# Low-Rank Solutions of Matrix Inequalities With Applications to Polynomial Optimization and Matrix Completion Problems

Ramtin Madani, Ghazal Fazelnia, Somayeh Sojoudi and Javad Lavaei

***Abstract*— This paper is concerned with the problem of finding a low-rank solution of an arbitrary sparse linear matrix inequality (LMI). To this end, we map the sparsity of the LMI problem into a graph. We develop a theory relating the rank of the minimum-rank solution of the LMI problem to the sparsity of its underlying graph. Furthermore, we propose two graph-theoretic convex programs to obtain a low-rank solution. The first convex optimization needs a tree decomposition of the sparsity graph. The second one does not rely on any computationally-expensive graph analysis and is always polynomial-time solvable. The results of this work can be readily applied to three separate problems of minimum-rank matrix completion, conic relaxation for polynomial optimization, and affine rank minimization. The results are finally illustrated on two applications of optimal distributed control and nonlinear optimization for electrical networks.**

## I. Introduction

Consider the linear matrix inequality (LMI) problem

$$\operatorname{trace}\{\mathbf{M}_k\mathbf{X}\} \leq a_k, \quad k = 1\dots p \tag{1a}$$

$$\mathbf{X} \succeq 0 \tag{1b}$$

with the variable $\mathbf{X} \in \mathbb{F}^n$, where $\succeq$ represents the positive semidefinite sign and $\mathbf{M}_1, \dots, \mathbf{M}_p \in \mathbb{F}^n$ are sparse matrices in $\mathbb{F}^n$ which is either the set of $n \times n$ real symmetric matrices $\mathbb{S}^n$ or complex Hermitian matrices $\mathbb{H}^n$. The objective of this paper is twofold. First, it is aimed to find a low-rank solution $\mathbf{X}^{\mathrm{opt}}$ of the above LMI (feasibility) problem using a convex program. Second, it is intended to study the relationship between the rank of such a low-rank solution and the sparsity level of the matrices $\mathbf{M}_1, \dots, \mathbf{M}_k$.

Consider the real-valued case where $\mathbf{X}$ and $\mathbf{M}_k$'s are all real matrices. Let $P \subseteq \mathbb{S}^n$ denote the convex polytope characterized by the linear inequalities given in (1a). In this work, the goal is to design an efficient algorithm to identify a low-rank matrix $\mathbf{X}^{\mathrm{opt}}$ in the set $\mathbb{S}^n_+ \cap P$, where $\mathbb{S}^n_+$ denotes the cone of positive semidefinite matrices in $\mathbb{S}^n$. The special case where $P$ is an affine subspace of $\mathbb{S}^n$ has been extensively studied in the literature [1]–[3]. In particular, the work [2] derives an upper bound on the rank of $\mathbf{X}^{\mathrm{opt}}$, which depends on the dimension of $P$ as opposed to the sparsity level of the problem. The paper [3] develops a polynomial-time algorithm to find a solution satisfying the bound condition given in [2]. However, since the bound obtained in [2] is independent of the sparsity of the LMI problem (1), it is known not to be tight for several practical examples [4]–[6].

The investigation of the above-mentioned LMI problem has direct applications in three fundamental problems: (i) minimum-rank positive semidefinite matrix completion, (ii) conic relaxation for polynomial optimization, and (iii) affine rank minimization. In what follows, these problems will be introduced in three separate subsections, followed by an outline of our contribution for each problem. Note that due to space restrictions, the proofs of the theorems developed in this work have been moved to the technical report [7].

### A. Low-rank Positive Semidefinite Matrix Completion

The LMI problem (1) encapsulates the low-rank positive semidefinite matrix completion problem, which is described below. Given a partially completed matrix with some known entries, the positive semidefinite matrix completion problem aims to design the unknown (free) entries of the matrix in such a way that the completed matrix becomes positive semidefinite. As a classical result, this problem has been fully addressed in [8], provided that the graph capturing the locations of the known entries of the matrix is chordal. The positive semidefinite matrix completion problem plays a critical role in reducing the complexity of large-scale semidefinite programs [9]–[11]. In the case where a minimum-rank completion is sought, the problem is referred to as *minimum-rank positive semidefinite matrix completion*. To formalize this problem, consider a simple graph $\mathcal{G} = (\mathcal{V}_\mathcal{G}, \mathcal{E}_\mathcal{G})$ with the vertex set $\mathcal{V}_\mathcal{G}$ and the edge set $\mathcal{E}_\mathcal{G}$. Let $\operatorname{gd}(\mathcal{G})$ denote the Gram dimension of $\mathcal{G}$ defined as the smallest positive integer $r$ such that for every $\widehat{\mathbf{X}} \in \mathbb{S}^+_{|\mathcal{V}_\mathcal{G}|}$, there exists a matrix $\mathbf{X} \in \mathbb{S}^+_{|\mathcal{V}_\mathcal{G}|}$ satisfying the inequality $\operatorname{rank}\{\mathbf{X}\} \leq r$ and the equations

$$X_{ij} = \widehat{X}_{ij}, \qquad (i,j) \in \mathcal{E}_\mathcal{G} \tag{2a}$$

$$X_{kk} = \widehat{X}_{kk}, \qquad k \in \mathcal{V}_\mathcal{G} \tag{2b}$$

According to the above definition, every arbitrary positive semidefinite matrix $\widehat{\mathbf{X}}$ can be turned into a matrix $\mathbf{X}$ with rank at most $\operatorname{gd}(\mathcal{G})$ by manipulating those off-diagonal entries of $\widehat{\mathbf{X}}$ that correspond to the non-existent edges of $\mathcal{G}$. The paper [12] introduces the notion of Gram dimension and shows that $\operatorname{gd}(\mathcal{G}) \leq \operatorname{tw}(\mathcal{G}) + 1$ (for real-valued problems), where $\operatorname{tw}(\mathcal{G})$ denotes the treewidth of the graph $\mathcal{G}$.

There is a large body of literature on a graph-theoretic parameter named the minimum semidefinite rank of a graph [13]. This parameter, denoted as $\operatorname{msr}(\mathcal{G})$, is equal to the smallest rank of all positive semidefinite matrices with the

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same support as the adjacency matrix of $\mathcal{G}$. The notion of OS-vertex number of $\mathcal{G}$, denoted by $\mathrm{OS}(\mathcal{G})$, has been recently proposed in [14] that serves as a lower bound on $\mathrm{msr}(\mathcal{G})$. The paper [14] also shows that $\mathrm{OS}(\mathcal{G}) = \mathrm{msr}(\mathcal{G})$ for a chordal graph $\mathcal{G}$ and conjectures the validity of this relation for an arbitrary graph.

The matrix completion problem (2) can be cast as the LMI problem (1). Hence, the minimum-rank positive semidefinite matrix completion problem amounts to finding a minimum-rank matrix in the convex set $\mathbb{S}_+^n \cap P$. In this work, we will utilize the notions of tree decomposition, minimum semidefinite rank of a graph, and OS-vertex number to find low-rank matrices in $\mathbb{S}_+^n \cap P$ using convex optimization. Let $\mathcal{G}$ denote a graph capturing the sparsity of the LMI problem (1). Consider the convex problem of minimizing a weighted sum of an arbitrary subset of the free entries of $\mathbf{X}$ subject to the matrix completion constraint (2). We show that the rank of every solution of this problem can be upper bounded in terms of the OS and msr of some supergraphs of $\mathcal{G}$. Our bound depends only on the locations of the free entries minimized in the objective function rather than their coefficients. In particular, given an arbitrary tree decomposition of $\mathcal{G}$ with width $t$, we show that the minimization of a weighted sum of certain free entries of $\mathbf{X}$ guarantees that every solution $\mathbf{X}^{\mathrm{opt}}$ of this problem belongs to $\mathbb{S}_+^n \cap P$ and satisfies the relation $\mathrm{rank}\{\mathbf{X}^{\mathrm{opt}}\} \leq t+1$, for all possible nonzero coefficients of the objective function. This result holds for both real and complex-valued problems.

