Conic Optimization for Robust Quadratic Regression

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Abstract

This paper is concerned with the robust quadratic regression problem, where the goal is to find the unknown parameters (state) of a system modeled by a set of quadratic equations. In this problem, a subset of equations are subject to errors (noise values) of arbitrary magnitudes. We propose two penalized conic optimization methods based on semidefinite programming and second-order cone programming to address this problem. Different conditions are derived to guarantee the exact recovery of a solution of the nonlinear regression problem. The results are first developed for arbitrary deterministic models, and then refined for Gaussian models. The obtained conditions consist in bounds on the number of bad measurements each method can tolerate without producing a nonzero estimation error. It is proved that the proposed methods allow up to a half of the total number of measurements to be grossly erroneous. The efficacy of the developed methods is demonstrated in different case studies, including data analytics for a European power grid.
Keywords: Robust Regression, Conic Programming, Bad Data Detection

1. Introduction

Nonlinear regression aims to find the parameters of a given model based on observational data. One may assume the existence of a potentially nonlinear function $f(\mathbf{x};\mathbf{a})$ defined over the set of all possible models $\mathbf{x} \in \mathcal{X}$ and all possible inputs $\mathbf{a} \in \mathcal{A}$, where the goal is to estimate the true model given a set of imperfect measurements y_i's:

$$y_i = f(\mathbf{x}, \mathbf{a}_i) + \eta_i, \qquad \forall i \in \{1, ..., m\}$$

In this formulation, the unknown error vector $\boldsymbol{\eta}$ could be the measurement noise with modest values. However, a more drastic scenario corresponds to the case where the vector $\boldsymbol{\eta}$ is sparse and its nonzero entries are allowed to be arbitrarily large. Under this circumstance, *a priori* information about the probability distribution of the sparse vector $\boldsymbol{\eta}$ may be available, in

addition to an upper bound on the cardinality of $\boldsymbol{\eta}$. This important problem is referred to as *robust regression* and appears in real-world situations when some observations, named outliers, are completely wrong in an unpredictable way. This could occur during an image acquisition with several corrupted pixels, or result from communication issues during data transmission for sensor networks. Such problems arise in different domains of applications and have been studied in the literature. In the context of electric power grid, the regression problem is known as state estimation, where the goal is to find the operating point of the system based on the voltage signals measured at buses and power signals measured over lines and at buses (Abur and Exposito (2004); Madani et al. (2017b); Zhang et al. (2018b)). Outliers in this case are associated with faulty sensors, cyber attacks, or regional data manipulation to impact the electricity market (Jin et al. (2017); Madani et al. (2017b)).

There are several classical works on robust regression and outliers detection. The book by Rousseeuw and Leroy (2005) offers an overview of many fundamental results in this area dating back to 1887 when Edgeworth proposed the least-absolute-value regression estimator. Modern techniques for handling sparse errors of arbitrary magnitudes vary with respect to different features: statistical properties of the error, class of the regression model $f(\mathbf{x}; \mathbf{a})$, set of possible true models, type of theoretical guarantees, and characteristics of the adversary model generating errors (Candès et al. (2011); Nasrabadi et al. (2011); Bhatia et al. (2015); Zhang et al. (2016); Klopp et al. (2017)). There is a plethora of papers on this topic for the well-known linear regression problem (Candes and Tao (2005); Wright and Ma (2010); Studer et al. (2012); Chen et al. (2013a); Bhatia et al. (2017)). In this case, the function $f(\mathbf{x}; \mathbf{a})$ is linear in the model vector $\mathbf{x}$, and can be written as $\mathbf{a}^*\mathbf{x}$. Nevertheless, there are far less results known for nonlinear regression. This is due to the fact that linear regression amounts to a system of linear equations with a cubic solution complexity if the measurements are error-free, whereas nonlinear regression is NP-hard and its complexity further increases with the inclusion of premeditated errors. However, very special cases of nonlinear regression have been extensively studied in the literature. In particular, the robust phase retrieval problem that can be formulated with $f(\mathbf{x}; \mathbf{a}_i) = |\mathbf{a}_i^*\mathbf{x}|^2$ has received considerable attention (Zhang et al. (2016); Hand and Voroninski (2016); Chen et al. (2017)).

To model a general nonlinear regression problem, notice that every smooth nonlinear function can be approximated arbitrarily precisely with a polynomial function, and that every polynomial function can be converted to a quadratic function subject to quadratic equality constraints (playing the role of error-free quadratic measurements) after introducing specific auxiliary variables (Sojoudi et al. (2014)). This implies that every nonlinear regression could be approximated up to any arbitrary precision with a quadratic regression where the augmented model of the system is quadratic. As a far more general case of phase retrieval, a quadratic regression problem with the variable $\mathbf{x}$ can be modeled as $f(\mathbf{x}; \mathbf{A}_i) = \mathbf{x}^*\mathbf{A}_i\mathbf{x}$. The state estimation problem for power systems belongs to the above model due to the quadratic laws of physics (i.e., the quadratic relationship between voltage and power), where each matrix $\mathbf{A}_i$ has rank 1 or 2. Robust regression in power systems is referred to as *bad data detection*. This problem was first studied in 1971 by Merrill and Schweppe (1971), and there are many recent progresses on this topic (Deka et al. (2015); Weng et al. (2015); Madani et al. (2017b)).

