Study of Nonlinear Power Optimization Problems using Algebraic Graph Theory

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***Abstract*—This work is concerned with solving non-convex power optimization problems by introducing the concept of "nonlinear optimization over graph". To this end, the structure of a given nonlinear real/complex optimization with quadratic arguments is mapped into a generalized weighted graph, where each edge is associated with a weight set constructed from the known parameters of the optimization (e.g., the coefficients). This generalized weighted graph captures both the sparsity of the optimization and possible patterns in the coefficients. The notion of "sign definite sets" is introduced for both real and complex weight sets, and it is then shown that the polynomial-time solvability of the optimization may be inferred from the topology of its associated graph together with the sign definiteness of its weight sets. As an application of this result, it is finally proved that a broad class of optimization problems over power networks are polynomial-time solvable via two convex relaxations due to the passivity of transmission lines.**

I. Introduction

Several classes of optimization problems, including polynomial optimization and quadratically-constrained quadratic program (QCQP) as a special case, are nonlinear/non-convex and NP-hard in the worst case. Due to the complexity of such problems, various convex relaxations based on semidefinite programming (SDP) and second-order cone programming (SOCP) have gained popularity [1]. These techniques enlarge the possibly non-convex feasible set into a convex set characterizable via convex functions, and then provide the exact or a lower bound on the optimal objective value.

This paper is motivated by the fact that real-world optimization problems are highly structured in many ways and their structures could in principle help reduce the computational complexity. For example, transmission lines and transformers used in power networks are passive devices, and as a result optimizations defined over electrical power networks have certain structures which distinguish them from abstract optimizations with random coefficients. The high-level objective is to understand how the computational complexity of a given nonlinear optimization is related to its (hidden) structure. This work is concerned with a broad class of nonlinear real/complex optimization problems, including QCQP. The main feature of this class is that the argument of each objective and constraint function is quadratic (as opposed to linear) in the optimization variable, and the goal is to use two conic relaxations (SDP and SOCP) to convexify the argument of the optimization.

In this work, the structure of a given nonlinear real/complex optimization is mapped into a generalized weighted graph, where each edge is associated with a weight set constructed from the known parameters of the optimization (e.g., the coefficients). This generalized weighted graph captures both the sparsity of the optimization and possible patterns in the coefficients. Several conditions are derived, each of which guarantees the solvability of the real/complex-valued optimization via either an SDP or an SOCP relaxation. These conditions are in terms of some weak properties of the underlying graph of the optimization, e.g., its topology and weight sets. The notion of "sign-definite real/complex weight sets" introduced in this work is central to the analysis of the weighted graph.

As an application, it is shown that power optimization problems naturally fit into the framework of *nonlinear optimization over graph* and indeed the power network serves as the graph of these optimizations. It is then proved that a broad class of energy optimizations can be convexified due to the physics of power networks. The results of this paper extend the recent work on energy optimization [2], [3], [4], [5], [6], [7] and general quadratic optimization [8], [9].

The plan of the paper is as follows. The concept of "nonlinear optimization over graph" is developed in Section II, and is applied to power optimization problems in Section III. The notations used throughout this paper will be provided below.

Notations: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{S}^n$, and $\mathbb{H}^n$ denote the sets of real numbers, complex numbers, $n \times n$ symmetric matrices, and $n \times n$ Hermitian matrices, respectively. $\text{Re}\{M\}$, $\text{Im}\{M\}$, M^H, and $\text{Rank}\{M\}$ denote the real part, imaginary part, conjugate transpose, and rank of a given scalar/matrix M, respectively. The notation $M \succeq 0$ means that M is symmetric/Hermitian and positive semidefinite. $\measuredangle(x)$ represents the phase of a complex number x. The imaginary unit is denoted as "i", while "i" is used for indexing. Given an undirected graph $\mathcal{G}$, the notation $i \in \mathcal{G}$ means that i is a vertex of $\mathcal{G}$. Moreover, the notation $(i,j) \in \mathcal{G}$ means that (i,j) is an edge of $\mathcal{G}$ and besides $i<j$. Given a number (vector) $\mathbf{x}$, $|\mathbf{x}|$ denotes its absolute value (2-norm).