In the case where a good decomposition of $\mathcal{G}$ with small width is not known, we propose a polynomial-time solvable optimization that is able to find a matrix in $\mathbb{S}_+^n \cap P$ with rank at most $2(n - \mathrm{msr}(\mathcal{G}))$. Note that this solution can be found in polynomial time, whereas our theoretical upper bound on its rank is hard to compute. The upper bound $2(n - \mathrm{msr}(\mathcal{G}))$ is a small number for a wide class of sparse graphs [15].

### *B. Sparse Quadratically-Constrained Quadratic Program*

The problem of searching for a low-rank matrix in the convex set $\mathbb{S}_+^n \cap P$ is important due in part to its application in obtaining suboptimal solutions of quadratically-constrained quadratic programs (QCQPs). Consider the standard nonconvex QCQP problem

$$\underset{x\in\mathbb{R}^{n-1}}{\text{minimize}} \quad f_0(x) \tag{3a}$$

$$\text{subject to} \quad f_k(x) \leq 0, \quad k = 1,\dots,p \tag{3b}$$

where $f_k(x) = x^{\mathrm{T}}\mathbf{A}_k x + 2b_k^{\mathrm{T}}x + c_k$ for $k = 0,1,\dots,p$. Every polynomial optimization can be cast as problem (3) and this also includes all combinatorial optimization problems [16]. Thus, the above nonconvex QCQP "covers almost everything" [16]. To tackle this NP-hard problem, define

$$\mathbf{F}_k \triangleq \begin{bmatrix} c_k & b_k^{\mathrm{T}} \\ b_k & \mathbf{A}_k \end{bmatrix}. \tag{4}$$

Each $f_k$ has the linear representation $f_k(x) = \mathrm{trace}\{\mathbf{F}_k\mathbf{X}\}$ for the following choice of $\mathbf{X}$:

$$\mathbf{X} \triangleq [1 \quad x^{\mathrm{T}}]^{\mathrm{T}}[1 \quad x^{\mathrm{T}}]. \tag{5}$$

It is obvious that an arbitrary matrix $\mathbf{X} \in \mathbb{S}^n$ can be factorized as (5) if and only if it satisfies the three properties $X_{11} = 1$, $\mathbf{X} \succeq 0$ and $\mathrm{rank}\{\mathbf{X}\} = 1$. Therefore, problem (3) can be reformulated as follows:

$$\underset{\mathbf{X}\in\mathbb{S}^n}{\text{minimize}} \quad \mathrm{trace}\{\mathbf{F}_0\mathbf{X}\} \tag{6a}$$

$$\text{subject to} \quad \mathrm{trace}\{\mathbf{F}_k\mathbf{X}\} \leq 0, \quad k = 1,\dots,p \tag{6b}$$

$$X_{11} = 1 \tag{6c}$$

$$\mathbf{X} \succeq 0 \tag{6d}$$

$$\mathrm{rank}\{\mathbf{X}\} = 1. \tag{6e}$$

In the above representation of QCQP, the constraint (6e) carries all the nonconvexity. Neglecting this constraint yields a convex problem, known as the semidefinite programming (SDP) relaxation of QCQP [17], [18]. The existence of a rank-1 solution for the SDP relaxation guarantees the equivalence of the original QCQP and its relaxed problem. The SDP relaxation technique provides a lower bound on the minimum cost of the original problem. If the QCQP problem and its SDP relaxation result in the same optimal objective value, then the relaxation is said to be exact. The SDP relaxation of a sparse QCQP problem often has infinitely many solutions and the conventional numerical algorithms would find a solution with the highest rank [19]. Hence, a question arises as to whether a low-rank solution of the SDP relaxation of a sparse QCQP can be found efficiently. To address this problem, let $\widehat{\mathbf{X}}$ denote an arbitrary solution of the SDP relaxation. If the QCQP problem (3) is sparse and associated with a sparsity graph $\mathcal{G}$, then every positive semidefinite matrix $\mathbf{X}$ satisfying the matrix completion constraint (2) is another solution of the SDP relaxation of the QCQP problem. Now, the results spelled out in the preceding subsection can be used to find a low-rank SDP solution.

### *C. Affine Rank Minimization Problem*

Consider the problem

$$\underset{\mathbf{W}\in\mathbb{R}^{m\times r}}{\text{minimize}} \quad \mathrm{rank}\{\mathbf{W}\} \tag{7a}$$

$$\text{subject to} \quad \mathrm{trace}\{\mathbf{N}_k\mathbf{W}\} \leq a_k, \ k = 1,\dots,p \tag{7b}$$

where $\mathbf{N}_1, ..., \mathbf{N}_p \in \mathbb{R}^{r\times m}$ are sparse matrices. This is an affine rank minimization problem without any positive semidefinite constraint. A popular convexification method for the above optimization is to replace its objective with the nuclear norm of $\mathbf{W}$ [20]. This is due to the fact that the nuclear norm $\|\mathbf{W}\|_*$ is the convex envelope for the function $\mathrm{rank}\{\mathbf{W}\}$ on the set $\{\mathbf{W} \in \mathbb{R}^{m\times r} \,|\, \|\mathbf{W}\| \leq 1\}$ [21]. A special case of Optimization (7), known as low-rank matrix completion problem, has been extensively studied in the literature due to its wide range of applications [22]. In this problem, the constraint (7) determines what entries of $\mathbf{W}$ are known.

A closely related problem is the following: can a matrix $\mathbf{W}$ be recovered by observing only a subset of its entries? Interestingly, $\mathbf{W}$ can be successfully recovered by means of a nuclear norm minimization as long as the matrix is non-structured and the number of observed entries of $\mathbf{W}$

is large enough [22]. The performance of the nuclear norm minimization method for the problem of rank minimization subject to general linear constraints has also been assessed in [23]. Based on empirical studies, the nuclear norm technique is very inefficient in the case where the number of free (unconstrained) entries of $\mathbf{W}$ is relatively large. In the present work, we propose a graph-theoretic approach that is able to generate low-rank solutions for a sparse problem of the form (7) and for a matrix completion problem with many unknown entries.

Optimization (7) can be embedded in a bigger problem of the form (1) by associating the matrix $\mathbf{W}$ with a positive semidefinite matrix variable $\mathbf{X}$ defined as

$$\mathbf{X} \triangleq \left[ \begin{array}{cc} \mathbf{X}_1 & \mathbf{W} \\ \mathbf{W}^{\mathrm{T}} & \mathbf{X}_2 \end{array} \right] \tag{8}$$

where $\mathbf{X}_1$ and $\mathbf{X}_2$ are two auxiliary matrices. Note that $\mathbf{W}$ acts as a submatrix of $\mathbf{X}$ corresponding to its first $m$ rows and last $r$ columns. More precisely, consider the nonconvex problem

$$\begin{aligned} \underset{\mathbf{X}\in\mathbb{S}^{r+m}}{\text{minimize}} \quad & \operatorname{rank}\{\mathbf{X}\} && \text{(9a)} \\ \text{subject to} \quad & \operatorname{trace}\{\mathbf{M}_k\mathbf{X}\} \leq a_k,\ k=1,\ldots,p && \text{(9b)} \\ & \mathbf{X} \succeq 0 && \text{(9c)} \end{aligned}$$

where

$$\mathbf{M}_k \triangleq \left[ \begin{array}{cc} \mathbf{0}_{m\times m} & \frac{1}{2}\mathbf{N}_k^{\mathrm{T}} \\ \frac{1}{2}\mathbf{N}_k & \mathbf{0}_{r\times r} \end{array} \right], \tag{10}$$

For every feasible solution $\mathbf{X}$ of the above problem, its associated submatrix $\mathbf{W}$ is feasible for (7) and satisfies

$$\operatorname{rank}\{\mathbf{W}\} \leq \operatorname{rank}\{\mathbf{X}\}. \tag{11}$$