The existing approaches for robust regression include the analysis of the unconstrained case (Candes and Tao (2005); Studer et al. (2012); Bhatia et al. (2015, 2017); Josz et al.

(2018)), the constrained scenario with conditions on the sparsity of the solution vector $\mathbf{x}$ (Wright and Ma (2010); Nasrabadi et al. (2011); Nguyen and Tran (2013); McWilliams et al. (2014)), and more sophisticated scenarios in the context of matrix completion (Candès et al. (2011); Chen et al. (2013b); Klopp et al. (2017); Zhang et al. (2018a)). Motivated by applications in inverse covariance estimation (Wang and Lin (2014)), the papers Xu et al. (2009); Yang and Xu (2013); McWilliams et al. (2014) consider sparse noise in the input vector $\mathbf{a}_i$ as opposed to the additive error considered in the present paper. The work Candes and Tao (2005) is based on l_1-minimization, whereas Nasrabadi et al. (2011) solve an extended Lasso formulation defined as the minimization of $\|\mathbf{y}-\mathbf{A}\mathbf{x}+\boldsymbol{\nu}\|_2^2+\mu_1\|\mathbf{x}\|_1+\mu_2\|\boldsymbol{\nu}\|_1$. The work Dalalyan and Chen (2012) proposes to solve a second-order cone programming (SOCP) for robust linear regression, which is related to the current paper with a focus on robust nonlinear regression. In contrast to the above-mentioned papers that aim to develop a single optimization problem to estimate the solution of a regression, there are iterative-based methods as well. For instance, Chen et al. (2013a); Bhatia et al. (2015, 2017) propose iterative algorithms via hard thresholding. The current paper significantly promotes and extends the ideas proposed in Molybog et al. (2018).

Due to the diversity in the problem formulation and approaches taken by different papers, it is difficult to compare the existing results since there is no single dominant method. However, the most common measures of performance for robust regression algorithms are the traditional algorithmic complexity and the permissible number of gross measurements $\|\boldsymbol{\eta}\|_0$ compared to the total number of measurements m. In this paper, the objective is to design a polynomial-time algorithm, in contrast with potentially exponential-time approaches (Víšek (2006)), with guaranteed convergence under technical assumptions. As far as the robustness of an algorithm is concerned, the existing works often provide probabilistic guarantees on the recoverability of the original parameter vector $\mathbf{x}$ for linear Gaussian stochastic systems under various assumptions on the relationship between $\|\boldsymbol{\eta}\|_0$ and m. In this case, the ratio $\frac{\|\boldsymbol{\eta}\|_0}{m}$, named breakdown point, is limited by a constant and could even approach 1 if the unknown solution $\mathbf{x}$ is sparse.

1.1 Contributions and Organization

The main objective of this paper is to analyze a robust regression problem for an arbitrary quadratic model that includes power system state estimation and phase retrieval as special cases. The focus is on the calculation of the maximum number of bad measurements that does not compromise the exact reconstruction of the model vector $\mathbf{x}$. In Section 2, we formally state the problem. In Section 3, we propose two conic optimization methods and study their properties. In particular, we obtain conditions that guarantee the exact reconstruction of $\mathbf{x}$. In Section 4, we develop the main results of this paper. Under certain technical assumptions, we derive the dependence between the number of perfect measurements and the maximum admissible number of wrong measurements. After that, we consider a stochastic setting based on Gaussian distributions. In this case, we show that the number of bad measurements can safely be up to an order of the square root of the total number of measurements, and moreover the breakpoint approaches 1/2 if there is enough prior information. Discussions on these results are provided in Section 5. To provide a broader range of possible approaches to the problem, Section 6 designs an alternative

iterative-based method. Numerical results are presented in Section 7, which includes a case study on a European power grid.

