Convex Relaxation for Optimal Power Flow Problem: Mesh Networks

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Abstract—This paper is concerned with a fundamental resource allocation problem for electrical power networks. This problem, named optimal power flow (OPF), is nonconvex due to the nonlinearities imposed by the laws of physics, and has been studied since 1962. We have recently shown that a convex relaxation based on semidefinite programming (SDP) is able to find a global solution of OPF for IEEE benchmark systems, and moreover this technique is guaranteed to work over acyclic (distribution) networks. The present work studies the potential of the SDP relaxation for OPF over cyclic (transmission) networks. Given an arbitrary weakly-cyclic network with cycles of size 3, it is shown that the injection region is convex in the lossless case and that the Pareto front of the injection region is convex in the lossy case. It is also proved that the SDP relaxation of OPF is exact for this type of network. Moreover, it is shown that if the SDP relaxation is not exact for a general mesh network, it would still have a low-rank solution whose rank depends on the structure of the network. Finally, a heuristic method is proposed to recover a rank-1 solution for the SDP relaxation whenever the relaxation is not exact.

I. INTRODUCTION

The optimal power flow (OPF) problem is concerned with finding an optimal operating point of a power system, which minimizes a certain objective function (e.g., power loss or generation cost) subject to network and physical constraints [1], [2]. Due to the nonlinear interrelation among active power, reactive power and voltage magnitude, OPF is described by nonlinear equations and may have a nonconvex/disconnected feasibility region [3]. Since 1962, the nonlinearity of the OPF problem has been studied, and various heuristic and local-search algorithms have been proposed [4], [5], [6].

The paper [7] proposes two methods for solving OPF: (i) to use a convex relaxation based on semidefinite programming (SDP), (ii) to solve the SDP-type Lagrangian dual of OPF. That work shows that the SDP relaxation is exact if and only if the duality gap is zero. More importantly, [7] makes the observation that OPF has a zero duality gap for IEEE benchmark systems with 14, 30, 57, 118 and 300 buses, in addition to several randomly generated power networks. This technique is the first method proposed since the introduction of the OPF problem, which is able to find a provably global solution for practical OPF problems. The SDP relaxation for OPF has attracted much attention due to its ability to find a global solution in polynomial time, and it has been applied to various applications in power systems including: voltage regulation in distribution systems [8], state estimation [9], calculation of voltage stability margin [10], economic

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dispatch in unbalanced distribution networks [11], and power management under time-varying conditions [12].

The paper [13] shows that the SDP relaxation is exact in two cases: (i) for acyclic networks, (ii) for cyclic networks after relaxing the angle constraints (similar result was derived in [14] and [15] for acyclic networks). This exactness was related to the passivity of transmission lines and transformers. A question arises as to whether the SDP relaxation remains exact for mesh (cyclic) networks (without any angle relaxations). To address this problem, the paper [16] shows that the relaxation may not be exact even for a three-bus cyclic network. Motivated by this negative result, we aim to explore the limitations of the SDP relaxation for mesh networks.

In this work, we first consider the three-bus system studied in [16] and prove that the SDP relaxation is exact if the OPF problem is modeled properly. More precisely, we show that there are four (almost) equivalent ways to model the capacity of a power line but only one of these models gives rise to the exactness of the SDP relaxation. We also prove that the relaxation remains exact for weakly-cyclic networks with cycles of size 3. Furthermore, we substantiate that this type of network has a convex injection region in the lossless case and a non-convex injection region with a convex Pareto front in the lossy case. The importance of this result is that the SDP relaxation works on certain cyclic networks, for example the ones generated from three-bus subgraphs (this type of network is related to three-phase systems).

In the case when the SDP relaxation does not work, an upper bound is provided on the rank of the minimum-rank solution of the SDP relaxation. This bound is related only to the structure of the power network and this number is expected to be very small for real-world power networks. Finally, a heuristic method is proposed to enforce the SDP relaxation to produce a rank-1 solution for general networks (by somehow killing the undesirable eigenvalues of the low-rank solution).

Notations: $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{S}^n_+$ and $\mathbb{H}^n_+$ denote the sets of real numbers, positive real numbers, $n \times n$ positive semidefinite symmetric matrices, and $n \times n$ positive semidefinite Hermitian matrices, respectively. $\text{Re}\{M\}$, $\text{Im}\{M\}$, $\text{rank}\{M\}$ and $\text{trace}\{M\}$ denote the real part, imaginary part, rank and trace of a given scalar/matrix M, respectively. The notation $M \succeq 0$ means that M is Hermitian and positive semidefinite. The notation $\angle x$ denotes the angle of a complex number x. The notation "i" is reserved for the imaginary unit. The symbol "$*$" represents the conjugate transpose operator. Given a matrix $\mathbf{W}$, its (l, m) entry is denoted as W_{lm}. The superscript $(\cdot)^{\text{opt}}$ is used to show the optimal value of an optimization parameter.

II. OPTIMAL POWER FLOW

Consider a power network with the set of buses $\mathcal{N} := \{1, 2, ..., n\}$, the set of generator buses $\mathcal{G} \subseteq \mathcal{N}$ and the set of flow lines $\mathcal{L} \subseteq \mathcal{N} \times \mathcal{N}$, where:

- A known constant-power load with the complex value $P_{D_k} + Q_{D_k}i$ is connected to each bus $k \in \mathcal{N}$.
- A generator with an unknown complex output $P_{G_k} + Q_{G_k}i$ is connected to each bus $k \in \mathcal{G}$.
- Each line $(l, m) \in \mathcal{L}$ of the network is modeled as an admittance y_{lm} (shunt elements are ignored with no loss of generality).

The goal is to design the unknown outputs of all generators in such a way that the load constraints are satisfied. This resource allocation problem is called *optimal power flow (OPF)*. To formulate this problem, define:

- V_k: Unknown complex voltage at bus $k \in \mathcal{N}$.
- P_{lm}: Unknown active power transferred from bus $l \in \mathcal{N}$ to the rest of the network through the line $(l, m) \in \mathcal{L}$.
- S_{lm}: Unknown complex power transferred from bus $l \in \mathcal{N}$ to the rest of the network through the line $(l, m) \in \mathcal{L}$.
- $f_k(P_{G_k})$: Known convex function representing the generation cost for generator $k \in \mathcal{G}$.