In particular, it is well known that the rank minimization problem (7) with linear constraints is equivalent to the rank minimization (9) with LMI constraints [21], [24]. Let $\widehat{\mathbf{X}}$ denote an arbitrary feasible point of optimization (9). Depending on the sparsity level of the problem (7), some entries of $\widehat{\mathbf{X}}$ are free and do not affect any constraints of (9) except for $\mathbf{X} \succeq 0$. Let the locations of those entries be captured by a bipartite graph. More precisely, define $\mathcal{B}$ as a bipartite graph whose first and second parts of vertices are associated with the rows and columns of $\mathbf{W}$, respectively. Suppose that each edge of $\mathcal{B}$ represents a constrained entry of $\mathbf{W}$. In this work, we propose two convex problems with the following properties:

1) The first convex program is constructed from an arbitrary tree decomposition of $\mathcal{B}$. The rank of every solution to this problem is upper bounded by $t+1$, where $t$ is the width of its tree decomposition. Given the decomposition, the low-rank solution can be found in polynomial time.
2) Since finding a tree decomposition of $\mathcal{B}$ with a low treewidth may be hard in general, the second convex program does not rely on any decomposition and is obtained by relaxing the real-valued problem (9) to a complex-valued convex program. The rank of every solution to the second convex problem is bounded by the number $2(r+m-\operatorname{msr}\{\mathcal{B}\})$ and such a solution can always be found in polynomial time.

## II. Notations and Definitions

For every nonnegative integer $n$, the notation $[n]$ represents the set $\{1,\ldots,n\}$. The symbols $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. $\mathbb{S}^n$ denotes the space of $n\times n$ real symmetric matrices and $\mathbb{H}^n$ denotes the space of $n\times n$ complex Hermitian matrices. Also, $\mathbb{S}^n_+ \subset \mathbb{S}^n$ and $\mathbb{H}^n_+ \subset \mathbb{H}^n$ represent the convex cones of real and complex positive semidefinite matrices, respectively. The notations $\mathbb{F}^n$, $\mathbb{F}^n_+$ and $\mathbb{F}$ refer to either $\mathbb{S}^n$, $\mathbb{S}^n_+$ and $\mathbb{R}$ or $\mathbb{H}^n$, $\mathbb{H}^n_+$ and $\mathbb{C}$ depending on the context (i.e., whether the real or complex domain is under study). $\operatorname{Re}\{\cdot\}$, $\operatorname{Im}\{\cdot\}$, $\operatorname{rank}\{\cdot\}$, and $\operatorname{trace}\{\cdot\}$ denote the real part, imaginary part, rank, and trace of a given scalar/matrix. Matrices are shown by capital and bold letters. The symbols $(\cdot)^{\mathrm{T}}$ and $(\cdot)^*$ denote transpose and conjugate transpose, respectively. Also, "$\mathbf{i}$" is reserved to denote the imaginary unit. The notation $\measuredangle x$ denotes the angle of a complex number $x$. The notation $\mathbf{W} \succeq 0$ means that $\mathbf{W}$ is a Hermitian and positive semidefinite matrix. Also $\mathbf{W} \succ 0$ means that it is Hermitian and strictly positive definite. The $(i,j)$ entry of $\mathbf{W}$ is denoted as $W_{ij}$. The vertex set and edge set of a simple graph $\mathcal{G}$ are shown by the notations $\mathcal{V}_{\mathcal{G}}$ and $\mathcal{E}_{\mathcal{G}}$, and the graph $\mathcal{G}$ is identified by the pair $(\mathcal{V}_{\mathcal{G}},\mathcal{E}_{\mathcal{G}})$. The set of all neighbors of the vertex $k$ in $\mathcal{G}$ is denoted by $\mathcal{N}_{\mathcal{G}}(k)$. The symbol $|\mathcal{G}|$ shows the number of vertices of $\mathcal{G}$.

**Definition 1.** *For two simple graphs $\mathcal{G}_1=(\mathcal{V}_1,\mathcal{E}_1)$ and $\mathcal{G}_2=(\mathcal{V}_2,\mathcal{E}_2)$, the notation $\mathcal{G}_1 \subseteq \mathcal{G}_2$ means that $\mathcal{V}_1 \subseteq \mathcal{V}_2$ and $\mathcal{E}_1 \subseteq \mathcal{E}_2$. $\mathcal{G}_1$ is called a subgraph of $\mathcal{G}_2$ and $\mathcal{G}_2$ is called a supergraph of $\mathcal{G}_1$. A subgraph $\mathcal{G}_1$ of $\mathcal{G}_2$ is said to be an induced subgraph if for every pair of vertices $v_l, v_m \in \mathcal{V}_1$, the relation $(v_l,v_m)\in\mathcal{E}_1$ holds if and only if $(v_l,v_m)\in\mathcal{E}_2$. In this case, $\mathcal{G}_1$ is said to be induced by the vertex subset $\mathcal{V}_1$.*

**Definition 2.** *For two simple graphs $\mathcal{G}_1=(\mathcal{V}_1,\mathcal{E}_1)$ and $\mathcal{G}_2=(\mathcal{V}_2,\mathcal{E}_2)$, the subgraph of $\mathcal{G}_2$ induced by the vertex set $\mathcal{V}_2\setminus\mathcal{V}_1$ is shown by the notation $\mathcal{G}_2\setminus\mathcal{G}_1$.*

**Definition 3.** *For two simple graphs $\mathcal{G}_1=(\mathcal{V},\mathcal{E}_1)$ and $\mathcal{G}_2=(\mathcal{V},\mathcal{E}_2)$ with the same set of vertices, their union is defined as $\mathcal{G}_1\cup\mathcal{G}_2=(\mathcal{V},\mathcal{E}_1\cup\mathcal{E}_2)$ while the notion $\setminus$ shows their subtraction edge-wise, i.e., $\mathcal{G}_1\setminus\mathcal{G}_2=(\mathcal{V},\mathcal{E}_1\setminus\mathcal{E}_2)$.*

**Definition 4.** *The representative graph of an $n\times n$ symmetric matrix $\mathbf{W}$, denoted by $\mathcal{G}(\mathbf{W})$, is a simple graph with $n$ vertices whose vertices $i$ and $j$ are connected if and only if $W_{ij}$ is nonzero.*

## III. Connection Between OS and Treewidth

In this section, we study the relationship between the graph parameters of OS and treewidth. For the sake of completeness, we first review these two graph notions.

**Definition 5** (OS)**.** *Given a graph $\mathcal{G}$, let $\mathcal{O}=\{o_k\}_{k=1}^s$ be a sequence of vertices of $\mathcal{G}$. Define $\mathcal{G}_k$ as the subgraph induced*

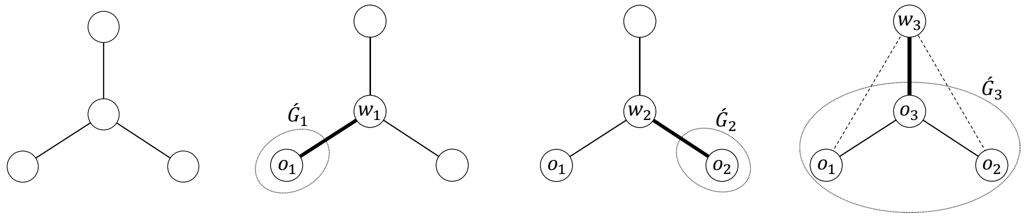


Fig. 1. A maximal OS-vertex sequence for a tree

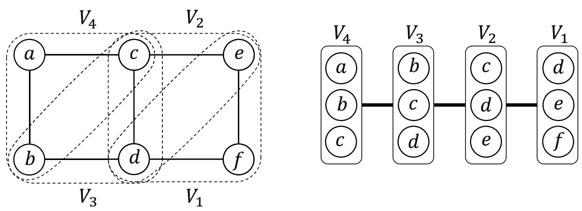


Fig. 2. A minimal tree decomposition for a ladder graph

*by the vertex set* $\{o_1, \ldots, o_k\}$ *for* $k = 1, ..., s$. *Let* $\mathcal{G}'_k$ *be the connected component of* $\mathcal{G}_k$ *containing* $o_k$. $\mathcal{O}$ *is called an OS-vertex sequence of* $\mathcal{G}$ *if for every* $k \in \{1, ..., s\}$, *the vertex* $o_k$ *has a neighbor* $w_k$ *with the following two properties:*

