be formalized later in this work.

The equations given in Theorem 1 do not necessarily ensure the stability of the distributed closed-loop system. In fact, it follows from the above discussion that whenever the centralized and distributed control systems have the same input and state trajectories, $x[0]$ resides in the stable manifold of the system $x[\tau + 1] = (A + BK_d)x[\tau]$, but the closed-loop system is not necessarily stable. To address this issue, we incorporate a regularization term into the objective of Optimization A to account for the stability of the system.

Optimization B. Given a constant number $\alpha \in [0, 1]$, this problem is defined as the minimization of the function

$$C(K) = \alpha \times C_1(K) + (1 - \alpha) \times C_2(K) \qquad (10)$$

with respect to the matrix variable $K \in \mathcal{K}$, where

$$C_1(K) = \text{trace}\left\{(K_c - K)P(K_c - K)^T\right\} \qquad (11a)$$

$$C_2(K) = \text{trace}\left\{(K_c - K)^T B^T B(K_c - K)\right\} \qquad (11b)$$

Note that $C_1(K)$ accounts for the performance of the distributed controller and $C_2(K)$ indirectly enforces a closed-loop stability. Furthermore, Optimization B has an explicit formula for its solution, which can be obtained via a system of linear equations. The details may be found in [18].

A. Lower Bound on Optimality Guarantee

So far, a convex optimization problem has been designed whose explicit solution produces a distributed controller with the right sparsity pattern such that it yields the same performance as the optimal centralized controller if the optimal objective value of this optimization problem is zero. Then, it has been argued that if the objective value is not zero but small enough at optimality, then its corresponding distributed controller may have a high optimality guarantee. In this section, this statement will be formalized by finding a lower bound on the global optimality guarantee of the designed distributed controller. In particular, it is aimed to show that this lower bound is in terms of the value of $C_1(K_d)$ in (10), and that a small $C_1(K_d)$ translates into a high optimality guarantee. To this end, we first derive an upper bound on the deviation of the state and input trajectories generated by the distributed controller from those of the centralized controller.

Lemma 1: Consider the optimal centralized gain K_c and an arbitrary stabilizing gain $K_d \in \mathcal{K}$ for which $A + BK_d$ is diagonalizable. The relations

$$\sum_{\tau=0}^{\infty} \|x_d[\tau] - x_c[\tau]\|_2^2 \leq \left(\frac{\kappa(V)\|B\|_2}{1 - \rho(A + BK_d)}\right)^2 C_1(K_d) \qquad (12a)$$

$$\sum_{\tau=0}^{\infty} \|u_d[\tau] - u_c[\tau]\|_2^2 \leq \left(1 + \frac{\kappa(V)\|K_d\|_2\|B\|_2}{1 - \rho(A + BK_d)}\right)^2 C_1(K_d) \qquad (12b)$$

hold, where $\kappa(V)$ is the condition number in 2-norm of the eigenvector matrix V of $A + BK_d$.

Proof: Please refer to [19]. ■

An important observation can be made on the connection between Optimization B and the upper bounds in (12a) and (12b). Note that Optimization B minimizes a combination of $C_1(K)$ and $C_2(K)$. While the second term indirectly accounts for stability, the first term $C_1(K)$ directly appears in the upper bounds in (12a) and (12b). Hence, Optimization B aims to minimize the deviation between the trajectories of the distributed and centralized control systems.