Define $\mathbf{V}$, $\mathbf{P}_G$, $\mathbf{Q}_G$, $\mathbf{P}_D$ and $\mathbf{Q}_D$ as the vectors $\{V_k\}_{k \in \mathcal{N}}$, $\{P_{G_k}\}_{k \in \mathcal{G}}$, $\{Q_{G_k}\}_{k \in \mathcal{G}}$, $\{P_{D_k}\}_{k \in \mathcal{N}}$ and $\{Q_{D_k}\}_{k \in \mathcal{N}}$, respectively. Given the known vectors $\mathbf{P}_D$ and $\mathbf{Q}_D$, OPF minimizes the total generation cost $\sum_{k \in \mathcal{G}} f_k(P_{G_k})$ over the unknown parameters $\mathbf{V}$, $\mathbf{P}_G$ and $\mathbf{Q}_G$ subject to the power balance equations at all buses and some network constraints. To simplify the formulation of OPF, with no loss of generality assume that $\mathcal{G} = \mathcal{N}$. The mathematical formulation of OPF is given in (1), where:

- (1a) and (1b) are the power balance equations accounting for the conservation of energy at bus k.
- (1c), (1d) and (1e) restrict the active power, reactive power and voltage magnitude at bus k, for the given limits $P_k^{\min}, P_k^{\max}, Q_k^{\min}, Q_k^{\max}, V_k^{\min}, V_k^{\max}$.
- Each line of the network is subject to a capacity constraint to be introduced later.
- $\mathcal{N}(k)$ denotes the set of all neighboring nodes of bus $k \in \mathcal{N}$.

A. Convex relaxation for optimal power flow

Regardless of the unspecified capacity constraint, the above formulation of the OPF problem is non-convex due to the nonlinear terms $|V_k|$'s and $V_k V_l^*$'s. Since this problem is NP-hard in the worst case, the paper [7] suggests solving a convex relaxation of OPF. To this end, notice that the constraints of OPF can all be expressed as linear functions of the entries of the quadratic matrix $\mathbf{V}\mathbf{V}^*$. This implies that if the matrix $\mathbf{V}\mathbf{V}^*$ is replaced by a new matrix variable $\mathbf{W} \in \mathbb{H}^n$, then the constraints of OPF become convex in $\mathbf{W}$. Since $\mathbf{W}$ plays the role of $\mathbf{V}\mathbf{V}^*$, two constraints must be added to the reformulated OPF problem in order to preserve the equivalence of the two formulations: (i) $\mathbf{W} \succeq 0$, (ii) rank$\{\mathbf{W}\} = 1$. Observe that Constraint (ii) is the only non-convex constraint

of the reformulated OPF problem. Motivated by this fact, the SDP relaxation of OPF is defined as the OPF problem reformulated in terms of $\mathbf{W}$ under the additional constraint $\mathbf{W} \succeq 0$, which is given in (2). If the SDP relaxation gives rise to a rank-1 solution $\mathbf{W}^{\text{opt}}$, then it is said that *the relaxation is exact*. The exactness of the SDP relaxation is a desirable property being sought, because it implies the equivalence of the convex SDP relaxation and the non-convex OPF problem.

B. Four types of capacity constraints

In this part, the capacity constraint in the formulation of the OPF problem given in (1) will be specified. Line flows are restricted by thermal limits, stability limits and possibly more constraints. Hence, each line $(l, m) \in \mathcal{L}$ must be associated with a capacity constraint. This constraint can be defined in terms of various quantities, including: (i) active flow P_{lm}, (ii) apparent power S_{lm}, (iii) angle difference $\angle V_l - \angle V_m$, (iv) voltage difference $V_l - V_m$, (v) line current. Notice that (iv) and (v) are equivalent in the context of this work, because each line has been modeled as a simple admittance and therefore $V_l - V_m$ is proportional to the line current. Hence, there are at least four meaningful ways (i)-(iv) to impose a capacity constraint on each line (l, m). These four types of constraints are provided in equation (3) for given upper bounds $\theta_{lm}^{\max} = \theta_{ml}^{\max}$, $P_{lm}^{\max} = P_{ml}^{\max}$, $S_{lm}^{\max} = S_{ml}^{\max}$ and $\Delta V_{lm}^{\max} = \Delta V_{ml}^{\max}$, where θ_{lm} denotes the angle difference $\angle V_l - \angle V_m$. Note that $\theta_{lm}^{\max}$ is considered to be less than $90°$ in this work due to the current practice in power networks.

The capacity constraints given in (3) can all be cast as convex inequalities in $\mathbf{W}$, leading to the reformulated constraints in (4). Therefore, the SDP relaxation of OPF remains convex after adding each of these capacity constraints. Since there are at least four types of capacity constraints, a question arises as to which one should be deployed in practice. To address this problem, observe that:

- If $|V_k|$ is equal to the nominal value 1 per unit for every $k \in \mathcal{N}$, then the capacity constraints (3a)-(3d) are all equivalent. In other words, for any given $\theta_{lm}^{\max}$, the three other limits $P_{lm}^{\max}$, $S_{lm}^{\max}$ and $\Delta V_{lm}^{\max}$ can be chosen in such a way that all four capacity constraints yield the same feasible set for the pair (V_l, V_m) (this will be demonstrated in the next subsection).
- For physically meaningful bounds $V_k^{\min}$ and $V_k^{\max}$ (close to 1), the four capacity constraints in (3) are "approximately" equivalent. In particular, if one of the bounds $\theta_{lm}^{\max}$, $P_{lm}^{\max}$, $S_{lm}^{\max}$ and $\Delta V_{lm}^{\max}$ is given, the remaining three bounds can be conservatively chosen in such a way that three of the capacity constraints imply the last constraint.