II. Nonlinear Optimization over Graph

Consider an undirected graph $\mathcal{G}$ with n vertices (nodes), where each edge $(i,j) \in \mathcal{G}$ has been assigned a nonzero edge weight set $\{c_{ij}^{(1)}, c_{ij}^{(2)}, ..., c_{ij}^{(k)}\}$ with k real/complex numbers (note that the superscripts in the weights are not exponents). This graph is called a *generalized weighted graph* as every edge is associated with a set of weights as opposed to a single weight. Consider an unknown vector $\mathbf{x} = \begin{bmatrix} x_1 & \cdots x_n \end{bmatrix}$ belonging to $\mathbb{D}^n$, where $\mathbb{D}$ is either $\mathbb{R}$ or $\mathbb{C}$. For every $i \in \mathcal{G}$, x_i is a variable associated with node i of the graph $\mathcal{G}$. Define:

$$\mathbf{y} = \left\{ |x_i|^2 \;\middle|\; \forall i \in \mathcal{G} \right\},$$

$$\mathbf{z} = \left\{ \text{Re}\left\{ c_{ij}^{(t)} x_i x_j^H \right\} \;\middle|\; \forall (i,j) \in \mathcal{G},\ t \in \{1, ..., k\} \right\}$$

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The sets $\mathbf{y}$ and $\mathbf{z}$ can be regarded as two vectors, where

- $\mathbf{y}$ collects the quadratic terms $|x_i|^2$'s (one term for each vertex).
- $\mathbf{z}$ collects the cross terms $\text{Re}\{c_{ij}^{(t)} x_i x_j^H\}$'s ($k$ terms for each edge).

Although the above formulation deals with $\text{Re}\{c_{ij}^{(t)} x_i x_j^H\}$ whenever $(i,j) \in \mathcal{G}$, it can handle terms of the form $\text{Re}\{\alpha x_j x_i^H\}$ and $\text{Im}\{\alpha x_i x_j^H\}$ for a complex weight α. This can be carried out using the transformations:

$$\begin{aligned} \text{Re}\{\alpha x_j x_i^H\} &= \text{Re}\{(\alpha^H) x_i x_j^H\}, \\ \text{Im}\{\alpha x_i x_j^H\} &= \text{Re}\{(-\alpha \mathrm{i}) x_i x_j^H\} \end{aligned}$$

This section is concerned with the nonlinear optimization

$$\begin{aligned} \min_{\mathbf{x} \in \mathbb{D}^n} \quad & f_0(\mathbf{y}, \mathbf{z}) \\ \text{subject to} \quad & f_j(\mathbf{y}, \mathbf{z}) \leq 0, \quad j = 1, 2, ..., m \end{aligned} \tag{1}$$

for given functions $f_0, ..., f_m$. Assume that $f_j(\mathbf{y}, \mathbf{z})$ is an increasing function with respect to all entries of $\mathbf{z}$, for $j = 0, ..., m$. The computational complexity of the above optimization depends in part on the structure of the functions f_j's. Regardless of these functions, Optimization (1) is intrinsically hard to solve (NP-hard in the worst case) because $\mathbf{y}$ and $\mathbf{z}$ are both nonlinear functions of $\mathbf{x}$. The objective is to convexify the second-order nonlinearity embedded in $\mathbf{y}$ and $\mathbf{z}$. To this end, notice that there exist two linear functions $l_1 : \mathbb{C}^{n \times n} \to \mathbb{R}^n$ and $l_2 : \mathbb{C}^{n \times n} \to \mathbb{R}^{k\tau}$ such that $\mathbf{y} = l_1(\mathbf{x}\mathbf{x}^H)$ and $\mathbf{z} = l_2(\mathbf{x}\mathbf{x}^H)$, where τ denotes the number of edges in $\mathcal{G}$. Motivated by the above observation, if $\mathbf{x}\mathbf{x}^H$ is replaced by a new matrix variable X, then $\mathbf{y}$ and $\mathbf{z}$ both become linear in X. This implies that the non-convexity induced by the quadratic terms $\text{Re}\{c_{ij}^{(t)} x_i x_j\}$'s and $|x_i|$'s all disappear if Optimization (1) is reformulated in terms of X. However, the optimal solution X may not be decomposable as $\mathbf{x}\mathbf{x}^H$ unless some additional constraints are imposed on X. It is straightforward to verify that Optimization (1) is equivalent to