Theorem 2: Assume that $Q = I_n$ and $R = I_m$. Given the optimal centralized gain K_c and an arbitrary stabilizing gain $K_d \in \mathcal{K}$ for which $A + BK_d$ is diagonalizable, the relations

$$\left(1 + \mu\sqrt{C_1(K_d)}\right)^2 J(K_c) \geq J(K_d) \geq J(K_c) \tag{13}$$

hold, where

$$\mu = \max\left\{ \frac{\kappa(V)\|B\|_2}{(1-\rho(A+BK_d))\sqrt{\sum_{\tau=0}^{\infty}\|x_c[\tau]\|_2^2}}, \frac{1-\rho(A+BK_d)+\kappa(V)\|K_d\|_2\|B\|_2}{(1-\rho(A+BK_d))\sqrt{\sum_{\tau=0}^{\infty}\|u_c[\tau]\|_2^2}} \right\} \tag{14}$$

Proof: According to Lemma 1, one can write:

$$\sum_{\tau=0}^{\infty}\|x_d[\tau]\|_2^2 + \sum_{\tau=0}^{\infty}\|x_c[\tau]\|_2^2 \leq \left(\frac{\kappa(V)\|B\|_2}{1-\rho(A+BK_d)}\right)^2 C_1(K_d) + 2\sum_{\tau=0}^{\infty}\|x_c[\tau]\|_2\|x_d[\tau]\|_2 \tag{15}$$

Dividing both sides of (15) by $\sum_{\tau=0}^{\infty}\|x_c[\tau]\|_2^2$ and using the Cauchy-Schwarz inequality for $\sum_{\tau=0}^{\infty}\|x_c[\tau]\|_2\|x_d[\tau]\|_2$ yield that

$$\frac{\sum_{\tau=0}^{\infty}\|x_d[\tau]\|_2^2}{\sum_{\tau=0}^{\infty}\|x_c[\tau]\|_2^2} \leq \left(1 + \frac{\kappa(V)\|B\|_2}{(1-\rho(A+BK_d))\sqrt{\sum_{\tau=0}^{\infty}\|x_c[\tau]\|_2^2}}\sqrt{C_1(K_d)}\right)^2 \tag{16}$$

Likewise,

$$\frac{\sum_{\tau=0}^{\infty}\|u_d[\tau]\|_2^2}{\sum_{\tau=0}^{\infty}\|u_c[\tau]\|_2^2} \leq \left(1 + \frac{1-\rho(A+BK_d)+\kappa(V)\|K_d\|_2\|B\|_2}{(1-\rho(A+BK_d))\sqrt{\sum_{\tau=0}^{\infty}\|u_c[\tau]\|_2^2}}\sqrt{C_1(K_d)}\right)^2 \tag{17}$$

Combining (16) and (17) leads to

$$\frac{J(K_d)}{J(K_c)} = \frac{\sum_{\tau=0}^{\infty}\|x_d[\tau]\|_2^2 + \sum_{\tau=0}^{\infty}\|u_d[\tau]\|_2^2}{\sum_{\tau=0}^{\infty}\|x_c[\tau]\|_2^2 + \sum_{\tau=0}^{\infty}\|u_c[\tau]\|_2^2} \leq (1 + \mu\sqrt{C_1(K_d)})^2 \tag{18}$$

This completes the proof. ■

Notice that whenever the optimal solution of Optimization B does not satisfy the equation $C_1(K_d) = 0$, Theorem 1 cannot be used to show the equivalence of the distributed and centralized controllers. Instead, Theorem 2 quantifies the similarity between the two control systems. It also states that one may find a distributed controller with a high performance guarantee by minimizing the objective of Optimization B. More precisely, it follows from (13) that

$$\frac{J(K_c)}{J(K_d)} \geq \frac{1}{(1 + \mu\sqrt{C_1(K_d)})^2} \tag{19}$$

Since a small $C_1(K_d)$ in (19) results in a high optimality guarantee for the designed distributed controller, this theorem justifies why it is beneficial to minimize (10), which in turn minimizes $C_1(K_d)$ while striving to find a stabilizing controller. Another implication of Theorem 4 is as follows: if there exists a better linear static distributed controller with the given structure, it outperforms K_d by at most a factor of $(1 + \mu\sqrt{C_1(K_d)})^2$.