1) $w_k \neq o_r$ *for* $1 \leq r \leq k$
2) $(w_k, o_r) \notin \mathcal{E}_{\mathcal{G}}$ *for every* $o_r \in \mathcal{V}_{G'_k} \setminus \{o_k\}$,

*Denote the maximum cardinality among all OS-vertex sequences of* $\mathcal{G}$ *as* $\mathrm{OS}(\mathcal{G})$ *[14] .*

Figure 1 illustrates the procedure for finding a maximal OS-vertex sequence for a tree. Dashed lines and bold lines highlight nonadjacency and adjacency, respectively, to demonstrate that each $w_i$ satisfies the conditions of Definition 5. The connected component of each $o_k$ in the subgraph induced by $\{o_1, \ldots, o_k\}$ is also shown. Notice that although $w_2$ is connected to $o_1$, it is a valid choice since $o_1$ and $o_2$ do not share the same connected component in $G_2$.

**Definition 6** (Treewidth)**.** *Given a graph* $\mathcal{G}$, *a tree* $\mathcal{T}$ *is called a tree decomposition of* $\mathcal{G}$ *if it satisfies the following properties:*

1) *Every node of* $\mathcal{T}$ *corresponds to and is identified by a subset of* $\mathcal{V}_{\mathcal{G}}$. *Alternatively, each node of* $\mathcal{T}$ *is regarded as a group of vertices of* $\mathcal{G}$.
2) *Every vertex of* $\mathcal{G}$ *is a member of at least one node of* $\mathcal{T}$.
3) $\mathcal{T}_k$ *is a connected graph for every* $k \in \mathcal{V}_{\mathcal{G}}$, *where* $\mathcal{T}_k$ *denotes the subgraph of* $\mathcal{T}$ *induced by all nodes of* $\mathcal{T}$ *containing the vertex* $k$ *of* $\mathcal{G}$.
4) *The subgraphs* $\mathcal{T}_i$ *and* $\mathcal{T}_j$ *have a node in common for every* $(i, j) \in \mathcal{E}_{\mathcal{G}}$.

*The width of a tree decomposition is the cardinality of its biggest node minus one (recall that each node of* $\mathcal{T}$ *is indeed a set containing a number of vertices of* $\mathcal{G}$*). The treewidth of* $\mathcal{G}$ *is the minimum width over all possible tree decompositions of* $\mathcal{G}$ *and is denoted by* $\mathrm{tw}(\mathcal{G})$.

Note that the treewidth of a tree is equal to 1. Figure 2 shows a graph $\mathcal{G}$ with 6 vertices named $a, b, c, d, e, f$, together with its minimal tree decomposition $\mathcal{T}$. Every node of $\mathcal{T}$ is a set containing three members of $\mathcal{V}_{\mathcal{G}}$. The width of this decomposition is therefore equal to 2.

**Definition 7** (Enriched Supergraph)**.** *Given a graph* $\mathcal{G}$ *accompanied by a tree decomposition* $\mathcal{T}$ *of width* $t$, $\overline{\mathcal{G}}$ *is called an enriched supergraph of* $\mathcal{G}$ *derived by* $\mathcal{T}$ *if it is obtained according to the following procedure:*

1) *Add a sufficient number of (redundant) vertices to the nodes of* $\mathcal{T}$, *if necessary, in such a way that every node includes exactly* $t+1$ *vertices. Also, add the same vertices to* $\mathcal{G}$ *(without incorporating new edges). Denote the new graphs associated with* $\mathcal{T}$ *and* $\mathcal{G}$ *as* $\tilde{\mathcal{T}}$ *and* $\tilde{\mathcal{G}}$, *respectively.*
2) *Index the nodes of the tree* $\tilde{\mathcal{T}}$ *as* $V_1, V_2, \ldots, V_{|\mathcal{T}|}$ *in such a way that for every* $r \in \{1, ..., |\mathcal{T}|\}$, *the node* $V_r$ *becomes a leaf of* $\mathcal{T}^r$ *defined as the subgraph of* $\tilde{\mathcal{T}}$ *induced by* $\{V_1, \ldots, V_r\}$. *Denote the neighbor of* $V_r$ *in* $\mathcal{T}^r$ *as* $V_{r'}$ *(note that* $V_r \subseteq \mathcal{V}_{\mathcal{G}}$*).*
3) *Set* $\mathcal{G}^{|\mathcal{T}|} := \tilde{\mathcal{G}}$ *and set* $\mathcal{O}^{|\mathcal{T}|}$ *as the empty sequence. Also set* $k = |\mathcal{T}|$.
4) *Let* $V_k \setminus V_{k'} = \{o_1, \ldots, o_s\}$ *and* $V_{k'} \setminus V_k = \{w_1, \ldots, w_s\}$. *Set*

$$\mathcal{G}^{k-1} := (\mathcal{V}_{\mathcal{G}^k}, \mathcal{E}_{\mathcal{G}^k} \cup \{(o_1, w_1), \ldots, (o_s, w_s)\}) \quad (12)$$

$$\mathcal{O}^{k-1} := \mathcal{O}^k \cup (o_1, \ldots, o_s) \quad (13)$$

$$k := k - 1 \quad (14)$$

5) *If* $k = 1$ *set* $\overline{\mathcal{G}} := \mathcal{G}^1$, $\mathcal{O} := \mathcal{O}^1$ *and terminate; otherwise go to step 4.* $\overline{\mathcal{G}}$ *is referred to as an enriched suppergraph of* $\mathcal{G}$ *derived by* $\mathcal{T}$.

Figure 3 delineates the process of obtaining an enriched supergraph $\overline{\mathcal{G}}$ of the graph $\mathcal{G}$ depicted in Figure 2. Bold lines show the added edges at each step of the algorithm. The last graph in Figure 3 sketches the resulting OS-vertex sequence $\mathcal{O}$. Observe that whether or not each non-bold edge exists in the graph, $\mathcal{O}$ still remains an OS-vertex sequence. The next theorem reveals the relationship between OS and treewidth.

**Theorem 1.** *Given a graph* $\mathcal{G}$ *accompanied by a tree decomposition* $\mathcal{T}$ *of width* $t$, *consider an enriched supergraph* $\overline{\mathcal{G}}$ *of* $\mathcal{G}$ *derived by* $\mathcal{T}$ *together with the sequence* $\mathcal{O}$ *constructed in Definition 7. Then,* $\mathcal{O}$ *is an OS-vertex sequence of every graph* $\mathcal{G}_s$ *in the set* $\left\{\mathcal{G}_s \,\middle|\, (\overline{\mathcal{G}} \setminus \mathcal{G}) \subseteq \mathcal{G}_s \subseteq \overline{\mathcal{G}}\right\}$ *and furthermore* $|\overline{\mathcal{G}}| - |\mathcal{O}| = t + 1$.