The above discussion sheds light on the fact that although there are at least four ways to limit a line flow, these methods are more or less equivalent. Nevertheless, the corresponding constraints in (4) are not necessarily the same in the SDP relaxation of OPF. In fact, this paper aims to show that these capacity constraints behave very differently in the SDP relax-

OPF Problem	SDP Relaxation of OPF

Minimize $\sum_{k\in\mathcal{G}} f_k(P_{G_k})$ over $\mathbf{P}_G$, $\mathbf{Q}_G$, $\mathbf{V}$
Subject to:
 	
  1- A capacity constraint for each line $(l,m)\in\mathcal{L}$
 	
  2- The following constraints for each bus $k\in\mathcal{N}$:
 	
$$P_{G_k} - P_{D_k} = \sum_{l\in\mathcal{N}(k)} \mathrm{Re}\left\{V_k(V_k^* - V_l^*)y_{kl}^*\right\} \quad (1a)$$
$$Q_{G_k} - Q_{D_k} = \sum_{l\in\mathcal{N}(k)} \mathrm{Im}\left\{V_k(V_k^* - V_l^*)y_{kl}^*\right\} \quad (1b)$$
$$P_k^{\min} \le P_{G_k} \le P_k^{\max} \quad (1c)$$
$$Q_k^{\min} \le Q_{G_k} \le Q_k^{\max} \quad (1d)$$
$$V_k^{\min} \le |V_k| \le V_k^{\max} \quad (1e)$$
	Minimize $\sum_{k\in\mathcal{G}} f_k(P_{G_k})$ over $\mathbf{P}_G$, $\mathbf{Q}_G$, $\mathbf{W}\in\mathbb{H}_+^n$
	Subject to:
	
	  1- A convexified capacity constraint for each line
	
	  2- The following constraints for each bus $k\in\mathcal{N}$:
	
	$$P_{G_k} - P_{D_k} = \sum_{l\in\mathcal{N}(k)} \mathrm{Re}\left\{(W_{kk} - W_{kl})y_{kl}^*\right\} \quad (2a)$$
	$$Q_{G_k} - Q_{D_k} = \sum_{l\in\mathcal{N}(k)} \mathrm{Im}\left\{(W_{kk} - W_{kl})y_{kl}^*\right\} \quad (2b)$$
	$$P_k^{\min} \le P_{G_k} \le P_k^{\max} \quad (2c)$$
	$$Q_k^{\min} \le Q_{G_k} \le Q_k^{\max} \quad (2d)$$
	$$(V_k^{\min})^2 \le W_{kk} \le (V_k^{\max})^2 \quad (2e)$$

Capacity constraint for line $(l,m)\in\mathcal{L}$	Convexified capacity constraint for line $(l,m)\in\mathcal{L}$
$$\|\theta_{lm}\| = \|\angle V_l - \angle V_m\| \le \theta_{lm}^{\max} \quad (3a)$$ $$\|P_{lm}\| = \|\mathrm{Re}\left\{V_l(V_l^* - V_m^*)y_{lm}^*\right\}\| \le P_{lm}^{\max} \quad (3b)$$ $$\|S_{lm}\| = \|V_l(V_l^* - V_m^*)y_{lm}^*\| \le S_{lm}^{\max} \quad (3c)$$ $$\|V_l - V_m\| \le \Delta V_{lm}^{\max} \quad (3d)$$	$$\mathrm{Im}\{W_{lm}\} \le \mathrm{Re}\{W_{lm}\}\tan(\theta_{lm}^{\max}) \quad (4a)$$ $$\mathrm{Re}\{(W_{ll} - W_{lm})y_{lm}^*\} \le P_{lm}^{\max} \quad (4b)$$ $$\|(W_{ll} - W_{lm})y_{lm}^*\|^2 \le (S_{lm}^{\max})^2 \quad (4c)$$ $$W_{ll} + W_{mm} - W_{lm} - W_{ml} \le (\Delta V_{lm}^{\max})^2 \quad (4d)$$

ation (i.e., after removing the rank constraint $\mathrm{rank}\{W\} = 1$). This surprising result will be elaborated in the next subsection.

C. SDP relaxation for a three-bus network

It has been shown in [16] that the SDP relaxation is not exact for a specific three-bus power network with a triangular topology, provided one line has a very limited capacity. The capacity constraint in [16] has been formulated with respect to apparent power. It is imperative to study the interesting observation made in [16] because if the SDP relaxation cannot handle very simple cyclic networks, its application to mesh networks would be questionable. The result of [16] implies that the SDP relaxation is not necessarily exact for cyclic networks if the capacity constraint (3c) is employed. The high-level objective of this part is to make the surprising observation that the SDP relaxation becomes exact if the capacity constraint (3d) is used instead (this result will be proved later in the paper). To this end, we explore a scenario for which all four types of capacity constraints provided in (3) are equivalent but their convexified counterparts behave very differently. The goal is to show that the SDP relaxation is always exact only for one of these capacity constraints.

Consider the three-bus system depicted in Figure 1(a),

which has been adopted from [16]. The parameters of this cyclic network are provided in Table I. Assume that lines $(1,2)$ and $(2,3)$ have very high capacities, i.e.,

$$\theta_{12}^{\max} = P_{12}^{\max} = S_{12}^{\max} = \Delta V_{12}^{\max} = \infty, \quad (5a)$$
$$\theta_{23}^{\max} = P_{23}^{\max} = S_{23}^{\max} = \Delta V_{23}^{\max} = \infty, \quad (5b)$$

while line $(1,3)$ has a very limited capacity. Since there are four ways to limit the flow over this line, we study four problems, each using only one of the capacity constraints given in (3). To this end, given an angle α belonging to the interval $[0, 90°]$, consider the following limits for these four problems:

$$\text{Problem A}: \quad \theta_{13}^{\max} = \alpha \quad (6a)$$
$$\text{Problem B}: \quad P_{13}^{\max} = \mathrm{Re}\{(1 - e^{\alpha i})y_{13}^*\} \quad (6b)$$
$$\text{Problem C}: \quad S_{13}^{\max} = |(1 - e^{\alpha i})y_{13}^*| \quad (6c)$$
$$\text{Problem D}: \quad \Delta V_{13}^{\max} = \sqrt{2(1 - \cos(\alpha))} \quad (6d)$$