It is worthwhile to mention that the bounds in Theorem 2 are derived to substantiate the reason behind the minimization of $C(K)$ in Optimization B for the controller design. However, these bounds are rather conservative compared to the actual performance of the designed distributed controller K_d. It will be shown through simulations that the actual optimality guarantee is high in several examples. Finding tighter bounds on the optimality guarantee is left as future work. Furthermore, One may speculate that there is no guarantee that the parameter μ remains small if $C_1(K)$ is minimized via Optimization C. However, note that μ is implicitly controlled by the term $C_2(K)$ in the objective function of Optimization C.

Remark 1: Notice that Theorem 2 is developed for the case of $Q = I_n$ and $R = I_m$. However, its proof can be adopted to derive similar bounds for the general case. Alternatively, for arbitrary positive-definite matrices Q and R, one can transform them into identity matrices through a reformulation of the ODC problem in order to use the bounds in Theorem 2. Define Q_d and R_d as $Q = Q_d^T Q_d$ and $R = R_d^T R_d$, respectively. The ODC problem with the tuple $(A, B, x[\cdot], u[\cdot])$ can be reformulated with respect to a new tuple $(\bar{A}, \bar{B}, \bar{x}[\cdot], \bar{u}[\cdot])$ defined as

$$\bar{A} \triangleq Q_d A Q_d^{-1}, \quad \bar{B} \triangleq Q_d B R_d^{-1},$$
$$\bar{x}[\tau] \triangleq Q_d x[\tau], \quad \bar{u}[\tau] \triangleq R_d u[\tau],$$

B. Sparsity Pattern

Consider a general discrete Lyapunov equation

$$MPM^T - P + HH^T = 0 \tag{20}$$

for constant matrices M and H. It is well known that if M is stable, the above equation has a unique positive semidefinite solution P. Extensive amount of work has been devoted to describing the behavior of the eigenvalues of the solution of (20) whenever HH^T has a low rank [20]–[22]. These results show that if HH^T possesses a small rank compared to the size of P, the eigenvalues of P satisfying (20) would tend to decay quickly. Supported by this explanation, one can notice that since $x[0]x[0]^T$ has rank 1 in the Lyapunov equation (8), the matrix P tends to have a few dominant eigenvalues. To illustrate this property, we will later show that only 15% of the eigenvalues of P are dominant for certain random highly-unstable systems. In the extreme case, if the closed-loop matrix $A + BK_c$ is 0 (the most stable discrete system) or alternatively $x[0]$ is chosen to be one of the eigenvectors of $A + BK_c$, the matrix P becomes rank-1.

On the other hand, Theorem 1 states that there exists a distributed controller with the global optimality guarantee of 100% if the optimal objective value of Optimization A is zero. In [18], we have shown that if the number of free elements of K_d is higher than the number of dominant eigenvalues of P, this optimal value becomes approximately zero. Given a natural number r, let $\hat{P}$ denote a rank-r approximation of P that is obtained by setting the $n-r$ smallest eigenvalues of P equal to zero in its eigenvalue decomposition.

Remark 2: There is a controller K_d whose global optimality degree is close to 100% if the number of free elements in each row of K_d is greater than or equal to the approximate rank of P (i.e., the number of clearly dominant eigenvalues). If the degree of freedom of K_d in each row is higher than r, then there are infinitely many distributed controllers with a high optimality degree, and then the chance of existence of a stabilizing controller among those candidates would be higher.

Motivated by Remark 2, we introduce an optimization problem next.

Optimization C. The problem is defined as

$$\min_{K} \ \text{trace}\left\{(K_c-K)B^TB(K_c-K)^T\right\} \tag{21a}$$

$$\text{s.t.} \ K\in\mathcal{K} \tag{21b}$$

$$(K_c-K)W=0 \tag{21c}$$

where W is an $n\times r$ matrix whose columns are those eigenvectors of P corresponding to its r dominant eigenvalues.