1.2 Notation

$\mathbb{R}^n$ and $\mathbb{C}^n$ denote the sets of real and complex n-dimensional vectors, respectively. Bold letters are reserved for vectors and matrices. $[\mathbf{A}]_{ij}$ or A_{ij} is the (i,j)-th element of a matrix $\mathbf{A}$. The symbols $\mathbb{H}^n$ and $\mathbb{S}^n$ denote the sets of $n \times n$ Hermitian and symmetric matrices. $\text{tr}(\mathbf{A})$ and $\langle \mathbf{A}, \mathbf{B} \rangle$ are the trace of a matrix $\mathbf{A}$ and the Frobenius inner product of two matrices $\mathbf{A}$ and $\mathbf{B}$. The conjugate transpose and Moore-Penrose pseudoinverse of $\mathbf{A}$ are shown as $\mathbf{A}^*$ and $\mathbf{A}^+$. The notation $\mathbf{A} \circ \mathbf{B}$ refers to the Hadamard (entrywise) multiplication. The eigenvalues of a matrix $\mathbf{M} \in \mathbb{H}^n$ are denoted as $\lambda_1(\mathbf{M}), ..., \lambda_n(\mathbf{M})$ in descending order. The smallest and the largest singular values of $\mathbf{A}$ are shown as $\sigma_{\min}$ and $\sigma_{\max}$, respectively. $\mathbf{e}_i$ stands for the i-th column of the unit matrix $\mathbf{I}$ of appropriate dimension. Given a matrix $\mathbf{A} \in \mathbb{C}^{n \times m}$ and a set $\mathcal{S} \subset \{1, \ldots, m\}$, the matrix $\mathbf{A}_{\mathcal{S}}$ is defined to be a matrix obtained by adjoining the columns of $\mathbf{A}$ with indexes in $\mathcal{S}$. Given a vector $\mathbf{a} \in \mathbb{C}^n$ and a set $\mathcal{S} \subset \{1, \ldots, n\}$, the vector $\mathbf{a}_{\mathcal{S}}$ is defined to be a subvector of $\mathbf{a}$ obtained by stacking components of $\mathbf{a}$ with indexes in $\mathcal{S}$. For a sequence of indexes $\mathcal{S}$, the symbol $\{\alpha_i\}_{i \in \mathcal{S}}$ denotes a sequence indexed by $\mathcal{S}$. Whenever the notation is obvious from the context, we drop the indexing subscript for notational simplicity. The symbol $\|\mathbf{v}\|_0$ shows the cardinality of a vector $\mathbf{v}$, i.e., the number of its nonzero elements. Given a matrix $\mathbf{A}$, the symbols $\|\mathbf{A}\|_1$, $\|\mathbf{A}\|_\infty$, $\|\mathbf{A}\|_2$, and $\|\mathbf{A}\|_F$ denote the maximum absolute column sum, maximum absolute row sum, maximum singular value, and Frobenius norm of $\mathbf{A}$, respectively. The cardinality of a set $\mathcal{M}$ is indicated as $|\mathcal{M}|$. The notation $a \sim \mathcal{N}(\alpha, \beta)$ means that a is a normally distributed random variable with the parameters α and β.

2. Problem Formulation and Preliminaries

The Robust Quadratic Regression aims to find a vector $\mathbf{x}$ in $\mathbb{R}^n$ or $\mathbb{C}^n$ such that

$$y_r = \mathbf{x}^* \mathbf{M}_r \mathbf{x} + \eta_r, \quad \forall r \in \{1, \ldots, m\}, \tag{1}$$

where

- $y_1, \ldots, y_m$ are some known real-valued measurements.
- $\eta_1, \ldots, \eta_m$ are unknown but sparsely occurring real-valued noise of arbitrary magnitudes.
- $\mathbf{M}_1, \ldots \mathbf{M}_m$ are some known $n \times n$ Hermitian matrices.

The regression problem could have two solutions $\pm\mathbf{x}$ in the real-valued case, which increases to infinitely many in the form of $\mathbf{x} \times e^{\sqrt{-1}\theta}$ in the complex case. To avoid this ambiguity, the objective of this work is to find the matrix $\mathbf{x}\mathbf{x}^*$ rather than $\mathbf{x}$ since this matrix is invariant under the rotation of $\mathbf{x}$. At the same time, the recovery of $\mathbf{x}$ from $\mathbf{x}\mathbf{x}^*$ is a simple problem that can be solved using the spectral decomposition. If m is large enough, then $\mathbf{x}\mathbf{x}^*$ is expected to be unique. This paper aims to recover any solution $\mathbf{x}\mathbf{x}^*$ in case there are multiple ones.

Consider the complex-valued case and write $\mathbf{x} = \mathbf{a} + \sqrt{-1}\mathbf{b} \in \mathbb{C}^n$ and $\mathbf{M} = \mathbf{A} + \sqrt{-1}\mathbf{B} \in \mathbb{H}^n$, where $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, $\mathbf{A} \in \mathbb{S}^n$ and $\mathbf{B} = -\mathbf{B}^T \in \mathbb{R}^{n\times n}$. It is straightforward to verify that:

$$\mathbf{x}^*\mathbf{M}\mathbf{x} = \mathbf{a}^T\mathbf{A}\mathbf{a} + 2\mathbf{b}^T\mathbf{B}\mathbf{a} + \mathbf{b}^T\mathbf{A}\mathbf{b} = \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}^T \begin{bmatrix} \mathbf{A} & -\mathbf{B} \\ \mathbf{B} & \mathbf{A} \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}$$

Notice that the matrices in the right-side of the above equation are real-valued. As a result, we will only develop the theoretical results of this work in the real-valued case $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{M}_r \in \mathbb{S}^n$ since they can be easily carried over to the complex-valved case. However, we will offer a case study on power systems where the unknown state is a complex vector.

In the robust regression problem, the vector $\boldsymbol{\eta}$ is assumed to be sparse. To distinguish between error-free and erroneous measurements, we partition the set of measurements into two subsets of *good* and *bad* measurements:

$$\mathcal{G} = \{r | \eta_r = 0\}, \quad \mathcal{B} = \{1, \ldots, m\} \backslash \mathcal{G}$$

To streamline the derivation of the analytical results of this paper, we assume that $\mathcal{G} = \{1, \ldots, |\mathcal{G}|\}$ and $\mathcal{B} = \{|\mathcal{G}|+1, \ldots, m\}$. However, the algorithms to be designed are completely oblivious to the type of each measurement and its membership in either $\mathcal{G}$ or $\mathcal{B}$.

The objective of this paper is to develop efficient algorithms for finding $\mathbf{x}$ precisely as long as $\boldsymbol{\eta}$ is sufficiently sparse. This statement will be formalized in the next sections.