$$\min_{X} \quad f_0(l_1(X), l_2(X)) \tag{2a}$$
$$\text{s.t.} \quad f_j(l_1(X), l_2(X)) \leq 0, \quad j = 1, ..., m \tag{2b}$$
$$X \succeq 0, \tag{2c}$$
$$\text{Rank}\{X\} = 1 \tag{2d}$$

where there is an implicit constraint that $X \in \mathbb{S}^n$ if $\mathbb{D} = \mathbb{R}$ and $X \in \mathbb{H}^n$ if $\mathbb{D} = \mathbb{C}$. To reduce the computational complexity of the above problem, two actions can be taken: (i) removing the non-convex constraint (2d), and (ii) relaxing the convex, but computationally-expensive, constraint (2c) to a set of simpler constraints on certain low-order submatrices of X. Based on this methodology, two relaxations will be proposed for Optimization (1) next.

SDP relaxation: This optimization is defined as

$$\min_{X} \quad f_0(l_1(X), l_2(X)) \tag{3a}$$
$$\text{s.t.} \quad f_j(l_1(X), l_2(X)) \leq 0, \quad j = 1, ..., m \tag{3b}$$
$$X \succeq 0 \tag{3c}$$

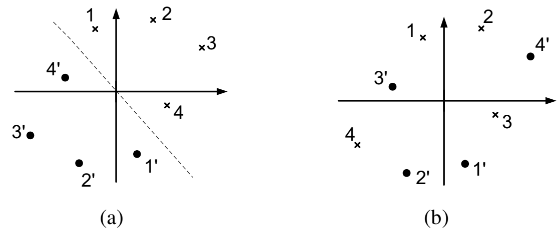

Fig. 1. In Figure (a), there exists a line separating x's (elements of $\mathcal{T}$) from o's (elements of $-\mathcal{T}$) so the set $\mathcal{T}$ is sign definite. In Figure (b), this is not the case.

SOCP relaxation: This optimization is defined as

$$\min_{X} \quad f_0(l_1(X), l_2(X)) \tag{4a}$$
$$\text{s.t.} \quad f_j(l_1(X), l_2(X)) \leq 0, \qquad j = 1, ..., m \tag{4b}$$
$$X\{(i,j)\} \succeq 0, \qquad \forall (i,j) \in \mathcal{G} \tag{4c}$$

where $X\{(i,j)\}$ denotes a 2×2 submatrix of X in the intersection of rows i, j and columns i, j of X.

The above SDP and SOCP relaxations are targeted at the non-convexity caused by the nonlinear relationship between $\mathbf{x}$ and $(\mathbf{y}, \mathbf{z})$. Note that these optimizations are convex relaxations only when the functions $f_0, ..., f_m$ are convex. If any of these functions is non-convex, additional relaxations may be needed to convexify the SDP and SOCP optimizations. Define f^*, f^*_{SDP}, and f^*_{SOCP} as the optimal solutions of Optimizations (2), (3), and (4), respectively. By comparing the feasible sets of these optimizations, it can be concluded that

$$f^*_{\text{SOCP}} \leq f^*_{\text{SDP}} \leq f^* \tag{5}$$

Given a particular optimization of the form (1), if any of the above inequalities for f^* turns into an equality, the associated relaxation would be able to find the solution of the original optimization. In this case, it is said that the relaxation is "tight" or "exact". The objective is to relate the exactness of the proposed relaxations to the topology of the graph $\mathcal{G}$ and some properties of its weights sets $\{c_{ij}^{(1)}, c_{ij}^{(2)}, ..., c_{ij}^{(k)}\}$'s. This will be carried out next via the nation of sign-definite sets.