Optimization C aims to make the closed-loop system stable while imposing a constraint on the performance of the distributed controller. In particular, the designed optimization problem searches for a stabilizing distributed controller in the subspace of high-performance controllers with the prescribed sparsity pattern. We show that Optimization C indeed has a closed-form solution. More precisely, one can derive an explicit formula for K_d through a system of linear equations.

For each row i of K_d, let r_i denote the number of free elements at row i. Since the rank of $\hat{P}$ is equal to r, in order to assure that the system of equations (21c) is not overdetermined, assume that $r_i\geq r$ for every $i\in\{1,2,...,m\}$. Furthermore, define l_i^j as a 0-1 row vector of size n such that $l_i^j(k)=1$ if the j^{th} free element of row i in K_d resides in column k of K_d and $l_i^j(k)=0$ otherwise, for every $j\in\{1,2,...,r_i\}$. Define

$$l_i=\left[{l_i^1}^T,{l_i^2}^T,...,{l_i^{r_i}}^T\right]^T \tag{22}$$

The set of all permissible vectors for the i^{th} row of K_d can be characterized in terms of the left null space of l_iW and an initial vector as

$$K_d^i=K_0^i+\beta_i\ \text{null}\{l_iW\}\ l_i \tag{23}$$

where β_i is an arbitrary row vector with size equal to the number of rows in $\text{null}\{l_iW\}$ and K_0^i is equal to

$$K_0^i=K_c^iW(\bar{l}_iW)^{-1}\bar{l}_i \tag{24}$$

where

$$\bar{l}_i=\left[{l_i^1}^T,{l_i^2}^T,...,{l_i^{r}}^T\right]^T \tag{25}$$

Therefore, the set of permissible distributed controllers with the structure imposed by (21) can be characterized as

$$\hat{\mathcal{K}}=\{K_0+\beta N|\beta\in\mathcal{B}\} \tag{26}$$

where

$$K_0=\left[{K_0^1}^T,{K_0^2}^T,...,{K_0^m}^T\right]^T \tag{27a}$$

$$N=\left[l_1^T\text{null}\{l_1W\}^T,...,l_m^T\text{null}\{l_mW\}^T\right] \tag{27b}$$

and $\mathcal{B}$ is the set of all matrices in the form of

$$\begin{bmatrix} \beta_1 & 0 & \cdots & 0 \\ 0 & \beta_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & \beta_m \end{bmatrix} \tag{28}$$

for arbitrary vectors β_i with size equal to the number of rows in $\text{null}\{l_iW\}$. Denote $M_1,...,M_l$ as 0-1 matrices such that, for every $t\in\{1,2,...,l\}$, $M_t(i,j)$ is equal to 1 if (i,j) is the location of the t^{th} element of the vector $[\beta_1,\beta_2,...,\beta_m]^T$ in $\mathcal{B}$ and is 0 otherwise.

Theorem 3: Consider the matrix $X\in\mathbb{R}^{l\times l}$ and vector $\mathbf{y}\in\mathbb{R}^l$ with the entries

$$X(i,j)=\text{trace}\left\{N^TM_i^TB^TBM_jN\right\} \tag{29a}$$

$$\mathbf{y}(i)=\text{trace}\left\{N^TM_i^TB^TB(K_c-K_0)\right\} \tag{29b}$$

for every $i,j\in\{1,2,...,l\}$. The matrix K_d is an optimal solution of Optimization C if and only if it can be expressed as $K_d=K_0+\beta N$, where β is defined as the matrix in (28) and the parameters $\beta_1,\beta_2...,\beta_m$ satisfy the linear equation $X[\beta_1,\beta_2,...,\beta_m]^T=\mathbf{y}$.