3. Conic Optimization Methods

Consider a variable matrix $\mathbf{W}$ playing the role of $\mathbf{x}\mathbf{x}^T$. This matrix is positive semidefinite and has rank 1. By dropping the rank constraint, one can cast the quadratic regression as a linear matrix regression. Motivated by this relaxation, consider the optimization problem

$$\begin{aligned} \underset{\mathbf{W}\in\mathbb{S}^n,\, \boldsymbol{\nu}\in\mathbb{R}^m}{\text{minimize}} \quad & \langle \mathbf{W}, \mathbf{M} \rangle + \mu \|\boldsymbol{\nu}\|_1 \\ \text{subject to} \quad & \langle \mathbf{W}, \mathbf{M}_r \rangle + \nu_r = y_r, \quad \forall r \in \{1 \ldots m\} && \text{(2a)} \\ & \mathbf{W} = \mathbf{W}^T \succeq_{\mathcal{C}} 0 && \text{(2b)} \end{aligned}$$

where the notation $\succeq_{\mathcal{C}}$ is the generalized inequality sign with respect to $\mathcal{C}$, which is either the cone of symmetric positive semidefinite (PSD) matrices or the 2×2 principal sub-matrices PSD cone (see Permenter and Parrilo (2014)). The above cones will be formally defined in Subsection 3.2.

Since no rank constraint is imposed on $\mathbf{W}$, a regularization term is included in the objective function. We refer to this problem as *penalized conic*. The regularizer involves a matrix $\mathbf{M}$ that will be designed based on an initial guess $\hat{\mathbf{x}} \in \mathbb{R}^n$ for the solution of the quadratic regression serving as prior knowledge about the vector $\mathbf{x}$. This is a convex problem and can be solved in polynomial time up to any given accuracy.

3.1 Penalized Semidefinite Programming

Consider Problem (2) with $\mathcal{C}$ equal to the PSD cone, and refer to it as penalized semidefinite programming (SDP). Given a matrix $\mathbf{X} \in \mathbb{S}^n$, define $\kappa(\mathbf{X})$ to be the sum of the two smallest

eigenvalues of $\mathbf{X}$, i.e.,

$$\kappa(\mathbf{X}) := \lambda_n(\mathbf{X}) + \lambda_{n-1}(\mathbf{X})$$

Let the matrix $\mathbf{M}$ in the objective function of the penalized SDP be chosen to have the following properties:

$$\begin{aligned} &\mathbf{M}\hat{\mathbf{x}} = \mathbf{0}, \\ &\text{rank}(\mathbf{M}) = n-1 \\ &\kappa(\mathbf{M}) > 0 \end{aligned}$$

There are infinitely many choices for $\mathbf{M}$, and it is not important which one to select as far as the analysis of this paper is concerned. Observe that the dual of Problem (2) can be obtained as:

$$\begin{aligned} \underset{\lambda \in \mathbb{R}^m}{\text{maximize}} \quad & -\mathbf{y}^T \boldsymbol{\lambda} \\ \text{subject to} \quad & \mathbf{M} + \sum_{r=1}^{m} \lambda_r \mathbf{M}_r \succeq 0 && \text{(3a)} \\ & \|\boldsymbol{\lambda}\|_\infty \leq \mu && \text{(3b)} \end{aligned}$$

where $\succeq 0$ is the positive semidefinite sign. Define the matrix $\bar{\mathbf{J}}$ and the vector $\bar{\mathbf{d}}$ as:

$$\bar{\mathbf{J}} = [\mathbf{M}_1\mathbf{x} \ \ldots \ \mathbf{M}_m\mathbf{x}] \quad \text{(4a)}$$

$$\bar{\mathbf{d}} = \mathbf{M}\mathbf{x} \quad \text{(4b)}$$

A preliminary result will be provided below, which will be used to study the penalized SDP.

Theorem 1 *Assume that there exists an index $r \in \{1, \ldots, m\}$ such that $\hat{\mathbf{x}}^T \mathbf{M}_r \hat{\mathbf{x}} \neq 0$ and*

$$\mu > \|\bar{\mathbf{J}}_{\mathcal{G}}^{+} \left(\bar{\mathbf{d}} - \mu \bar{\mathbf{J}}_{\mathcal{B}} \ \text{sign}(\boldsymbol{\eta}_{\mathcal{B}})\right) \|_\infty \quad \text{(5a)}$$

$$\frac{\kappa(\mathbf{M})}{2 \max_r \|\mathbf{M}_r\|_2} > \|\bar{\mathbf{J}}_{\mathcal{G}}^{+} \left(\bar{\mathbf{d}} - \mu \bar{\mathbf{J}}_{\mathcal{B}} \ \text{sign}(\boldsymbol{\eta}_{\mathcal{B}})\right) \|_1 + \mu |\mathcal{B}| \quad \text{(5b)}$$

Then, $(\mathbf{x}\mathbf{x}^T, \boldsymbol{\eta})$ is the unique solution of the penalized SDP. Moreover, $\hat{\boldsymbol{\lambda}} = \begin{bmatrix} \hat{\boldsymbol{\lambda}}_{\mathcal{G}} \\ \hat{\boldsymbol{\lambda}}_{\mathcal{B}} \end{bmatrix}$ defined as

$$\begin{aligned} \hat{\boldsymbol{\lambda}}_{\mathcal{B}} &= -\mu \ \text{sign}(\boldsymbol{\eta}_{\mathcal{B}}) \\ \hat{\boldsymbol{\lambda}}_{\mathcal{G}} &= -\bar{\mathbf{J}}_{\mathcal{G}}^{+} \left(\bar{\mathbf{d}} + \bar{\mathbf{J}}_{\mathcal{B}} \hat{\boldsymbol{\lambda}}_{\mathcal{B}}\right) \end{aligned}$$

is a dual solution.