**Corollary 1.** *For every graph* $\mathcal{G}$, *there exists a supergraph*

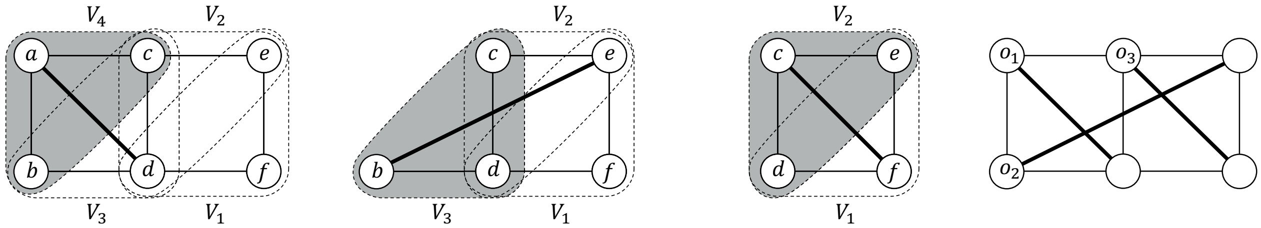


Fig. 3. The first three figures show the process of deriving an enriched supergraph $\overline{\mathcal{G}}$ of the graph $\mathcal{G}$ given in Figure 2. The last figure shows an OS-vertex sequence $\mathcal{O}$ for the graph $\mathcal{G}$.

*$\overline{\mathcal{G}}$ with the property that*

$$|\overline{\mathcal{G}}|-\min_{\mathcal{G}_s}\left\{\mathrm{OS}(\mathcal{G}_s)\,\middle|\,(\overline{\mathcal{G}}\setminus\mathcal{G})\subseteq\mathcal{G}_s\subseteq\overline{\mathcal{G}}\right\}\leq \mathrm{tw}(\mathcal{G})+1 \quad (15)$$

## IV. Low-Rank Solutions Via Graph Decomposition

In this section, we develop a graph-theoretic technique to find a low-rank feasible solution of the LMI problem (1).

**Optimization A:** *Let $\mathcal{G}$ and $\mathcal{G}'$ be two graphs such that $\mathcal{V}_{\mathcal{G}}=[n]$, $\mathcal{V}_{\mathcal{G}'}=[m]$, and $\mathcal{E}_{\mathcal{G}}\subseteq\mathcal{E}_{\mathcal{G}'}$. Consider arbitrary matrices $\widehat{\mathbf{X}}\in\mathbb{F}^n_+$ and $\mathbf{Z}\in\mathbb{F}^m$ with the property that $\mathcal{G}(\mathbf{Z})=\mathcal{G}'$, where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$. The problem*

$$\underset{\overline{\mathbf{X}}\in\mathbb{F}^m}{\textit{minimize}} \quad \mathrm{trace}\{\mathbf{Z}\overline{\mathbf{X}}\} \quad (16a)$$
$$\textit{subject to} \quad \overline{X}_{kk}=\widehat{X}_{kk}, \quad k\in\mathcal{V}_{\mathcal{G}} \quad (16b)$$
$$\overline{X}_{kk}=1, \quad k\in\mathcal{V}_{\mathcal{G}'}\setminus\mathcal{V}_{\mathcal{G}} \quad (16c)$$
$$\overline{X}_{ij}=\widehat{X}_{ij}, \quad (i,j)\in\mathcal{E}_{\mathcal{G}} \quad (16d)$$
$$\overline{\mathbf{X}}\succeq 0 \quad (16e)$$

*is referred to as* "Optimization A with the input $(\mathcal{G},\mathcal{G}',\mathbf{Z},\widehat{\mathbf{X}})$*"*.

Optimization A is a convex program with a non-empty feasible set. Let $\overline{\mathbf{X}}^{\mathrm{opt}}\in\mathbb{F}^m$ denote an arbitrary solution of Optimization A with the input $(\mathcal{G}_1,\mathcal{G}_2,\mathbf{Z},\widehat{\mathbf{X}})$ and $\mathbf{X}^{\mathrm{opt}}\in\mathbb{F}^n$ represent a matrix obtained from $\overline{\mathbf{X}}^{\mathrm{opt}}$ by removing its last $m-n$ rows and $m-n$ columns. Then, $\mathbf{X}^{\mathrm{opt}}$ is called the *subsolution to Optimization A associated with* $\overline{\mathbf{X}}^{\mathrm{opt}}$. Note that $\mathbf{X}^{\mathrm{opt}}$ and $\widehat{\mathbf{X}}$ share the same diagonal and values for the entries corresponding to the edges of $\mathcal{G}$. Hence, Optimization A is intrinsically a positive semidefinite matrix completion problem with the input $\widehat{\mathbf{X}}$ and the output $\mathbf{X}^{\mathrm{opt}}$.

**Definition 8** (msr)**.** *Given a simple graph $\mathcal{G}$, define the minimum semidefinite rank of $\mathcal{G}$ as*

$$\mathrm{msr}(\mathcal{G})\triangleq\min\left\{\mathrm{rank}(\mathbf{W})\,|\,\mathcal{G}(\mathbf{W})=\mathcal{G},\ \mathbf{W}\succeq\mathbf{0}\right\} \quad (17)$$

**Theorem 2.** *Assume that $\mathbf{M}_1,\ldots,\mathbf{M}_p$ are arbitrary matrices in $\mathbb{F}^n$ which is equal to either $\mathbb{S}^n$ or $\mathbb{H}^n$. Suppose that $a_1,\ldots,a_p$ are real numbers such that the feasibility problem* (1) *has a solution $0\prec\widehat{\mathbf{X}}\in\mathbb{F}^n$. Let $\mathcal{G}=\mathcal{G}(\mathbf{M}_1)\cup\cdots\cup\mathcal{G}(\mathbf{M}_p)$.*

a) *Consider an arbitrary supergraph $\mathcal{G}'$ of $\mathcal{G}$. Every subsolution $\mathbf{X}^{\mathrm{opt}}$ to Optimization A with the input $(\mathcal{G},\mathcal{G}',\mathbf{Z},\widehat{\mathbf{X}})$ is a solution to the LMI problem* (1) *and satisfies the relation*

$$\mathrm{rank}\{\mathbf{X}^{\mathrm{opt}}\}\leq|\mathcal{G}'|-\min_{\mathcal{G}_s}\left\{\mathrm{msr}(\mathcal{G}_s)\,\middle|\,(\mathcal{G}'\setminus\mathcal{G})\subseteq\mathcal{G}_s\subseteq\mathcal{G}'\right\} \quad (18)$$

b) *Consider an arbitrary tree decomposition $\mathcal{T}$ of $\mathcal{G}$ with width $t$. Let $\overline{\mathcal{G}}$ be an enriched supergraph of $\mathcal{G}$ derived by $\mathcal{T}$. Every subsolution $\mathbf{X}^{\mathrm{opt}}$ to Optimization A with the input $(\mathcal{G},\overline{\mathcal{G}},\mathbf{Z},\widehat{\mathbf{X}})$ is a solution to* (1) *and satisfies the relation*

$$\mathrm{rank}\{\mathbf{X}^{\mathrm{opt}}\}\leq t+1 \quad (19)$$

Observe that the objective function of Optimization A is a weighted sum of certain entries of the matrix $\overline{\mathbf{X}}$, where the weights come from the matrix $\mathbf{Z}$. Part (a) of Theorem 2 proposes an upper bound on the rank of all subsolutions of this optimization, which is contingent upon the graph of the weighting matrix $\mathbf{Z}$ without making use of the nonzero values of the weights.

**Remark 1.** *Suppose that $\widehat{\mathbf{X}}$ is a solution to the feasibility problem* (1) *that is not strictly positive definite. For every $\varepsilon>0$, we have $\widehat{\mathbf{X}}+\varepsilon\mathbf{I}\succ 0$. Hence, according to Parts (a) and (b) of Theorem 2, every subsolution $\mathbf{X}^{\mathrm{opt}}_{\varepsilon}$ to Optimization A with the inputs $(\mathcal{G},\mathcal{G}',\mathbf{Z},\widehat{\mathbf{X}}+\varepsilon\mathbf{I})$ and $(\mathcal{G},\overline{\mathcal{G}},\mathbf{Z},\widehat{\mathbf{X}}+\varepsilon\mathbf{I})$ satisfies the respective rank conditions* (18) *and* (19)*. Since the matrix $\mathbf{X}^{\mathrm{opt}}_{\varepsilon}$ is feasible for* (1) *whenever $\varepsilon=0$, choosing a sufficiently small $\epsilon$ enables us to obtain a low-rank solution $\mathbf{X}^{\mathrm{opt}}_{\varepsilon}$ that violates the constraints of the LMI problem* (1) *less than any prescribed number.*

The following result follows immediately from Theorem 2 and Remark 1.