It is straightforward to verify that Problems A-D are equivalent due to the fact that they all lead to the same feasible set for the pair (V_1, V_3). After removing the rank constraint from the OPF problem, these four problems become very distinct. To illustrate this property, we solve four relaxed SDP problems
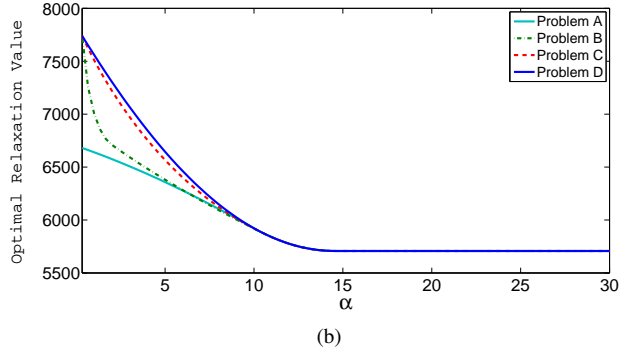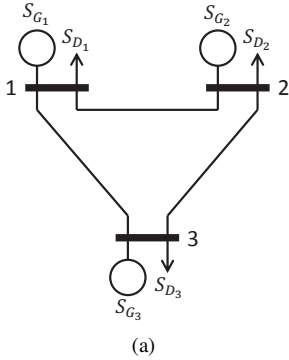
Fig. 1: (a) Three-bus system studied in Section II-C ; (b) Optimal objective value of the SDP relaxation for Problems A-D.

for the network depicted in Figure 1(a), corresponding to the equivalent Problems A-D. Figure 1(b) plots the optimal objective value of each of the four SDP relaxations as a function of α over the period $\alpha \in [0, 30°]$. Let $f^*(\alpha)$ denote the solution of the original OPF problem. Each of the curves in Figure 1(b) is theoretically a lower bound on the function $f^*(\alpha)$ in light of removing the non-convex constraint rank$\{W\} = 1$. A few observations can be made here:

- The SDP relaxation for Problem D yields a rank-1 solution for all values of α. Hence, the curve drawn in Figure 1(b) associated with Problem D represents the function $f^*(\alpha)$, leading to the true solution of OPF.
- The curves for the SDP relaxations of Problems A-C do not overlap with $f^*(\alpha)$ if $\alpha \in (0, 7°)$. Moreover, the gap between these curves and the function $f^*(\alpha)$ is significant for certain values of α.

In summary, three types of capacity constraints make the SDP relaxation inexact in general, while the last type of capacity constraint makes the SDP relaxation always exact. It should be mentioned that the current practice in power systems is to use either Problem B or Problem C, but this example signifies that Problem D is the only one making the SDP relaxation a successful technique.

Based on the methodology developed in [7] and [13], the above result can be interpreted in terms of the duality gap for OPF: *there are four equivalent non-convex formulations of the OPF problem with the property that three of them have a nonzero duality gap in general while the last one always has a zero duality gap.*

The conclusion is that the capacity constraint for mesh networks should be formulated in terms of voltage difference as opposed to active power, apparent power or angle difference.

III. INJECTION REGION FOR WEAKLY-CYCLIC NETWORKS

A power network under operation has a pair of flows (P_{lm}, P_{ml}) over each line $(l, m) \in \mathcal{L}$ and a net injection P_k at each bus $k \in \mathcal{N}$, where P_k is indeed equal to $P_{G_k} - P_{D_k}$. This means that two vectors can be attributed to the network: (i) injection vector $\mathbf{P} = [\ P_1\ P_2\ \cdots\ P_n\ ]$, (ii) flow vector $\mathbf{F} = [P_{lm}|\ (l, m) \in \mathcal{L}]$. Due to the relation $P_k = \sum_{l \in \mathcal{N}(k)} P_{kl}$, there exists a matrix M such that $\mathbf{P} = M \times \mathbf{F}$.

$$f_1(P_{G_1}) \triangleq 0.11 P_{G_1}^2 + 5.0 P_{G_1}$$
$$f_2(P_{G_2}) \triangleq 0.085 P_{G_2}^2 + 1.2 P_{G_2}$$
$$f_3(P_{G_3}) \triangleq 0$$

$Z_{23} = 0.025 + 0.750i,$	$S_{D_1} = 110$ MW
$Z_{31} = 0.065 + 0.620i,$	$S_{D_2} = 110$ MW
$Z_{12} = 0.042 + 0.900i,$	$S_{D_3} = 95$ MW

$$V_k^{\min} = V_k^{\max} = 1 \quad \text{for} \quad k = 1, 2, 3$$
$$(Q_k^{\min}, Q_k^{\max}) = (-\infty, \infty) \quad \text{for} \quad k = 1, 2, 3$$
$$(P_k^{\min}, P_k^{\max}) = (-\infty, \infty) \quad \text{for} \quad k = 1, 2$$
$$P_3^{\min} = P_3^{\max} = 0$$

TABLE I: Parameters of the three-bus system drawn in Figure 1(a) with the base value 100 MVA.

In order to understand the computational complexity of OPF, it is beneficial to explore the feasible set for the injection vector. To this end, two notions of *flow region* and *injection region* will be defined in line with [17].

Definition 1: Define the flow region $\mathcal{F}$ as the set of all flow vectors $\mathbf{F} = [P_{lm}\ |\ (l, m) \in \mathcal{L}]$ for which there exists a voltage phasors vector $[\ V_1\ V_2\ \cdots\ V_n\ ]$ such that

$$P_{lm} = \text{Re}\{V_l(V_l^* - V_m^*)y_{lm}^*\}, \quad (l, m) \in \mathcal{L} \quad (7a)$$
$$|V_l - V_m| \leq \Delta V_{lm}^{\max}, \quad (l, m) \in \mathcal{L} \quad (7b)$$
$$V_k^{\min} \leq |V_k| \leq V_k^{\max}, \quad k \in \mathcal{N} \quad (7c)$$

Define also the injection region $\mathcal{P}$ as $M \cdot \mathcal{F}$.

The above definition of the flow and injection regions captures the laws of physics, capacity constraints and voltage constraints. One can make this definition more comprehensive by incorporating reactive-power constraints.

Definition 2: Define the convexified flow region $\mathcal{F}_c$ as the set of all flow vectors $\mathbf{F} = [P_{lm}\ |\ (l, m) \in \mathcal{L}]$ for which there exists a matrix $\mathbf{W} \in \mathbb{H}_+^n$ such that

$$P_{lm} = \text{Re}\{(W_{ll} - W_{lm})y_{lm}^*\} \quad (8a)$$
$$W_{ll} + W_{mm} - W_{lm} - W_{ml} \leq (\Delta V_{lm}^{\max})^2 \quad (8b)$$
$$(V_k^{\min})^2 \leq W_{kk} \leq (V_k^{\max})^2 \quad (8c)$$

for every $(l, m) \in \mathcal{L}$ and $k \in \mathcal{N}$. Define also the convexified injection region $\mathcal{P}_c$ as $M \cdot \mathcal{F}_c$.