Proof The proof is provided in Appendix A. ■

The conditions given in Theorem 1 will be refined in Section 4 to uncover useful properties of the penalized SDP.

3.2 Penalized Second-Order Cone Programming

In this section, we will introduce the penalized second-order cone programming (SOCP). This optimization problem is obtained by building the cone $\mathcal{C}$ based on the 2×2 principal submatrices of $\mathbf{W}$, as explained below.

Definition 2 ($2\mathcal{PSM}$) *A matrix $\mathbf{X} \in \mathbb{S}^n$ belongs to the 2×2 principal sub-matrices PSD cone if each 2×2 principal sub-matrix of $\mathbf{X}$ is positive semidefinite, i.e.,*

$$[\mathbf{e}_i \;\; \mathbf{e}_j]^T \mathbf{X} [\mathbf{e}_i \;\; \mathbf{e}_j] \succeq 0, \qquad \forall\; i < j$$

Since the $2\mathcal{PSM}$ cone is not self-dual, we introduce the scaled diagonally dominant cone below.

Definition 3 ($\mathcal{SDD}$) *A matrix $\mathbf{X} \in \mathbb{R}^{n \times n}$ belongs to the scaled diagonally dominant cone if there exist a set of 2×2 positive semidefinite matrices $\{\mathbf{X}^{ij}\}_{i<j}^{j \leq n}$ such that*

$$\sum_{j=2}^{n} \sum_{i=1}^{j-1} [\mathbf{e}_i \;\; \mathbf{e}_j] \mathbf{X}^{ij} [\mathbf{e}_i \;\; \mathbf{e}_j]^T = \mathbf{X}$$

Note that the notation $\{\mathbf{X}^{ij}\}_{i<j}^{j \leq n}$ in the above definition means $\{\mathbf{X}^{ij} \big| j = 2, \ldots, n, i = 1, \ldots, j-1\}$. The next lemma explains the connection between $2\mathcal{PSM}$ and $\mathcal{SDD}$ cones.

Lemma 4 (Permenter and Parrilo (2014)) *The dual of the 2×2 principal sub-matrices PSD cone is the scaled diagonally dominant cone of the same dimension.*

In what follows, we will define and describe certain properties of a linear space of diagonal decompositions of matrices.

Definition 5 *The sequence $\{\mathbf{A}^{ij} \in \mathbb{S}^2\}_{i<j}$ is said to be a diagonal decomposition (or just decomposition) of $\mathbf{A} \in \mathbb{S}^n$ if*

$$\mathbf{A} = \sum_{j=2}^{n} \sum_{i=1}^{j-1} [\mathbf{e}_i, \mathbf{e}_j] \mathbf{A}^{ij} [\mathbf{e}_i, \mathbf{e}_j]^T$$

A decomposition that consists of PSD matrices is a certificate that a matrix belongs to the $\mathcal{SSD}$ cone. Similarly to the function κ defined for the penalized SDP, we introduce the function $\chi(\{\mathbf{X}^{ij}\}_{i<j})$ as follows:

$$\chi\left(\{\mathbf{X}^{ij}\}_{i<j}\right) := \min_{i<j} \operatorname{tr}(\mathbf{X}^{ij}) = \min_{i<j} \left(\lambda_1(\mathbf{X}^{ij}) + \lambda_2(\mathbf{X}^{ij})\right)$$

Consider a sequence $\{\mathbf{M}^{ij} \in \mathbb{S}^2\}_{i<j}^{j \leq n}$ such that

$$\begin{cases} \chi(\{\mathbf{M}^{ij}\}_{i<j}) > 0 \\ \mathbf{M}^{ij} [\hat{x}_i \;\; \hat{x}_j]^T = \mathbf{0} \end{cases}$$

Define penalized SOCP as Problem (2) with $\mathcal{C}$ equal to the $2\mathcal{PSM}$ cone and $\mathbf{M} = \sum_{i<j} [\mathbf{e}_i \;\; \mathbf{e}_j] \mathbf{M}^{ij} [\mathbf{e}_i \;\; \mathbf{e}_j]^T$. Since $\mathbf{M}^{ij} \succeq 0$, the matrix $\mathbf{M}$ belongs to the $\mathcal{SDD}$ cone.