**Corollary 2.** *If the real/complex LMI problem* (1) *is feasible, then it has a solution $\mathbf{X}^{\mathrm{opt}}$ with rank at most $\mathrm{tw}(\mathcal{G})+1$.*

## V. Low-Rank Solutions via Complex Analysis

Consider the problem of finding a low-rank solution $\mathbf{X}^{\mathrm{opt}}$ for the LMI problem (1). Theorem 2 can be used for this purpose, but it needs solving one of the following graph problems: (i) designing a supergraph $\mathcal{G}'$ minimizing the upper bound given in (18), or (ii) obtaining a tree decomposition of $\mathcal{G}$ with the minimum width. Although these graph problems are easy to solve for highly sparse and structured graphs, they are NP-hard for arbitrary graphs. A question arises as to whether a low-rank solution can be obtained using a polynomial-time algorithm without requiring an expensive graph analysis. This problem will be addressed in this section.

**Definition 9.** *Given a complex number $z$, define*

$$z^{\mathrm{ray}} \triangleq \{\lambda z \,|\, \lambda \in \mathbb{R},\ \lambda \geq 0\}. \tag{20}$$

**Optimization B:** *Let $\mathcal{G}$ be a simple graph with $n$ vertices and $\mathbb{F}$ be equal to either $\mathbb{S}$ or $\mathbb{H}$. Consider arbitrary matrices $\widehat{\mathbf{X}} \in \mathbb{F}^n_+$ and $\mathbf{Z} \in \mathbb{F}^n$ such that $\mathcal{G}(\mathbf{Z})$ is a supergraph of $\mathcal{G}$. The problem*

$$\underset{\mathbf{X}\in\mathbb{F}^n}{\text{minimize}} \quad \mathrm{trace}\{\mathbf{Z}\mathbf{X}\} \tag{21a}$$

$$\text{subject to} \quad X_{kk} = \widehat{X}_{kk}, \qquad k \in \mathcal{V}_{\mathcal{G}} \tag{21b}$$

$$X_{ij} - \widehat{X}_{ij} \in Z^{\mathrm{ray}}_{ij}, \qquad (i,j) \in \mathcal{E}_{\mathcal{G}} \tag{21c}$$

$$\mathbf{X} \succeq 0 \tag{21d}$$

*is referred to as "*Optimization B with the input *$(\mathcal{G}, \widehat{\mathbf{X}}, \mathbf{Z}, \mathbb{F})$".*

**Theorem 3.** *Assume that $\mathbf{M}_1, \ldots, \mathbf{M}_p$ are arbitrary matrices in $\mathbb{S}^n$. Suppose that $a_1, \ldots, a_p$ are real numbers such that the LMI problem* (1) *has a feasible solution $0 \prec \widehat{\mathbf{X}} \in \mathbb{S}^n$. Let $\mathbf{Z} \in \mathbb{H}^n$ be an arbitrary matrix such that $\mathrm{Re}\{\mathbf{Z}\} = 0_{n\times n}$ and $\mathcal{G}(\mathbf{Z})$ is a supergraph of $\mathcal{G}(\mathbf{M}_1) \cup \cdots \cup \mathcal{G}(\mathbf{M}_p)$.*

a) *Every solution $\mathbf{X}^{\mathrm{opt}} \in \mathbb{H}^n$ of Optimization B with the input $(\mathcal{G}, \widehat{\mathbf{X}}, \mathbf{Z}, \mathbb{H})$ is a solution of the LMI problem* (1) *and satisfies the relation*

$$\mathrm{rank}\{\mathbf{X}^{\mathrm{opt}}\} \leq n - \mathrm{msr}(\mathcal{G}(\mathbf{Z})). \tag{22}$$

b) *The matrix* $\mathrm{Re}\{\mathbf{X}^{\mathrm{opt}}\}$ *is a real-valued solution of the LMI problem* (1) *and satisfies the inequality*

$$\mathrm{rank}\{\mathbf{X}^{\mathrm{real}}\} \leq \min\{2(n - \mathrm{msr}(\mathcal{G}(\mathbf{Z})), n\}. \tag{23}$$

Consider an LMI problem with real-valued coefficients. Theorem 3 states that the complex-valued Optimization B can be exploited to find a real solution of the LMI problem under study with a guaranteed bound on its rank. This bound might be looser than the ones derived in Theorem 2, but is still small for very sparse graphs. Note that although the calculation of the bound given in (23) is an NP-hard problem, Optimization B is polynomial-time solvable without requiring any expensive graph preprocessing. The generalization of Theorem 3 to a complex LMI problem may be found in [7].

**Remark 2.** *Suppose that $\widehat{\mathbf{X}}$ is a solution to the feasibility problem* (1) *that is not strictly positive definite. For every $\varepsilon > 0$, we have $\widehat{\mathbf{X}} + \varepsilon\mathbf{I} \succ 0$. Hence, according to Theorem 3, every subsolution $\mathbf{X}^{\mathrm{opt}}_{\varepsilon}$ to Optimization B with the input $(\mathcal{G}, \widehat{\mathbf{X}} + \varepsilon I, \mathbf{Z}, \mathbb{H})$ satisfies the rank condition* (23)*. Since the matrix $\mathbf{X}^{\mathrm{opt}}_{\varepsilon}$ is feasible for* (1) *whenever $\varepsilon = 0$, choosing a sufficiently small $\epsilon$ enables us to obtain a low-rank solution $\mathbf{X}^{\mathrm{opt}}_{\varepsilon}$ that violates the constraints of the LMI problem* (1) *less than any prescribed number.*

## VI. Low-Rank Solutions for Affine Problems

In this section, we will extend the results derived earlier to the affine rank minimization problem.

**Definition 10.** *For an arbitrary matrix $\mathbf{W} \in \mathbb{C}^{m\times r}$, the notation $\mathcal{B}(\mathbf{W}) = (\mathcal{V}_{\mathcal{B}}, \mathcal{E}_{\mathcal{B}})$ denotes a bipartite graph defined as:*

1) *$\mathcal{V}_{\mathcal{B}} = \mathcal{V}_1 \cup \mathcal{V}_2$ where $|\mathcal{V}_1| = m$, $|\mathcal{V}_2| = r$ and $\mathcal{V}_1 \cap \mathcal{V}_2 = \varnothing$.*
2) *For every $(i,j) \in \mathcal{V}_1 \times \mathcal{V}_2$, we have $(i,j) \in \mathcal{E}_{\mathcal{B}}$ if and only if $W_{ij} \neq 0$.*

**Definition 11.** *Consider an arbitrary matrix $\mathbf{X} \in \mathbb{H}^n$ and two natural numbers $m$ and $r$ such that $n \geq m + r$. The matrix $\mathrm{sub}_{m,r}\{\mathbf{X}\}$ is defined as the $m \times r$ submatrix of $\mathbf{X}$ located in the intersection of rows $1, ..., m$ and columns $m+1, ..., m+r$ of $\mathbf{X}$.*

**Theorem 4.** *Consider the feasibility problem*

$$\mathrm{trace}\{\mathbf{N}_k\mathbf{W}\} \leq a_k,\ k = 1, \ldots, p \tag{24}$$