Proof: The proof follows by applying KKT conditions to Optimization C and incorporating the set of permissible controllers (26). The details are omitted for brevity. ■

IV. Distributed Controller Design: Stochastic Systems

In this section, the results developed earlier will be generalized to stochastic systems. For the input disturbance and measurement noise, define the covariance matrices

$$\Sigma_d=\mathcal{E}\left\{Ed[\tau]d[\tau]^TE^T\right\},\ \Sigma_v=\mathcal{E}\left\{Fv[\tau]v[\tau]^TF^T\right\} \tag{30}$$

for all $\tau\in\{0,1,...,\infty\}$. It is assumed that $d[\tau]$ and $v[\tau]$ are identically distributed and independent random vectors with Gaussian distribution and zero mean for all times τ. Let K_c denote the gain of the optimal static centralized controller $u[\tau]=K_cy[\tau]$ minimizing (5) for the stochastic system (4). Note that if $F=0$, the matrix K_c can be found using the Riccati equation. The goal is to design a stabilizing distributed controller $u[\tau]=K_dy[\tau]$ with a high global optimality guarantee such that $K_d\in\mathcal{K}$. For an arbitrary discrete-time random process $a[\tau]$ with $\tau\in\{0,1,...,\infty\}$, denote $\lim_{\tau\to+\infty}a[\tau]$ with the random variable $a[\infty]$ if the limit exists. Note that the closeness of the random tuples $(u_c[\infty],x_c[\infty])$ and $(u_d[\infty],x_d[\infty])$ is sufficient to guarantee that the centralized and distributed controllers lead to similar performances. This is due to the fact that only the limiting behaviors of the states and inputs determine the objective value of the optimal control problem in (5). Hence, it is not necessarily for the centralized and distributed control systems to have similar trajectories for states and inputs at all times as long as they have similar limiting behaviors.

In order to guarantee that the objective value (5) for the distributed control system is close to that of the centralized control system, the second moments of $x_c[\infty]$ and $x_d[\infty]$ should be similar. To this end, we propose an optimization problem to indirectly minimize $\mathcal{E}\left\{\|x_c[\infty]-x_d[\infty]\|_2^2\right\}$. Then, analogous to the deterministic scenario, we aim to show that a small value for a convex surrogate of $\mathcal{E}\left\{\|x_c[\infty]-x_d[\infty]\|_2^2\right\}$ leads to similar objective values for the distributed and centralized controllers.

Lemma 2: Given arbitrary stabilizing controllers K_c and K_d, the relation

$$\mathcal{E}\left\{\|x_c[\infty]-x_d[\infty]\|_2^2\right\} = \operatorname{trace}\{P_1+P_2-P_3-P_4\} \quad (31)$$

holds, where P_1, P_2, P_3 and P_4 are the unique solutions of the equations

$$(A+BK_d)P_1(A+BK_d)^T - P_1 + \Sigma_d + (BK_d)\Sigma_v(BK_d)^T = 0 \quad (32a)$$

$$(A+BK_c)P_2(A+BK_c)^T - P_2 + \Sigma_d + (BK_c)\Sigma_v(BK_c)^T = 0 \quad (32b)$$

$$(A+BK_d)P_3(A+BK_c)^T - P_3 + \Sigma_d + (BK_d)\Sigma_v(BK_c)^T = 0 \quad (32c)$$

$$(A+BK_c)P_4(A+BK_d)^T - P_4 + \Sigma_d + (BK_c)\Sigma_v(BK_d)^T = 0 \quad (32d)$$

Proof: Please refer to [19]. ■

Note that (32a) and (32b) are Lyapunov equations, whereas (32c) and (32d) are Stein equations. These equations all have unique solutions if $A+BK_d$ and $A+BK_c$ are stable. Lemma 2 implies that in order to minimize $\mathcal{E}\left\{\|x_c[\infty]-x_d[\infty]\|_2^2\right\}$, the trace of $P_1+P_2-P_3-P_4$ should be minimized subject to (32) and $K_d \in \mathcal{K}$. However, this is a hard problem in general. In particular, the minimization of the singleton $\operatorname{trace}\{P_1\}$ subject to (32a) and $K_d \in \mathcal{K}$ is equivalent to the ODC problem under study. Due to the possible intractability of the minimization of $\mathcal{E}\left\{\|x_c[\infty]-x_d[\infty]\|_2^2\right\}$, we aim to minimize an upper bound on this function (similar to the deterministic case).