Notice that the dual of the penalized SOCP takes the form:

$$\begin{aligned}
\text{maximize} \quad & -\mathbf{y}^T \boldsymbol{\lambda} \\
\text{subject to} \quad & \mathbf{M} + \sum_{r=1}^{m} \lambda_r \mathbf{M}_r = \mathbf{H} && \text{(6a)} \\
& \sum_{i<j} [\mathbf{e}_i \;\; \mathbf{e}_j] \mathbf{H}^{ij} [\mathbf{e}_i \;\; \mathbf{e}_j]^T = \mathbf{H} && \text{(6b)} \\
& \mathbf{H}^{ij} \succeq 0 && \text{(6c)} \\
& \|\boldsymbol{\lambda}\|_\infty \leq \mu && \text{(6d)}
\end{aligned}$$

where the variables are $\lambda \in \mathbb{R}^m$, $\mathbf{H} \in \mathbb{S}^n$ and $\{\mathbf{H}^{ij}\}_{i<j}^{j\leq n} \subset \mathbb{S}^2$. Now, it is easy to observe that each conic constraint $[\mathbf{e}_i \;\; \mathbf{e}_j]^T \mathbf{W} [\mathbf{e}_i \;\; \mathbf{e}_j] \succeq 0$ in Problem (2) corresponds to the dual variable matrix $\mathbf{H}^{ij}$. Hence, the complementary slackness condition can be written as

$$\langle [\mathbf{e}_i \;\; \mathbf{e}_j]^T \mathbf{W} [\mathbf{e}_i \;\; \mathbf{e}_j], \mathbf{H}^{ij} \rangle = 0, \qquad \forall \; i < j \leq n$$

Define $\mathbf{G}$ to be a symmetric matrix such that $\mathbf{M}^{ij} [x_i \;\; x_j]^T = [G_{ij} \;\; G_{ji}]^T$ for all $i < j \in \{1, \ldots, n\}$ and $G_{ii} = 0$ for all $i \in \{1, \ldots, n\}$. Furthermore, for every $r \in \{1, \ldots m\}$, define $\mathbf{R}^r$ as a matrix with the properties:

$$\begin{cases} \sum\limits_j R^r_{ij} = M^r_{ii} \\ R^r_{ii} = 0 \end{cases} \qquad \text{(e.g. } R^r_{ij} = \tfrac{M^r_{ii}}{n-1}\text{)}$$

Given $r \in \{1, \ldots, m\}$, define $\mathbf{G}^r$ as a symmetric matrix with the properties $G^r_{ij} = \frac{x_j}{x_i} M^r_{ij}$ and $G^r_{ii} = 0$ for all $i, j \in \{1, \ldots, n\}$. Similarly to (4), define:

$$\begin{aligned}
\tilde{\mathbf{J}} = & \;\; \left[\;\; \text{vecnd}\left(\mathbf{R}^1\right) \;\; \ldots \;\; \text{vecnd}\left(\mathbf{R}^m\right) \;\; \right] + \left[\;\; \text{vecnd}\left(\mathbf{G}^1\right) \;\; \ldots \;\; \text{vecnd}\left(\mathbf{G}^m\right) \;\; \right] && \text{(7a)} \\
\tilde{\mathbf{d}} = & \; \text{vecnd}\left(\mathbf{G}\right) && \text{(7b)}
\end{aligned}$$

where the vectorization operator vecnd : $\mathbb{R}^{n\times n} \to \mathbb{R}^{n^2-n}$ puts all elements excluding the diagonal of its matrix argument into a vector. The counterpart of Theorem 1 will be stated below for the penalized SOCP.

Theorem 6 *Assume that the components of the solution are nonzero (i.e., $x_i \neq 0$ for all $i \in \{1, \ldots, n\}$) and that there exists an index $r \in \{1, \ldots, m\}$ such that $\hat{\mathbf{x}}^* \mathbf{M}_r \hat{\mathbf{x}} \neq 0$ and*

$$\begin{aligned}
\mu > & \; \|\tilde{\mathbf{J}}^+_{\mathcal{G}} \left(\tilde{\mathbf{d}} - \mu \tilde{\mathbf{J}}_{\mathcal{B}} \; \text{sign}(\boldsymbol{\eta}_{\mathcal{B}}) \right) \|_\infty && \text{(8a)} \\
\frac{\chi(\{\mathbf{M}^{ij}\}_{i<j})}{\max\limits_{r, \; i<j} |\text{tr}(\{\mathbf{M}^{ij}_r\}_{i<j})|} > & \; \|\tilde{\mathbf{J}}^+_{\mathcal{G}} \left(\tilde{\mathbf{d}} - \mu \tilde{\mathbf{J}}_{\mathcal{B}} \; \text{sign}(\boldsymbol{\eta}_{\mathcal{B}}) \right) \|_1 + \mu |\mathcal{B}| && \text{(8b)}
\end{aligned}$$

Then, $(\mathbf{x}\mathbf{x}^T, \boldsymbol{\eta})$ *is the unique solution of the the penalized SOCP. Moreover,* $\hat{\boldsymbol{\lambda}} = \begin{bmatrix} \hat{\boldsymbol{\lambda}}_{\mathcal{G}} \\ \hat{\boldsymbol{\lambda}}_{\mathcal{B}} \end{bmatrix}$ *defined as*

$$\hat{\boldsymbol{\lambda}}_{\mathcal{B}} = -\,\mu\ \mathrm{sign}(\boldsymbol{\eta}_{\mathcal{B}})$$
$$\hat{\boldsymbol{\lambda}}_{\mathcal{G}} = -\,\tilde{\mathbf{J}}_{\mathcal{G}}^{+}\left(\tilde{\mathbf{d}} + \tilde{\mathbf{J}}_{\mathcal{B}}\hat{\boldsymbol{\lambda}}_{\mathcal{B}}\right)$$

can be completed to a dual optimal solution.