Definition 1: A finite set $\mathcal{T} \subset \mathbb{R}$ is said to be *sign definite with respect to* $\mathbb{R}$ if its elements are either all negative or all nonnegative. $\mathbb{T}$ is called *negative* if its elements are negative and is called *positive* if its elements are nonnegative. A finite set $\mathcal{T} \subset \mathbb{C}$ is said to be *sign definite with respect to* $\mathbb{C}$ if when the sets $\mathcal{T}$ and $-\mathcal{T}$ are mapped into two collections of points in $\mathbb{R}^2$, then there exists a line separating the two sets (the elements of the sets are allowed to lie on the line).

To illustrate Definition 1, consider a complex set $\mathcal{T}$ with four elements, whose corresponding points are labeled as 1, 2, 3, and 4 in Figure 1(a). The points corresponding to $-\mathcal{T}$ are labeled as 1', 2', 3', and 4' in the same picture. Since there exists a line separating x's (elements of $\mathcal{T}$) from o's (elements of $-\mathcal{T}$), the set $\mathcal{T}$ is sign definite. In contrast, if the elements of $\mathcal{T}$ are distributed according to Figure 1(b), the set will no longer be sign definite. Note that the definition of sign-definite complex sets is inspired by the fact that a real set $\mathcal{T}$ is sign definite with respect to $\mathbb{R}$ if $\mathcal{T}$ and $-\mathcal{T}$ are separable via a point (on the horizontal axis).

Based on the notion of sign-definite weight sets, the exactness of the SDP and SOCP relaxations will be studied in both real and complex cases (see [10] for the proofs).

Real-valued case ($\mathbb{D} = \mathbb{R}$): Choose a set of cycles $\mathcal{O}_1, ..., \mathcal{O}_p$ of the graph $\mathcal{G}$ such that they form a cycle basis, implying that every arbitrary cycle of the graph can be obtained by a combination of these p basic cycles. The SOCP and SDP relaxations are both tight, provided each weight set $\{c_{ij}^{(1)}, ..., c_{ij}^{(k)}\}$ is sign definite with respect to $\mathbb{R}$ and

$$\prod_{(i,j)\in\mathcal{O}_r} \sigma_{ij} = (-1)^{|\mathcal{O}_r|}, \qquad \forall r \in \{1, ..., p\}$$

where σ_{ij} shows the sign of the weight set associated with the edge $(i,j) \in \mathcal{G}$. This condition is naturally satisfied in three special cases:

- $\mathcal{G}$ is acyclic with arbitrary sign definite edge sets.
- $\mathcal{G}$ is bipartite with positive weight sets.
- $\mathcal{G}$ is arbitrary with negative weight sets.

Complex-valued case ($\mathbb{D} = \mathbb{C}$): Assume that each edge set $\{c_{ij}^{(1)}, ..., c_{ij}^{(k)}\}$ is sign definite with respect to $\mathbb{C}$. This assumption is trivially met if $k \leq 2$ or the weight set consists of only real (or imaginary) numbers. The following statements hold:

1) The SOCP and SDP relaxations are tight if $\mathcal{G}$ is acyclic.
3) The SDP relaxation is exact if $\mathcal{G}$ is bipartite and weakly cyclic with positive or negative real weight sets (a graph is called *weakly cyclic* if every edge of the graph belongs to at most one cycle of the graph).
4) The SDP relaxation is exact if $\mathcal{G}$ is a weakly cyclic graph with imaginary homogeneous weight sets.