In what follows, we will propose an optimization problem as the counterpart of Optimization B for stochastic systems.

Optimization D: Given a constant number $\alpha \in [0,1]$, this problem is defined as the minimization of the function

$$C_s(K) = \alpha \times C_1^s(K) + (1-\alpha) \times C_2^s(K) \quad (33)$$

with respect to the matrix variable $K \in \mathcal{K}$, where P_s is the unique solution to

$$\begin{aligned}(A+BK_c)P_s(A+BK_c)^T - P_s \\ + \Sigma_d + (BK_c)\Sigma_v(BK_c)^T = 0\end{aligned} \quad (34)$$

and

$$C_1^s(K) = \operatorname{trace}\left\{(K_c-K)(\Sigma_v+P_s)(K_c-K)^T\right\} \quad (35a)$$

$$C_2^s(K) = \operatorname{trace}\left\{(K_c-K)^T B^T B(K_c-K)\right\} \quad (35b)$$

It is straightforward to observe that the explicit solution derived for Optimization B in [18] can be adopted to find an explicit formula for all solutions of Optimization D.

Lemma 3: Consider the optimal centralized gain K_c and an arbitrary stabilizing gain $K_d \in \mathcal{K}$ for which $A+BK_d$ is diagonalizable. The relations

$$\mathcal{E}\left\{\|x_c[\infty]-x_d[\infty]\|_2^2\right\} \leq \left(\frac{\kappa(V)\|B\|_2}{1-\rho(A+BK_d)}\right)^2 C_1^s(K_d) \quad (36a)$$

$$\mathcal{E}\left\{\|u_c[\infty]-u_d[\infty]\|_2^2\right\} \leq \left(1+\frac{\kappa(V)\|K_d\|_2\|B\|_2}{1-\rho(A+BK_d)}\right)^2 C_1^s(K_d) \quad (36b)$$

hold, where $\kappa(V)$ is the condition number in 2-norm of the eigenvector matrix V of $A+BK_d$.

Proof: Using Lemma 2, one can verify that $\mathcal{E}\{(x_c[\infty]-x_d[\infty])(x_c[\infty]-x_d[\infty])^T\}$ converges to a finite constant matrix. The proof follows from this fact by adopting the argument made in the proof of Lemma 1. ■

In what follows, the counterpart of Theorem 2 will be presented for stochastic systems.

Theorem 4: Assume that $Q = I_n$ and $R = I_m$. Given the optimal centralized gain K_c and an arbitrary stabilizing gain $K_d \in \mathcal{K}$ for which $A+BK_d$ is diagonalizable, the relations

$$\left(1+\mu_s\sqrt{C_1^s(K_d)}\right)^2 J(K_c) \geq J(K_d) \geq J(K_c) \quad (37)$$

hold, where

$$\mu_s = \max\left\{\frac{\kappa(V)\|B\|_2}{(1-\rho(A+BK_d))\sqrt{\mathcal{E}\{\|x_c[\infty]\|_2^2\}}}, \frac{1-\rho(A+BK_d)+\kappa(V)\|K_d\|_2\|B\|_2}{(1-\rho(A+BK_d))\sqrt{\mathcal{E}\{\|u_c[\infty]\|_2^2\}}}\right\} \quad (38)$$

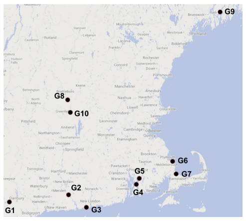
Proof: The proof is a consequence of Lemma 3 and the argument made in the proof of Theorem 2. ■

It can be inferred from Theorem 4 that Optimization D aims to indirectly maximize the optimality guarantee and assure stability.