*where $a_1, \ldots, a_p \in \mathbb{R}$ and $\mathbf{N}_1, \ldots, \mathbf{N}_p \in \mathbb{R}^{r\times m}$. Let $\widehat{\mathbf{W}} \in \mathbb{R}^{m\times r}$ denote a feasible solution of this feasibility problem and $\widehat{\mathbf{X}} \in \mathbb{S}^{r+m}_+$ be a matrix such that $\mathrm{sub}_{r,m}\{\widehat{\mathbf{X}}\} = \widehat{\mathbf{W}}$. Define $\mathcal{G} = \mathcal{B}(\mathbf{N}_1^{\mathrm{T}}) \cup \cdots \cup \mathcal{B}(\mathbf{N}_p^{\mathrm{T}})$. The following statements hold:*

a) *Consider an arbitrary supergraph $\mathcal{G}'$ of $\mathcal{G}$ with $n$ vertices, where $n \geq r+m$. Let $\mathbf{X}^{\mathrm{opt}}$ denote an arbitrary solution of Optimization A with the input $(\mathcal{G}, \mathcal{G}', \mathbf{Z}, \widehat{\mathbf{X}})$. Then, $\mathbf{W}^{\mathrm{opt}}$ defined as $\mathrm{sub}_{m,r}\{\mathbf{X}^{\mathrm{opt}}\}$ is a solution of the feasibility problem* (24) *and satisfies the relation*

$$\mathrm{rank}\{\mathbf{W}^{\mathrm{opt}}\} \leq |\mathcal{G}'| - \min\big\{\mathrm{msr}(\mathcal{G}_s) \,\big|\, (\mathcal{G}' \setminus \mathcal{G}) \subseteq \mathcal{G}_s \subseteq \mathcal{G}'\big\} \tag{25}$$

b) *Consider an arbitrary tree decomposition $\mathcal{T}$ of $\mathcal{G}$ with width $t$. If $\mathcal{G}'$ in Part (a) is considered as an enriched supergraph of $\mathcal{G}$ derived by $\mathcal{T}$, then*

$$\mathrm{rank}\{\mathbf{W}^{\mathrm{opt}}\} \leq t + 1 \tag{26}$$

c) *Let $\mathbf{X}^{\mathrm{opt}}$ denote an arbitrary solution of Optimization B with the input $(\mathcal{G}, \widehat{\mathbf{X}}, \mathbf{Z}, \mathbb{H})$. Then, $\mathbf{W}^{\mathrm{opt}}$ defined as $\mathrm{sub}_{m,r}\{\mathrm{Re}\{\mathbf{X}^{\mathrm{opt}}\}\}$ is a solution of the feasibility problem* (24) *and satisfies the relation*

$$\mathrm{rank}\{\mathbf{W}^{\mathrm{real}}\} \leq \min\{2(r + m - \mathrm{msr}(\mathcal{G}(\mathbf{Z})), r, m\}. \tag{27}$$

**Corollary 3.** *If the feasibility problem* (24) *has a non-empty feasible set, then it has a solution $\mathbf{W}^{\mathrm{opt}}$ with rank at most $\mathrm{tw}\big(\mathcal{B}(\mathbf{N}_1^{\mathrm{T}}) \cup \cdots \cup \mathcal{B}(\mathbf{N}_p^{\mathrm{T}})\big) + 1$.*

In what follows, we adapt Theorem 4 to improve upon the nuclear norm method by incorporating a weighted sum into this norm and then we obtain a guaranteed bound on the rank of every solution of the underlying convex optimization.

**Theorem 5.** *Suppose that $\mathcal{B}$ is a bipartite graph with $|\mathcal{V}_{\mathcal{B}_1}| = m$ and $|\mathcal{V}_{\mathcal{B}_2}| = r$. Given arbitrary matrices $\widehat{\mathbf{W}}$ and $\mathbf{Q}$ in $\mathbb{R}^{m\times r}$, consider the convex program*

$$\underset{\mathbf{W}\in\mathbb{R}^{m\times r}}{\text{minimize}} \quad \|\mathbf{W}\|_* + \mathrm{trace}\{\mathbf{Q}^{\mathrm{T}}\mathbf{W}\} \tag{28a}$$

$$\text{subject to} \quad W_{ij} = \widehat{W}_{ij},\ (i,j) \in \mathcal{E}_{\mathcal{B}} \tag{28b}$$

*Let $\mathcal{B}'$ be defined as the supergraph $\mathcal{B} \cup \mathcal{B}(\mathbf{Q})$. Then, every* *solution* $\mathbf{W}^{\mathrm{opt}}$ *of the optimization* (28) *satisfies the inequality*

$$\operatorname{rank}\{\mathbf{W}^{\mathrm{opt}}\} \leq m + r \\ - \min\left\{\operatorname{msr}(\mathcal{B}_s) \,\middle|\, (\mathcal{B}' \setminus \mathcal{B}) \subseteq \mathcal{B}_s \subseteq \mathcal{B}'\right\}. \quad (29)$$

The nuclear norm method reviewed in Subsection I-C corresponds to the case $\mathbf{Q} = 0$ in Theorem 5. However, this theorem discloses the role of the weighting matrix $\mathbf{Q}$. In particular, this matrix can be designed based on the results developed in Section III to yield a small number for the upper bound given in (29), provided $\mathcal{B}$ is a sparse graph.

## VII. Applications

### A. Optimal Power Flow Problem

Consider an $n$-bus power network with the topology described by a simple graph $\mathcal{G}$, meaning that each vertex belonging to $\mathcal{V}_\mathcal{G} = \{1, \ldots, n\}$ represents a node of the network and each edge $(i, j)$ belonging to $\mathcal{E}_\mathcal{G}$ represents a transmission line with the admittance $y_{ij}$. Define $\mathbf{V} \in \mathbb{C}^n$ as the voltage phasor vector, i.e., $V_k$ is the voltage phasor for node $k \in \mathcal{V}_\mathcal{G}$. Let $\mathbf{S} = \mathbf{P} + \mathbf{Q}\mathbf{i}$ represent the nodal complex power vector, where $\mathbf{P} \in \mathbb{R}^n$ and $\mathbf{Q} \in \mathbb{R}^n$ are the vectors of active and reactive powers injected at all buses. $S_k$ can be interpreted as the complex-power supply minus the complex-power demand at node $k$ of the network. The classical optimal power flow (OPF) problem is as follows:

$$\underset{\mathbf{V},\mathbf{P},\mathbf{Q}\in\mathbb{C}^n}{\text{minimize}} \sum_{k\,\in\,\mathcal{V}_\mathcal{G}} f_k(P_k) \quad (30a)$$

$$\text{subject to } P_k + Q_k\mathbf{i} = \sum_{i\in\mathcal{N}_\mathcal{G}(k)} V_k(V_k^* - V_i^*)y_{ki}^*, \; k \in \mathcal{V}_\mathcal{G} \quad (30b)$$

$$V_k^{\min} \leq |V_k| \leq V_k^{\max}, \qquad k \in \mathcal{V}_\mathcal{G} \quad (30c)$$

$$P_k^{\min} \leq P_k \leq P_k^{\max}, \qquad k \in \mathcal{V}_\mathcal{G} \quad (30d)$$

$$Q_k^{\min} \leq Q_k \leq Q_k^{\max}, \qquad k \in \mathcal{V}_\mathcal{G} \quad (30e)$$

$$\operatorname{Re}\{V_i(V_i^* - V_j^*)y_{ij}^*\} \leq P_{ij}^{\max}, \; (i,j) \in \mathcal{E}_\mathcal{G} \quad (30f)$$

$$|V_i - V_j| \leq \Delta V_{ij}^{\max}, \qquad (i,j) \in \mathcal{E}_\mathcal{G} \quad (30g)$$

where $V_k^{\min}$, $V_k^{\max}$, $P_k^{\min}$, $P_k^{\max}$, $Q_k^{\min}$, $Q_k^{\max}$, $P_{ij}^{\max}$, and $\Delta V_{ij}^{\max}$ are some given network limitations, $f_k(S_k)$ is a convex function accounting for the power generation cost at node $k$, and $\mathcal{N}_\mathcal{G}(k)$ denotes the set of neighbors of bus $k$.