In addition, if the graph $\mathcal{G}$ can be decomposed as a union of edge-disjoint subgraphs in an acyclic way such that each subgraph has one of the above three structural properties 1-3, then the SDP relaxation is exact.

The aforementioned results still hold after two generalizations: (i) allowance of weight sets with different cardinalities, and (ii) inclusion of linear terms (besides quadratic terms) into the arguments of the functions $f_0, ..., f_m$.

III. Application in Power Systems

A majority of real-world optimizations are naturally 'optimization over graph", meaning that the optimization is defined over the graph characterizing a physical system. For example, optimizations in circuits, antenna systems, and communication networks can easily be regarded as "optimization over graph". Then, the question of interest is: how does the computational complexity of an optimization relate to the structure of the system over which the optimization is performed? This question will be explored here in the context of AC electrical power grids (DC power systems can be treated similarly). Assume that the graph $\mathcal{G}$ corresponds to an AC power network, where:

- The power network has n nodes.

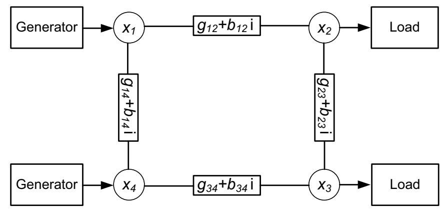

Fig. 2. An example of a power system studied in Section III.

- For every $(i,j) \in \mathcal{G}$, nodes i and j are connected to each other in the power network via a transmission line with the impedance $g_{ij} + b_{ij}\mathrm{i}$ (shunt elements are ignored to streamline the presentation).
- Each node $i \in \mathcal{G}$ of the network is connected to an external device, which exchanges electrical power with the power network.

Figure 2 exemplifies a sample power network in which two external devices generate power while the remaining ones consume power. Each line $(i,j) \in \mathcal{G}$ is associated with four power flows:

- p_{ij}: Active power entering the line from node i
- p_{ji}: Active power entering the line from node j
- q_{ij}: Reactive power entering the line from node i
- q_{ji}: Reactive power entering the line from node j

Note that $p_{ij}+p_{ji}$ and $q_{ij}+q_{ji}$ represent the active and reactive losses incurred in the line. Let x_i denote the complex voltage (phasor) for node $i \in \mathcal{G}$. One can write:

$$\begin{aligned}
p_{ij}(\mathbf{x}) &= \mathrm{Re}\left\{x_i(x_i - x_j)^H \frac{1}{g_{ij} - b_{ij}\mathrm{i}}\right\},\\
p_{ji}(\mathbf{x}) &= \mathrm{Re}\left\{x_j(x_j - x_i)^H \frac{1}{g_{ij} - b_{ij}\mathrm{i}}\right\},\\
q_{ij}(\mathbf{x}) &= \mathrm{Im}\left\{x_i(x_i - x_j)^H \frac{1}{g_{ij} - b_{ij}\mathrm{i}}\right\},\\
q_{ji}(\mathbf{x}) &= \mathrm{Im}\left\{x_j(x_j - x_i)^H \frac{1}{g_{ij} - b_{ij}\mathrm{i}}\right\}
\end{aligned}$$

Note that since the flows all depend on $\mathbf{x}$, the argument $\mathbf{x}$ has been added to the above equations (e.g., $p_{ij}(\mathbf{x})$ instead of p_{ij}). The flows $p_{ij}(\mathbf{x})$, $p_{ji}(\mathbf{x})$, $q_{ij}(\mathbf{x})$, and $q_{ji}(\mathbf{x})$ can all be expressed in terms of $|x_i|^2$, $|x_j|^2$, and $\mathrm{Re}\left\{c_{ij}^{(t)} x_i x_j^H\right\}$ for $t = 1, 2, 3, 4$, where

$$c_{ij}^{(1)} = \frac{-1}{g_{ij} - b_{ij}\mathrm{i}}, \quad c_{ij}^{(2)} = \frac{-1}{g_{ij} + b_{ij}\mathrm{i}},$$
$$c_{ij}^{(3)} = \frac{\mathrm{i}}{g_{ij} - b_{ij}\mathrm{i}}, \quad c_{ij}^{(4)} = \frac{-\mathrm{i}}{g_{ij} + b_{ij}\mathrm{i}}$$