Remark 3: It is well-known that finding the optimal centralized controller at the presence of noise is intractable in its worst case. However, using the method we introduced in [16], one can design a near-globally optimal centralized controller based on SDP relaxation and use that as a substitute for K_c in Optimization D (in order to design a distributed controller that performs similarly to the near-globally optimal centralized controller). Furthermore, the objective value of this SDP relaxation serves as a lower bound on the cost of the optimal centralized controller and can be deployed to find the optimality guarantee of the designed distributed controller.

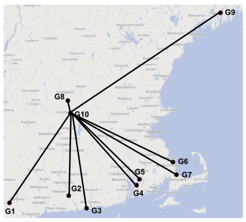
V. Numerical Results

Two examples will be offered in this section to demonstrate the efficacy of the proposed controller design technique.

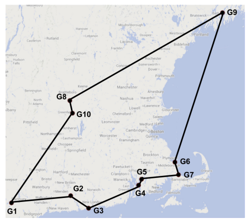


(a) Fully Distributed

(b) Localized



(c) Star Topology (G10 in center)



(d) Ring

Fig. 1: Communication structures studied in Example 1 for IEEE 39-Bus test System

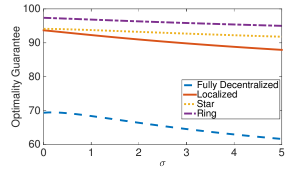
A. Example 1: Power Networks

In this example, the objective is to design a distributed controller for the primary frequency control of a power network. The system under investigation is the IEEE 39-Bus New England test System [23]. The state-space model of this system, after linearizing the swing equations, can be described as

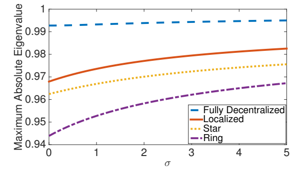
$$\dot{x}(\tau) = A_c x(\tau) + B_c u(\tau) \tag{39}$$

where $A_c \in \mathbb{R}^{20\times 20}$, $B_c \in \mathbb{R}^{20\times 10}$, and $x(\tau)$ contains the rotor angles and frequencies of the 10 generators in the system (see [24] for the details of this model). The input of the system is the mechanical power applied to each generator. The goal is to first discretize the system with the sampling time of 0.2 second, and then design a distributed controller to stabilize the system while achieving a high degree of optimality. We consider four different topologies for the structure of the controller: distributed, localized, star and ring. A visual illustration of these topologies is provided in Figure 1, where each node represents a generator and each line specifies what generators are allowed to communicate.

Suppose that the power system is subject to input disturbance and measurement noise. We assume that $\Sigma_d = I_n$ and $\Sigma_v = \sigma I_n$, with σ varying from 0 to 5. The matrices Q and R are selected as I_n and $0.1 \times I_m$, respectively, and α is chosen as 0.4. The simulation results are provided in Figure 2. It can be observed that the designed controllers are all stabilizing with no exceptions. Moreover, the global optimality guarantees for the ring, star, localized, and fully distributed topologies are above 95%, 91.8%, 88%, and 61.7%, respectively. Note that the optimality guarantee of the system using the designed fully decentralized controller is relatively low. This may be due to the possibly large gap between the optimal costs of the optimal centralized and fully decentralized controllers.