The OPF problem is a highly non-convex problem that is known to be difficult to solve in general. However, the constraints of optimization (30) can all be expressed as linear functions of the entries of the quadratic matrix $\mathbf{V}\mathbf{V}^*$. This implies that the constraints of OPF are linear in terms of a matrix variable $\mathbf{X} \triangleq \mathbf{V}\mathbf{V}^*$. One can reformulate OPF by replacing each $V_iV_j^*$ by $X_{ij}$ and represent the constraints in the form of (1a) with a union graph that is isomorphic to the network topology graph $\mathcal{G}$. In order to preserve the equivalence of the two formulations, two additional constraints must be added to the problem: (i) $\mathbf{X} \succeq 0$, (ii) $\operatorname{rank}\{\mathbf{X}\} = 1$. If we drop the rank condition as the only non-convex constraint of the reformulated OPF problem, we attain the SDP relaxation of OPF that is convex. On the other hand, the parameter $\operatorname{tw}(\mathcal{G})$ is perceived to be small for graphs that describe a practical network topology. We have verified the treewidth of $\mathcal{G}$ for several power systems and reported our findings in [25]. It can be seen that the treewidth of a Polish network with 3375 nodes is at most 28. As long as the treewidth is relatively small, Theorem 2 states that the convex relaxation method yields a low-rank SDP solution for the OPF problem that can be found using convex optimization. We have performed more than 7000 numerical simulations in [19] to elucidate the possibility of the retrieval of a near-optimal global solution of OPF from a low-rank SDP solution.

### B. Optimal Distributed Control Problem

Consider the discrete-time system

$$\begin{cases} x[\tau+1] = Ax[\tau] + Bu[\tau] \\ \quad\; y[\tau] = Cx[\tau] \end{cases} \qquad \tau = 0, 1, 2, \ldots \quad (31)$$

with the known parameters $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times r}$, $C \in \mathbb{R}^{m\times n}$ and $x[0] \in \mathbb{R}^n$, where $x[\tau]$, $u[\tau]$ and $y[\tau]$ represent the state, input and output of the system, respectively. The goal is to design a decentralized (distributed) controller minimizing a quadratic cost functional. With no loss of generality, we focus on the static case where the objective is to design a static controller of the form $u[\tau] = Ky[\tau]$ under the constraint that the controller gain $K$ must belong to a given linear subspace $\mathcal{K} \subseteq \mathbb{R}^{r\times m}$. The set $\mathcal{K}$ captures the sparsity structure of the unknown decentralized controller $u[\tau] = Ky[\tau]$ and, more specifically, it contains all $r \times m$ real-valued matrices with forced zeros in certain entries.

In this case, the optimal decentralized problem (ODC) aims to design a static controller $u[\tau] = Ky[\tau]$ to minimize the finite-horizon cost functional

$$\sum_{\tau=0}^{p} \left(x[\tau]^{\mathrm{T}}x[\tau] + u[\tau]^{\mathrm{T}}u[\tau]\right)$$

subject to the system dynamics (31) and the controller requirement $K \in \mathcal{K}$, for a terminal time $p$ [6], [26], [27]. To simplify this NP-hard problem, define the vectors

$$\begin{aligned} x &= \left[x[0]^{\mathrm{T}} \,\cdots\, x[p]^{\mathrm{T}}\right]^{\mathrm{T}}, \; u = \left[u[0]^{\mathrm{T}} \,\cdots\, u[p]^{\mathrm{T}}\right]^{\mathrm{T}}, \\ y &= \left[y[0]^{\mathrm{T}} \,\cdots\, y[p]^{\mathrm{T}}\right]^{\mathrm{T}}, \; v = \left[1 \; h^{\mathrm{T}} \; x^{\mathrm{T}} \; u^{\mathrm{T}} \; y^{\mathrm{T}}\right]^{\mathrm{T}} \end{aligned} \quad (32)$$

where $h$ denotes the vector of all nonzero (free) entries of $K$. The objective function and constraints of the ODC problem are all quadratic with respect to the vector $v$. However, they can be cast as linear functions of the entries of the matrix $vv^{\mathrm{T}}$. Thus, by replacing $vv^{\mathrm{T}}$ with a new variable $\mathbf{X}$, ODC can be expressed as a linear program with respect to this new variable. Nevertheless, in order to preserve the equivalence through reformulation, three additional constraints need to be imposed: (i) $\mathbf{X} \succeq 0$, (ii) $\operatorname{rank}\{\mathbf{X}\} = 1$, and (iii) $X_{11} = 1$. Note that the constraint (ii) carries all the non-convexity of the reformulated ODC problem. By dropping this rank constraint, an SDP relaxation of the ODC problem will be attained.

The ODC problem has a natural sparsity, which makes its SDP relaxation possess a low-rank solution. To pinpoint the underlying sparsity pattern of the problem, we construct a graph $\mathcal{G}$ as follows:

- Let $\eta$ denote the size of the vector $v$. The graph $\mathcal{G}$ has $\eta$ vertices corresponding to the entries of $v$. In particular, the vertex set of $\mathcal{G}$ can be partitioned into five vertex subsets, where subset 1 consist of a single vertex associated with the number 1 in the vector $v$ and subsets 2-5 correspond to the vectors $x$, $u$, $y$, and $h$, respectively.
- Given two distinct numbers $i, j \in \{1, ..., \eta\}$, vertices $i$ and $j$ are connected in the graph $\mathcal{G}$ if and only if the quadratic term $v_i v_j$ appears in the objective or one of the constraints of the reformulated ODC problem. As an example, vertex 1 is connected to the vertex subsets corresponding to the vectors $x$, $u$, and $y$. This is due to the fact that the linear terms $x[\tau]$, $u[\tau]$ and $y[\tau]$ appear in the optimization (notice that $x[\tau]$ can be regarded as $1 \times x[\tau]$, implying the product of 1 and $x[\tau]$).

The parameter $\mathrm{tw}(\mathcal{G})$ depends only on the sparsity pattern of $K$. For example, $\mathrm{tw}(\mathcal{G})$ is equal to 2 for a diagonal controller $K$. In this case, it follows from Theorem 2 that the SDP relaxation of the ODC problem has a solution $\mathbf{X}^{\mathrm{opt}}$ with rank at most 3. To obtain such a solution, we need to create a *supergraph* $\overline{\mathcal{G}}$ of $\mathcal{G}$ using Theorem 1 as follows. First, we connect the vertices corresponding to the $i^{\mathrm{th}}$ and $(i+1)^{\mathrm{th}}$ entries of $h$ for $i = 1, 2, ..., m-1$. Then, we add a new vertex to the resulting graph and connect it to all of the existing vertices. It can be shown that $|\overline{\mathcal{G}}| - \mathrm{OS}(\mathcal{G}_s) \leq 3$ for every $\mathcal{G}_s$ such that $(\overline{\mathcal{G}} \setminus \mathcal{G}) \subseteq \mathcal{G}_s \subseteq \overline{\mathcal{G}}$. Now, the supergraph $\overline{\mathcal{G}}$ can be fed into Theorem 2 to find a solution $\mathbf{X}^{\mathrm{opt}}$ with rank at most 3. We have performed several numerical simulations in [28] and observed that the resulting rank-3 matrix can be well approximated by a rank-1 matrix, leading to a near-optimal solution of ODC.

## VIII. Conclusions

This paper deals with obtaining a low-rank solution for an arbitrary linear matrix inequality problem. To this end, two graph-theoretic convex programs are derived, which rely on the graph notions of OS, msr, and treewidth. It is explained how the developed results can be deployed for three fundamental problems of minimum-rank matrix completion, conic relaxation for polynomial optimization, and affine rank minimization. The results are also applied to two applications in the areas of electrical power networks and control systems.

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