It is straightforward to verify that $\mathcal{P} \subseteq \mathcal{P}_c$ and $\mathcal{F} \subseteq \mathcal{F}_c$.
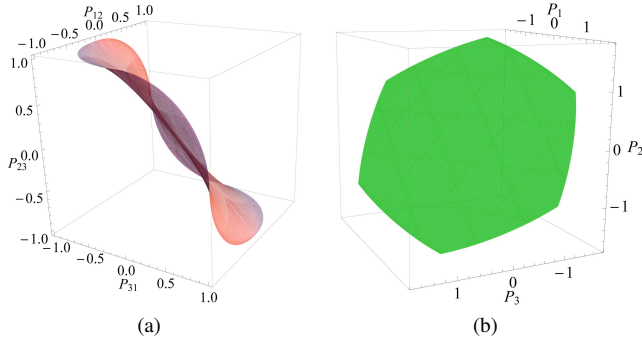
Fig. 2: (a) The reduced flow region $\mathcal{F}^r$ for a three-bus mesh network; (b) The injection region $\mathcal{P}$ for a three-bus mesh network.

A. Lossless cycles

A lossless network has the property that $P_{lm} + P_{ml} = 0$ for every $(l, m) \in \mathcal{L}$, or alternatively $\text{Re}\{y_{lm}\} = 0$. Since real-world transmission networks are very close to being lossless, we study lossless mesh networks here. The flow region $\mathcal{F}$ has been defined in terms of two flows P_{lm} and P_{ml} for each line $(l, m) \in \mathcal{L}$. Due to the relation $P_{ml} = -P_{lm}$ for lossless networks, one can define a reduced flow region $\mathcal{F}^r$ based on one flow P_{lm} for each line (l, m).

The reduced flow region $\mathcal{F}^r$ has been plotted in Figure 2(a) for a cyclic three-bus network, where $V_k^{\min} = V_k^{\max}$ for $k = 1, 2, 3$ and the capacity limits are chosen arbitrarily. This feasible set is a non-convex 2-dimensional curvy surface in $\mathbb{R}^3$. The corresponding injection region $\mathcal{P}$ can be obtained by applying an appropriate linear transformation to $\mathcal{F}^r$. Surprisingly, this set becomes convex, as depicted in Figure 2(b). More precisely, it can be shown that $\mathcal{P} = \mathcal{P}_c$ in this case. The goal of this part is to investigate the convexity of $\mathcal{P}$ for a single cycle. The results will be generalized in the next subsection.

Assume for now that the power network is composed of a single cycle with the links $(1, 2), \ldots, (n - 1, n), (n, 1)$.

Theorem 1: Consider a lossless n-bus cycle with $n \geq 3$. The reduced flow region $\mathcal{F}^r$ is non-convex in the special case $V_k^{\min} = V_k^{\max}$, $k = 1, 2, ..., n$.

Proof: The reduced flow region $\mathcal{F}^r$ consists of all vectors of the form $(\alpha_1 \sin(\theta_{12}), \alpha_2 \sin(\theta_{23}), \ldots \alpha_n \sin(\theta_{n1}))$, where $\theta_{12} + \theta_{23} + \cdots + \theta_{n1} = 0$ and $\alpha_k = |V_k||V_{k+1}|\text{Im}\{y_{k,k+1}^*\}$ for $k \in \mathcal{N}$. Therefore, $\mathcal{F}^r$ can be characterized in terms of $n - 1$ independent angle differences $\theta_{12}, ..., \theta_{(n-1),n}$. This implies that $\mathcal{F}^r$ is an $(n - 1)$-dimensional surface embedded in $\mathbb{R}^n$. On the other hand, this region cannot be embedded in $\mathbb{R}^{n-1}$ due to its non-zero curvature. Thus, $\mathcal{F}^r$ cannot be a convex subset of $\mathbb{R}^n$. ∎

Since $V_k^{\min} \simeq V_k^{\max}$ in practice, it follows from Theorem 1 that the reduced flow region is expected to be non-convex under a normal operation.

Theorem 2: Consider a lossless n-bus cycle. The following statements hold:

a) For $n = 2$ and $n = 3$, the injection region $\mathcal{P}$ is convex and in particular $\mathcal{P} = \mathcal{P}_c$.

b) For $n \geq 5$, the injection region $\mathcal{P}$ is non-convex in the special case

$$V_k^{\min} = V_k^{\max} = V^{\max}, \qquad k \in \mathcal{N}$$
$$\Delta V_{lm}^{\max} = \Delta V^{\max}, \qquad (l, m) \in \mathcal{L} \tag{9}$$

for arbitrary numbers $V^{\max}$ and $\Delta V^{\max}$.

Proof of Part (a): The proof is trivial for $n = 2$, hence only the case $n = 3$ will be studied here. Consider an arbitrary injection vector $(\bar{P}_1, \bar{P}_2, \bar{P}_3)$ belonging to the convexified injection region $\mathcal{P}_c$. In order to prove Part (a), it suffices to show that $(\bar{P}_1, \bar{P}_2, \bar{P}_3)$ is contained in $\mathcal{P}$. Alternatively, it is enough to prove that the feasibility problem

$$\bar{P}_k = \sum_{l \in \mathcal{N}(k)} \text{Re}\left\{(W_{kk} - W_{kl})y_{kl}^*\right\}, \qquad k \in \mathcal{N} \tag{10a}$$

$$W_{ll} + W_{mm} - W_{lm} - W_{ml} \leq (\Delta V_{lm}^{\max})^2, \ (l, m) \in \mathcal{L} \tag{10b}$$

$$(V_k^{\min})^2 \leq W_{kk} \leq (V_k^{\max})^2, \qquad k \in \mathcal{N} \tag{10c}$$

$$\mathbf{W} \succeq 0 \tag{10d}$$

has a rank-1 solution $\mathbf{W}$. To this end, we convert the above feasibility problem to an optimization by adding the objective function

$$\min_{\mathbf{W} \in \mathbb{H}^n} - \sum_{(l,m) \in \mathcal{L}} \text{Re}\{W_{lm}\} \tag{11}$$