Define

$$\begin{aligned}
\mathbf{p}(\mathbf{x}) &= \{p_{ij}(\mathbf{x}), p_{ji}(\mathbf{x}) \mid \forall (i,j) \in \mathcal{G}\},\\
\mathbf{q}(\mathbf{x}) &= \{q_{ij}(\mathbf{x}), q_{ji}(\mathbf{x}) \mid \forall (i,j) \in \mathcal{G}\}
\end{aligned}$$

Consider the optimization

$$\begin{aligned}
\min_{\mathbf{x}\in\mathbb{C}^n} \quad & h_0(\mathbf{p}(\mathbf{x}), \mathbf{q}(\mathbf{x}), \mathbf{y}(\mathbf{x}))\\
\text{s.t.} \quad & h_j(\mathbf{p}(\mathbf{x}), \mathbf{q}(\mathbf{x}), \mathbf{y}(\mathbf{x})) \leq 0, \quad j = 1, 2, ..., m
\end{aligned} \tag{6}$$

for given functions $h_0, ..., h_m$, where $\mathbf{y}(\mathbf{x})$ is the vector of $|x_i|^2$'s. This optimization aims to optimize the flows in a power network. The constraints of this optimization are meant to limit line flows, voltage magnitudes, power delivered to each load, and power supplied by each generator. Observe that $\mathbf{p}(\mathbf{x})$ and $\mathbf{q}(\mathbf{x})$ are both quadratic in $\mathbf{x}$. Assume that $h_j(\cdot,\cdot,\cdot)$ is increasing (or decreasing) in its first and second vector arguments. Since the above optimization can be cast as (1), the SDP and SOCP relaxations introduced before can be used to eliminate the effect of quadratic terms. To study under what conditions the relaxations are exact, note that each edge (i,j) of $\mathcal{G}$ has the weight set $\{c_{ij}^{(1)}, c_{ij}^{(2)}, c_{ij}^{(3)}, c_{ij}^{(4)}\}$. Due to the physics of a transmission line, g_{ij} and b_{ij} are nonnegative real numbers in practice. As a result of this property, the set $\{c_{ij}^{(1)}, c_{ij}^{(2)}, c_{ij}^{(3)}, c_{ij}^{(4)}\}$ turns out to be sign definite (see Definition 1). The following theorem follows from the results outlined in Section II and [4].

Theorem 1: The SDP and SOCP relaxations are both exact for the general power optimization (6) as long as $\mathcal{G}$ is acyclic or has a sufficient number of phase shifters in its cycles (one phase shifter for each basic cycle).

Optimizing power flows is a fundamental problem, which is solved every 5 to 15 minutes in practice for power grids. This problem, named Optimal Power Flow (OPF), has several variants, which are used for different purposes (real-time operation, electricity market, security assessment, etc.). Nevertheless, a more realistic form of this optimization often has two more constraints, which cannot be described in terms of $\mathbf{p}(\mathbf{x}), \mathbf{q}(\mathbf{x}), \mathbf{y}(\mathbf{x})$:

- *Line flow constraint:* For every $(i,j) \in \mathcal{G}$, the line current magnitude $\left|\frac{x_i - x_j}{g_{ij} + b_{ij}\mathrm{i}}\right|$ cannot exceed a maximum number $I_{\max}$. This constraint can be written as:

$$|x_i|^2 + |x_j|^2 - 2\mathrm{Re}\{x_i x_j^H\} \leq |g_{ij} + b_{ij}\mathrm{i}|^2 I_{\max}^2 \tag{7}$$