Proof The proof is provided in Appendix B. ■

We need to mention that while there is some freedom in the choice of $\hat{\boldsymbol{\lambda}}_{\mathcal{G}}$ in the proof of Theorem 6, the dual variables $\hat{\boldsymbol{\lambda}}_{\mathcal{B}}$ associated with the bad measurements are inflexible. This is elaborated below.

Lemma 7 $\hat{\boldsymbol{\lambda}}_{\mathcal{B}} = -\mu\ \mathrm{sign}(\boldsymbol{\eta}_{\mathcal{B}})$ *is the only possible setting of the optimal dual variables if the relaxation is exact.*

Proof The proof is provided in Appendix B ■

4. Main Results

In this section, we develop key theoretical results on the Robust Quadratic Regression solution via the conic methods presented in the preceding section. The common structure of the conditions in Theorems 1 and 6 allows us to derive results providing guarantees for both the SDP and the SOCP approaches simultaneously. To do that, we will use the universal notations $\mathbf{J}$ and $\mathbf{d}$ to denote $\bar{\mathbf{J}}$ and $\bar{\mathbf{d}}$ in the penalized SDP case and to denote $\tilde{\mathbf{J}}$ and $\tilde{\mathbf{d}}$ in the penalized SOCP case. Define

$$\alpha_{\mathrm{SDP}} = \frac{\kappa(\mathbf{M})}{2\|\bar{\mathbf{d}}\|_2 \max\limits_{r} \|\mathbf{M}_r\|_2} \quad \text{or} \quad \alpha_{\mathrm{SOCP}} = \frac{\chi(\{\mathbf{M}^{ij}\}_{i<j})}{\|\tilde{\mathbf{d}}\|_2 \max\limits_{r,\ i<j} |\mathrm{tr}(\mathbf{M}_r^{ij})|}$$

The parameter α is considered to be equal to α_{SDP} or α_{SOCP} depending on whether the penalized SDP or SOCP is analyzed. The same definition holds for l that takes one of the following values:

$$l_{\mathrm{SDP}} = n; \qquad l_{\mathrm{SOCP}} = n^2 - n$$

4.1 Deterministic Bound

In this subsection, we establish a uniform bound on the number of bad measurements that a penalized conic relaxation can tolerate. To do so, we make use of two matrix properties introduced in Bhatia et al. (2017).

Definition 8 (*SSC property) A matrix* $\mathbf{X} \in \mathbb{R}^{l\times m}$ *is said to satisfy the Subset Strong Convexity (SSC) Property at level* p *with constant* γ_p *if*

$$\gamma_p \leq \min_{|S|=p} \sqrt{\lambda_{min}(\mathbf{X}_S \mathbf{X}_S^T)}$$

Definition 9 (*SSS property) A matrix* $\mathbf{X} \in \mathbb{R}^{l\times m}$ *is said to satisfy the Subset Strong Smoothness (SSS) Property at level* p *with constant* Γ_p *if*

$$\max_{|S|=p} \sqrt{\lambda_{max}(\mathbf{X}_S \mathbf{X}_S^T)} \leq \Gamma_p$$

Note that the notation $|S| = p$ in the above definition specifies the index set of any p columns of the matrix $\mathbf{X}$. The ratio of the constants γ_p and Γ_{m-p} can be interpreted as a uniform condition number at level p. By leveraging the properties of these constants, the next theorem offers some conditions that will later be used to develop a bound on the permissible number of bad measurements.

Theorem 10 *Let* $\mathbf{J}$ *be a matrix in* $\mathbb{R}^{l\times m}$ *that satisfies the SSC and SSS properties with the constants* $\gamma_{|\mathcal{G}|}$ *and* $\Gamma_{|\mathcal{B}|}$ *respectively* ($\gamma_{|\mathcal{G}|} > \Gamma_{|\mathcal{B}|}$). *Also, let* $\mathbf{d}$ *be a vector in* $\mathbb{R}^l$ *and* $\boldsymbol{\lambda}$ *be a vector in* $\mathbb{R}^m$ *such that* $\boldsymbol{\lambda}_{\mathcal{B}} = \mu \cdot \mathbf{s}$, *where* μ *is a scalar and the entries of* $\mathbf{s}$ *are only* $+1$ *or* -1. *If*

$$\alpha\gamma_{|\mathcal{G}|}(\gamma_{|\mathcal{G}|} - \Gamma_{|\mathcal{B}|}) - |\mathcal{B}|\gamma_{|\mathcal{G}|} > \Big(\sqrt{|\mathcal{B}|}\Gamma_{|\mathcal{B}|} + (\gamma_{|\mathcal{G}|} - \Gamma_{|\mathcal{B}|})\Big)\sqrt{|\mathcal{G}|}$$

then the interval

$$\left[\frac{\|\mathbf{d}\|_2}{\gamma_{|\mathcal{G}|} - \Gamma_{|\mathcal{B}|}}, \frac{(\alpha\gamma_{|\mathcal{G}|} - \sqrt{|\mathcal{G}|})\|\mathbf{d}\|_2}{\sqrt{|\mathcal{B}||\mathcal{G}|}\Gamma_{|\mathcal{B}|} + |\mathcal{B}|\gamma_{|\mathcal{G}|}}\right] \tag{9}$$