to this problem. Let $\nu_k \in \mathbb{R}$, $\psi_{lm} \in \mathbb{R}_+$, $\underline{\mu}_k \in \mathbb{R}_+$, $\overline{\mu}_k \in \mathbb{R}_+$, and $\mathbf{A} \in \mathbb{H}_+^3$ denote the Lagrange multipliers corresponding to the constraints (10a), (10b), lower bound (10c), upper bound (10c), and (10d), respectively. It can be shown that

$$A_{lm} = -1 - \psi_{lm} - \psi_{ml} - \frac{\nu_l y_{lm}^* + \nu_m y_{lm}}{2} \tag{12}$$

for every $(l, m) \in \mathcal{L}$. Moreover, the complementary slackness condition yields that $\text{trace}\{\mathbf{W}^{\text{opt}}\mathbf{A}^{\text{opt}}\} = 0$ at optimality. To prove that $\mathbf{W}^{\text{opt}}$ has rank 1, it suffices to show that $\mathbf{A}^{\text{opt}}$ has rank 2. To prove the later statement by contradiction, assume that $\mathbf{A}^{\text{opt}}$ has rank 1. Therefore, the following relation must hold:

$$\angle A_{12} + \angle A_{23} + \angle A_{31} = 0 \tag{13}$$

On the other hand, it can be concluded from (12) that

$$\text{Re}\{A_{12}\}, \text{Re}\{A_{23}\}, \text{Re}\{A_{31}\} < 0 \tag{14a}$$

$$\frac{\text{Im}\{A_{12}\}}{|y_{12}^*|} + \frac{\text{Im}\{A_{23}\}}{|y_{23}^*|} + \frac{\text{Im}\{A_{31}\}}{|y_{31}^*|} = 0 \tag{14b}$$

If A_{12}, A_{23} and A_{31} are regarded as three vectors in $\mathbb{R}^3$, it is easy to verity that since these vectors need to satisfy (14), the angle relation (13) does no hold. This contradiction completes the proof for Part (a).

Sketch of Proof for Part (b): Define

$$\theta^{\max} = \cos^{-1}\left(1 - \frac{(\Delta V^{\max})^2}{2}\right) \tag{15}$$

As pointed out in the proof of Theorem 1, the reduced flow region $\mathcal{F}^r$ contains all vectors of the form $(\alpha_1 \sin(\theta_{12}), \alpha_2 \sin(\theta_{23}), \ldots \alpha_n \sin(\theta_{n1}))$, where $\theta_{12} + \theta_{23} +$

$\dots + \theta_{n1} = 0$ and $|\theta_{12}|, \dots, |\theta_{n1}| \leq \theta^{\max}$. Four observations can be made here:

i) The mapping from $\mathcal{F}^r$ to $\mathcal{P}$ is linear.

ii) The kernel of the map from $\mathcal{F}^r$ to $\mathcal{P}$ has dimension 1.

iii) Due to (i) and (ii), it can be proved that the restriction of $\mathcal{F}^r$ to the angle $\theta_{12} = \theta^{\max}$ and $\theta_{n1} = -\theta^{\max}$ is a convex set whenever $\mathcal{P}$ is convex.

iv) The restriction of $\mathcal{F}^r$ to the angles $\theta_{12} = \theta^{\max}$ and $\theta_{n1} = -\theta^{\max}$ amounts to the reduced flow region for a single cycle of size $n-2$. In light of Theorem 1, this set is nonconvex if $n - 2 \geq 3$.

The proof of Part (b) follows from the above facts. ∎

B. Weakly-cyclic networks

In this part, the objective is to study the convexity of the injection region for a class of mesh networks.

Definition 3: A graph (network) is called *weakly cyclic* if every edge of the graph belongs to at most one cycle in the graph.

Theorem 3: For a lossless weakly-cyclic network with cycles of size 3, the injection region $\mathcal{P}$ is convex and in addition $\mathcal{P} = \mathcal{P}_c$.

Proof: The proof has been omitted due to space restrictions. ∎

The injection region $\mathcal{P}$ is not necessarily convex for lossy networks. For example, the set $\mathcal{P}$ corresponding to a three-bus mesh network with nonzero loss is a curvy 2-dimensional surface in $\mathbb{R}^3$. The objective of this part is to show that the front of this non-convex feasible set is convex in some sense. To derive this result, we assume that the resistance of each line of the network is nonnegative. This passivity assumption is trivially met for overhead transmission lines and underground cables.

Definition 4: Given a set $A \subseteq \mathbb{R}^n$, define its Pareto front as the set of all points $(a_1, \dots, a_n) \in A$ for which there does not exist a different point $(b_1, \dots, b_n)$ in A such that $b_i \leq a_i$ for $i = 1, \dots, n$.

Pareto front is an important subset of A because the solution of an arbitrary optimization over A with an increasing objective function must lie on the Pareto front of A.

Theorem 4: For a lossy weakly-cyclic network with cycles of size 3, the injection region $\mathcal{P}$ and the convexified injection region $\mathcal{P}_c$ share the same Pareto front.

Proof: Assume for now that the network is composed of a single cycle. Along with the proof of Part (b) of Theorem 2, it suffices to prove that $\mathbf{A}^{\mathrm{opt}}$ has rank 2 as opposed to rank 1. This property is guaranteed to hold if

$$\angle A_{12} + \angle A_{23} + \angle A_{31} \neq 0 \qquad (16)$$

where

$$A_{lm} = -\left(1 + \psi_{lm} + \psi_{ml} + \mathrm{Re}\{y_{lm}^*\}\left(\frac{\nu_l + \nu_m}{2}\right)\right)$$
$$- \mathrm{Im}\{y_{lm}^*\}\left(\frac{\nu_l - \nu_m}{2}\right)\mathrm{i}, \quad (l,m) \in \mathcal{L}$$

In the case when $\nu_1, \nu_2, \nu_3 \geq 0$, the relations given in (14) hold, which make the equation (16) be satisfied. Hence, as long

as $\nu_1, \nu_2, \nu_3 \geq 0$, the matrix $\mathbf{A}$ has rank 2 at optimality and therefore $\mathbf{W}^{\mathrm{opt}}$ becomes rank 1. It is straightforward to verify that restricting ν_1, ν_2 and ν_3 to be nonnegative corresponds to focusing on the Pareto front of $\mathcal{P}$ as opposed to the entire injection region. The above argument can be generalized to a weakly-cyclic network with multiple cycles. ∎

IV. SDP RELAXATION FOR OPF

Consider the OPF problem (1) with the capacity constraint (3d). In this section, the SDP relaxation of this OPF problem will be studied.