(a) Optimality Guarantee



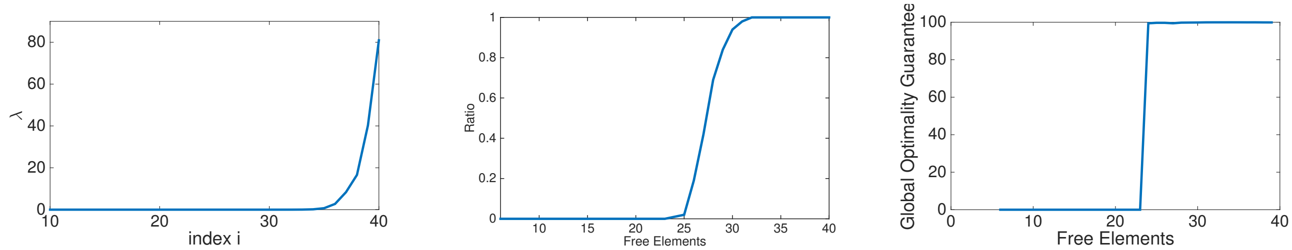
(b) Maximum Absolute Eigenvalues

Fig. 2: Global optimality guarantee and maximum absolute eigenvalue of $A + BK_d$ for four different topologies and different values of σ, under the assumptions $\Sigma_d = I_n$ and $\Sigma_v = \sigma I_n$.

B. Example 2: Highly-Unstable Random Systems

Consider $n = m = 40$ and $Q = R = I$. We generate random continuous-time systems and then discretize them. The entries of A_c are chosen randomly from a Gaussian distribution with the mean 0 and variance 25. Moreover, the entries of B_c are chosen randomly from a normal distribution. Finally, A and B are obtained by discretizing (A_c, B_c) using the zero-order hold method with the sampling time of 0.1 second.

As mentioned earlier, the eigenvalues of P in the Lyapunov equation (8) may decay rapidly. To support this statement, we compute the eigenvalues of P for 100 random unstable systems that are generated according to the above-mentioned rules. Subsequently, we arrange the absolute eigenvalues of P for each system in ascending order and label them as $\lambda_1, \lambda_2, ..., \lambda_{40}$. For every $i \in \{1, 2, ..., 40\}$, the mean of λ_i for these 100 independent random systems is drawn in Figure 3a (the variance is very low). It can be observed that only 15% of the eigenvalues (6 eigenvalues) are dominant and P can be well approximated by a low-rank matrix. In order to ensure a high optimality guarantee, each row of K_d should have at least 6 free elements. However, as discussed earlier, more free elements may be required to make the system stable via Optimization C. To study the minimum number of free elements needed to achieve a closed-loop stability and a high optimality guarantee, we find distributed controllers using Optimization C for different numbers of free elements at each row of K_d. In all of these random systems, P is approximated by a rank-6 matrix $\hat{P}$. For each sparsity level, we then calculate the percentage of closed-loop systems that become stable using Optimization C. Moreover, their average global optimality guarantee using the designed K_d is also obtained. The results are provided in Figures 3b and 3c. The number of stable closed-loop systems increases quickly as the number of free elements in each row of the distributed controller gain exceeds 25. As mentioned earlier, there could be a non-trivial gap between the minimum number of free elements satisfying the performance criterion and the minimum number of free elements required to make the closed-loop system stable. Furthermore, it can be observed in Figure 3c that the designed distributed controller has an optimality guarantee close to 100% for all stable closed-loop systems.



(a) Absolute eigenvalues of P (b) Percentage of systems (c) Optimality Degree

Fig. 3: The mean of absolute eigenvalues of P, the percentage of the stabilized systems, and their average optimality degree with respect to the number of free elements in each row of K_d for 100 highly-unstable random systems in Example 2.

VI. Conclusions

This paper studies the optimal distributed control problem for linear discrete-time systems. The goal is to design a stabilizing static distributed controller with a pre-defined structure, whose performance is close to that of the best centralized controller. In our previous work, we have designed a simple and easy-to-implement optimization problem whose solution is a near-optimal distributed controller with a pre-specified sparsity structure. In this paper, we derive a theoretical lower bound on the optimality guarantee of the designed distributed control, and prove that a small optimal objective value for this optimization problem brings about a high optimality guarantee for the designed distributed controller. Furthermore, we fully characterize the subspace of high-performance distributed controllers with the right sparsity structure and design an optimization problem that seeks a stabilizing controller within this subspace. The results are then extended to stochastic systems that are subject to input disturbance and measurement noise, and it is shown that these systems benefit from similar lower bounds on optimality guarantee. The proposed method is evaluated on a power network and several random systems.

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