- *Angle constraint:* For every $(i,j) \in \mathcal{G}$. the absolute angle difference $|\measuredangle x_i - \measuredangle x_j|$ should not exceed a maximum angle $\theta_{ij}^{\max} \in [0, 90°]$ (due to stability and thermal limits). This constraint can be written as

$$\mathrm{Im}\{x_i x_j^H\} \leq |\tan\theta_{ij}^{\max}| \times \mathrm{Re}\{x_i x_j^H\}$$

  or equivalently

$$\begin{aligned} -\tan\theta_{ij}^{\max} \times \mathrm{Re}\{x_i x_j^H\} + \mathrm{Re}\{(+\mathrm{i})\ x_i x_j^H\} \leq 0 \\ -\tan\theta_{ij}^{\max} \times \mathrm{Re}\{x_i x_j^H\} + \mathrm{Re}\{(-\mathrm{i})\ x_i x_j^H\} \leq 0 \end{aligned} \tag{8}$$

Since (7) and (8) are quadratic in $\mathbf{x}$, they can easily be incorporated into Optimization (6) and its relaxations. However, the edge set $\{c_{ij}^{(1)}, c_{ij}^{(2)}, c_{ij}^{(3)}, c_{ij}^{(4)}\}$ should be extended to $\{c_{ij}^{(1)}, c_{ij}^{(2)}, c_{ij}^{(3)}, c_{ij}^{(4)}, -1, \mathrm{i}, -\mathrm{i}\}$ for every $(i,j) \in \mathcal{G}$. It is interesting to note that this set is still sign definite and therefore the results of Theorem 1 are valid under this generalization.

Another interesting case is the optimization of active power flows for lossless networks. In this case, g_{ij} is equal to zero for every $(i,j) \in \mathcal{G}$. Hence, $p_{ji}(\mathbf{x})$ can be simply replaced by $-p_{ij}(\mathbf{x})$. Motivated by this observation, define the reduced vector of active powers as $\mathbf{p}_r(\mathbf{x}) = \{p_{ij}(\mathbf{x}) \mid \forall (i,j) \in \mathcal{G}\}$, and consider the optimization

$$\begin{aligned} \min_{\mathbf{x} \in \mathbb{C}^n} \quad & \bar{h}_0(\mathbf{p}_r(\mathbf{x}), \mathbf{y}(\mathbf{x})) \\ \text{s.t.} \quad & \bar{h}_j(\mathbf{p}_r(\mathbf{x}), \mathbf{y}(\mathbf{x})) \leq 0, \quad j = 1, 2, ..., m \end{aligned} \tag{9}$$

for some functions $\bar{h}_0(\cdot,\cdot), ..., \bar{h}_m(\cdot,\cdot)$, which are assumed to be increasing in their first vector argument. Now, each edge (i,j) of the graph $\mathcal{G}$ is accompanied by the singleton weight set $\left\{\frac{-\mathrm{i}}{b_{ij}}\right\}$, which is sign definite. The following theorem follows from the results outlined in Section II.

Theorem 2: The SDP relaxation is exact for Optimization (9) if $\mathcal{G}$ is weakly cyclic.

Note that the result of Theorem 2 does not necessarily hold for the SOCP relaxation.

IV. CONCLUSIONS

This work develops the notion of "optimization over graph" and applies it to nonlinear power optimization problems. First, a broad class of nonlinear real/complex optimization problems is considered, where the argument of each objective and constraint function is quadratic (as opposed to linear) in the optimization variable. To explore the polynomial-time solvability of each optimization problem via two convex relaxations, the structure of the optimization is mapped into a generalized weighted graph with a weight set assigned to each edge. It is shown that the exactness of the proposed convex relaxations may be deduced from the topology of the graph and the sign definiteness of its weight sets. As an application, it is shown that a nonlinear power optimization problem can be naturally considered as *optimization over graph*, where the weight sets of the graph are all sign definite due to the passivity of transmission lines. This property makes a broad class of energy optimization problems easy to solve.

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