A. Exactness of SDP relaxation for weakly-cyclic networks

In this part, the exactness of the SDP relaxation will be examined for weakly-cyclic networks.

Theorem 5: Consider the OPF problem (1) with the capacity constraint (3d) for a weakly-cyclic network with cycles of size 3. The following statements hold:

a) The SDP relaxation is exact in the lossless case, provided $Q_k^{\min} = -\infty$ for every $k \in \mathcal{N}$.

b) The SDP relaxation is exact in the lossy case, provided $P_k^{\min} = Q_k^{\min} = -\infty$ and $Q_k^{\max} = +\infty$ for every $k \in \mathcal{N}$. □

Sketch of Proof: The proof consists of two steps:

- First, assume that the network is composed of a single cycle. It is needed to prove that the rank of the Lagrangian multiplier $\mathbf{A}$ associated with the constraint $\mathbf{W} \succeq 0$ has rank 2. This can be carried out in line with the proof of Part (a) of Theorem 2, which shows that the SDP relaxation has a rank-1 solution $\mathbf{W}^{\mathrm{opt}}$.

- Second, consider a general weakly-cyclic network with cycles of size 3. The proof of Theorem 3 can be deployed to show that the SDP relaxation has a rank-1 solution $\mathbf{W}^{\mathrm{opt}}$. ∎

B. Low-rank solution for SDP relaxation

The SDP relaxation is not always exact for general mesh networks. A question arises as to whether the SDP relaxation possesses a low-rank solution whenever it does not have a rank-1 solution. This question will be addressed below.

Define η as the minimum number of vertices whose removal from the power network eliminates all cycles of the graph. To illustrate the definition of η, this number will be calculated for a few types of graphs:

- $\eta = 0$ if the power network is acyclic.
- $\eta = 1$ if all cycles of the power network share a common node (see Figure 3).

Theorem 6: Consider the OPF problem subject to the capacity constraints (3a), (3b) and (3d) under the assumption $P_k^{\min} = Q_k^{\min} = -\infty$ for every $k \in \mathcal{N}$. If this problem is feasible, then the relaxed OPF problem has a solution $(\mathbf{W}^{opt}, \mathbf{P}_G^{\mathrm{opt}}, \mathbf{Q}_G^{\mathrm{opt}})$ such that $\mathrm{rank}\{\mathbf{W}^{opt}\} \leq \eta + 1$.

Proof: The proof is based on the minimum rank of a graph [18] and the perturbation of the objective function of OPF by $\varepsilon \times \sum_{(l,m) \in \mathcal{N} \times \mathcal{N}} \mathrm{Re}\{W_{lm}\}$. The details have been omitted due to space restrictions. ∎
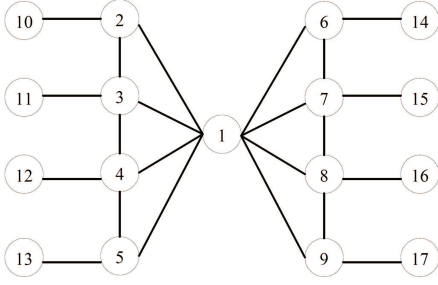
Fig. 3: The number η is equal to 1 for the above graph because the removal of the single node 1 kills all loops of the graph.

The parameters η is typically a very small number, especially for planar graphs. Hence, it can be inferred from Theorem 6 that the SDP relaxation has a low-rank solution in practice, whose rank is mostly related to the structure of the graph (e.g., being acyclic or weakly-cyclic) rather than the number of its vertices. The importance of this result will later be spelled out.

C. Approximate solution for OPF

As discussed in the preceding subsection, the SDP relaxation is expected to have a low-rank solution. This solution may be used to find an approximate rank-1 solution. Another technique is to enforce the SDP relaxation to kill the undesirable eigenvalues of the low-rank solution. To this end, notice that the proofs of the important results of this paper were partly based on the penalty term $\sum_{(l,m)\in\mathcal{L}} \mathrm{Re}\{W_{lm}\}$. Motivated by this fact, we modify the SDP relaxation below.

Perturbed SDP relaxation: This optimization is obtained from the SDP relaxation of the OPF problem by replacing its objective function with

$$\sum_{k\in\mathcal{G}} f_k(P_{G_k}) - \varepsilon \sum_{(l,m)\in\mathcal{L}} \mathrm{Re}\{W_{lm}\} \qquad (17)$$

for a given positive number ε.

The perturbed SDP relaxation aims to increase the real part of some entries of $\mathbf{W}$ with the hope of making this matrix rank-1 at optimality. The efficacy of this idea will be demonstrated in simulation results. Note that the recent literature of compressed sensing might suggest adding a penalty term based on the nuclear norm of $\mathbf{W}$. However, this idea fails to work for OPF because all feasible solutions of the SDP relaxation have almost equal nuclear norms (recall that $V_k^{\min}$ and $V_k^{\max}$ are close to each other for $k = 1, ..., n$).

V. SIMULATIONS

Example 1: Consider a ring network with 10 nodes and 10 links $(1,2), (2,3), ..., (9,10), (10,1)$. Suppose that the nodes $4, 5, 6, 9$ and 10 are generator buses while the remaining nodes are load buses. Let the cost function $f_k(P_{G_k})$ be chosen as $c_k P_{G_k}$, where

$$c_4 = c_5 = c_{10} = 1, \ c_6 = c_9 = 2$$

Consider also the line admittance values as

$$y_{lm} = -\mathrm{i}, \qquad \forall(l,m) \in \{(1,2),(3,4),(4,5),(10,1)\}$$
$$y_{lm} = -2\mathrm{i}, \qquad \forall(l,m) \in \{(2,3),(6,7),(9,10)\}$$
$$y_{56} = -3\mathrm{i}, \ y_{78} = -0.5\mathrm{i}, \ y_{89} = -0.7\mathrm{i}$$

Assume that the load values are equal to

$$P_{D_1} = -16, \ P_{D_2} = -14, \ P_{D_3} = -18, \ P_{D_7} = P_{D_8} = -20$$

The goal is to solve an OPF problem minimizing the total generation cost subject to the above load constraints and the following network requirements:

- *Voltage constraints:* $V_k^{min} = 0.95$ (per unit) and $V_k^{max} = 1.05$ for every $k \in \mathcal{N}$, where the base value is 100 MVA.
- *Flow constraints:* $\Theta_{lm}^{max} = 14°$ for every $(l,m) \in \mathcal{L}$.
- *Generator constraints:* P_{G_9} must be less than or equal to 20 MW.

Solving the SDP relaxation for this network yields the optimal cost $88, corresponding to the optimal outputs

$$P_{G_4} = 17.8, \quad P_{G_5} = 32, \quad P_{G_6} = P_{G_9} = 0, \quad P_{G_{10}} = 38.2$$

The obtained matrix $\mathbf{W}^{\mathrm{opt}}$ has the following eigenvalues:

$$0.0132, \ 0.0146, \ 0.0381, \ 0.0694, \ 0.0896,$$
$$0.2134, \ 0.3167, \ 0.5424, \ 1.4405, \ 7.3939$$

Although the motivation behind using the SDP relaxation was to hopefully obtain a rank-1 solution, the obtained matrix $\mathbf{W}^{\mathrm{opt}}$ is full rank with no zero eigenvalues. One may speculate that the relaxation is inexact in this case. To explore this issue, consider the perturbed SDP relaxation for a small nonzero ϵ (say $\epsilon = 10^{-5}$). Solving this optimization leads to the optimal cost $88, where its solution $\mathbf{W}_\epsilon^{opt}$ has only one nonzero eigenvalue (with value 10.5). The corresponding optimal productions are as follows:

$$P_{G_4} = 24.03, \ P_{G_5} = 26.28, \ P_{G_6} = P_{G_9} = 0, \ P_{G_{10}} = 37.69$$

Three conclusions can be made here:

- The SDP relaxation has a hidden rank-1 solution in the sense that a numerical algorithm might produce another solution of this optimization whose rank is high (rank 10 in this case).
- To find the hidden rank-1 solution of the SDP relaxation, the perturbed SDP relaxation can be used.
- The SDP relaxation is exact, but this property may not be easily detected from an optimal primal solution $\mathbf{W}^{\mathrm{opt}}$ or a dual matrix $\mathbf{A}^{\mathrm{opt}}$, unless a small perturbation is applied to the objective function.

Example 2: Consider again the ring network studied in Example 1, but with the following cost coefficients:

$$c_4 = 10, \ c_5 = 5, \ c_6 = 6, \ c_9 = 9, \ c_{10} = -3$$

In this case, the generator at Bus 10 is assumed to have a decreasing cost function. There are various motivations for considering a non-increasing cost function (other than technological constraints) such as the tendency to penetrate as much
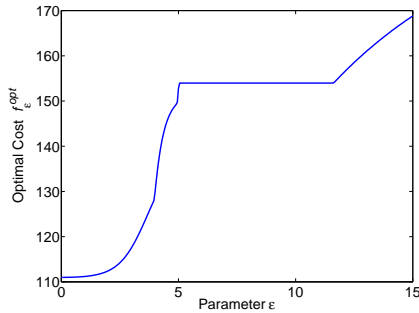
Fig. 4: The optimal cost for the perturbed SDP relaxation as a function of ϵ for the network studied in Example 2.

renewable energy as possible. Solving the SDP relaxation for this network yields the optimal cost \$110.99, corresponding to a rank-8 matrix $\mathbf{W}^{\text{opt}}$ with the nonzero eigenvalues:

$$0.0003, \ 0.0007, \ 0.0026, \ 0.0035,$$
$$0.0075, \ 0.0104, \ 0.4722, \ 9.5029$$

The associated optimal outputs of the generators at buses $4, 5, 6, 9$ and 10 are:

$$P_{G_4} = P_{G_6} = P_{G_9} = 0, \ P_{G_5} = 46.87, \ P_{G_{10}} = 41.13$$

Since it is not obvious whether or not the SDP relaxation has a hidden rank-1 solution, we solve the perturbed SDP relaxation for different values of ε. Define the optimal cost $f_\varepsilon^{\text{opt}}$ as the value of $\sum_{k \in \mathcal{G}} f_k(P_{G_k})$ (and not $\sum_{k \in \mathcal{G}} f_k(P_{G_k}) - \varepsilon \sum_{(l,m) \in \mathcal{L}} \text{Re}\{W_{lm}\}$) at optimality. Figure 4 depicts the optimal cost $f_\varepsilon^{\text{opt}}$ for ε from 0 to 15. The following observations can be made:

- The obtained numerical solution $\mathbf{W}_\varepsilon^{opt}$ has rank 2 for every ε in the range $[0, 5.03)$.
- The obtained numerical solution $\mathbf{W}_\varepsilon^{opt}$ has rank 1 for every ε in the range $[5.03, 15]$.
- The optimal cost $f_\varepsilon^{\text{opt}}$ is fixed at the value \$153.97 over the relatively wide range $[5.03, 11.59]$.
- The optimal cost $f_\varepsilon^{\text{opt}}$ has the same value as the optimal cost obtained for the original OPF problem by MATPOWER for every ε in the range $[5.03, 11.59]$.

This means that whenever the SDP relaxation is inexact, its non-trivial perturbation (for a relatively large value of ε) may find a local solution (if not global) of the original OPF problem.

VI. CONCLUSIONS

The optimal power flow (OPF) problem is a fundamental optimization for power networks, which aims to optimize the steady-state operating point of a power system. We have recently shown that the semidefinite programming (SDP) can be used to find a global solution of the OPF problem for IEEE benchmark power systems. Although the exactness of the SDP relaxation has been successfully proved for acyclic networks, a recent work has witnessed the failure of this technique for a three-bus cyclic network. Inspired by this observation, the

present paper is concerned with understanding the limitations of the SDP relaxation for cyclic power networks. First, it is shown that the injection region of a weakly-cyclic network with cycles of size 3 is convex in the lossless case and has a convex Pareto front in the lossy case. It is then proved that the SDP relaxation works for this type of network. This result implies that the failure of the SDP relaxation for a three-bus network recently reported in the literature can be fixed using a good modeling of the line capacity. As a more general result, it is then shown that whenever the SDP relaxation does not work, it would still have a low-rank solution in practice. Finally, a heuristic technique is proposed to make the SDP relaxation produce a rank-1 solution for general cyclic networks.

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