is not empty and the system of inequalities

$$\begin{cases} \mu > \|\boldsymbol{\lambda}_{\mathcal{G}}\|_\infty \\ \alpha\|\mathbf{d}\|_2 > \|\boldsymbol{\lambda}_{\mathcal{G}}\|_1 + \mu|\mathcal{B}| \end{cases} \tag{10}$$

can be satisfied with $\boldsymbol{\lambda}_{\mathcal{G}} = -\mathbf{J}_{\mathcal{G}}^{+}(\mathbf{J}_{\mathcal{B}}\boldsymbol{\lambda}_{\mathcal{B}} + \mathbf{d})$ *for every* μ *in the interval* (9).

Proof The proof directly follows from Definitions 8 and 9, as well as Lemma 28 proved in Appendix C. ■

The following corollary results from combining Theorems 1, 6 and 10.

Corollary 11 *Consider the penalized SDP problem. Suppose that* $\bar{\mathbf{J}}$ *satisfies the SSC and SSS properties with the constants* $\gamma_{|\mathcal{G}|}$ *and* $\Gamma_{|\mathcal{B}|}$, *respectively* ($\gamma_{|\mathcal{G}|} > \Gamma_{|\mathcal{B}|}$). *If*

$$\alpha_{SDP}\gamma_{|\mathcal{G}|}(\gamma_{|\mathcal{G}|} - \Gamma_{|\mathcal{B}|}) - |\mathcal{B}|\gamma_{|\mathcal{G}|} > \Big(\sqrt{|\mathcal{B}|}\Gamma_{|\mathcal{B}|} + (\gamma_{|\mathcal{G}|} - \Gamma_{|\mathcal{B}|})\Big)\sqrt{|\mathcal{G}|}$$

then there exists a constant μ *for which* $(\mathbf{x}\mathbf{x}^T, \boldsymbol{\eta})$ *is the unique solution of the penalized SDP problem.*

Corollary 12 *Consider the penalized SOCP problem. Assume that* $\tilde{\mathbf{J}}$ *satisfies the SSC and SSS properties with the constants* $\gamma_{|\mathcal{G}|}$ *and* $\Gamma_{|\mathcal{B}|}$*, respectively* ($\gamma_{|\mathcal{G}|} > \Gamma_{|\mathcal{B}|}$)*. If*

$$\alpha_{SOCP}\gamma_{|\mathcal{G}|}(\gamma_{|\mathcal{G}|} - \Gamma_{|\mathcal{B}|}) - |\mathcal{B}|\gamma_{|\mathcal{G}|} > \Big(\sqrt{|\mathcal{B}|}\Gamma_{|\mathcal{B}|} + (\gamma_{|\mathcal{G}|} - \Gamma_{|\mathcal{B}|})\Big)\sqrt{|\mathcal{G}|}$$

then there exists a constant μ *for which* $(\mathbf{x}\mathbf{x}^T, \boldsymbol{\eta})$ *is the unique solution of the penalized SOCP problem.*

The guarantees given in Corollaries 11 and 12 are established for the "worst-case scenario" when the adversary is adaptive and strategically selects the indexes of the error vector $\boldsymbol{\eta}$ based on the input of the problem.

4.2 Stochastic Bound

In the preceding section, we derived deterministic bounds on the permissible number of bad measurements. These bounds can be simplified and nicely interpreted for stochastic systems. In this section, we aim to study those problems that are, in some sense, drawn from the standard Gaussian distribution. This will be formalized below.

Definition 13 *A matrix* $\mathbf{X}$ *is called standard Gaussian over* $\mathbb{R}$ *if its entries are independent and identically distributed random variables with a standard normal distribution.*

Theorem 14 *Let* $\mathbf{J}$ *be a matrix in* $\mathbb{R}^{l\times m}$ *that is sampled from a normal standard Gaussian distribution. Also, let* $\mathbf{d}$ *be a vector in* $\mathbb{R}^l$ *and* $\boldsymbol{\lambda}$ *be a vector in* $\mathbb{R}^m$ *such that* $\boldsymbol{\lambda}_{\mathcal{B}} = \mu\cdot\mathbf{s}$*, where* μ *is a scalar and the entries of* $\mathbf{s}$ *are only* $+1$ *or* -1*. Consider arbitrary numbers* $\delta \in (0,1)$ *and* $\epsilon > 0$*. Denote* $\tau_{\delta,\varepsilon} = \frac{\sqrt{cl + c'\log\frac{2}{\delta}}}{1-2\varepsilon}$*, where* $c = 24e^2\log\frac{3}{\varepsilon}$ *and* $c' = 24e^2$*. If*

$$\sqrt{|\mathcal{G}|} > \sqrt{|\mathcal{B}|}\frac{\sqrt{1+\Delta_{|\mathcal{B}|}}}{\sqrt{1-\Delta_{|\mathcal{G}|}}} + \frac{|\mathcal{B}|}{\alpha\sqrt{1-\Delta_{|\mathcal{G}|}}-1}\frac{\sqrt{1+\Delta_{|\mathcal{B}|}}+\sqrt{1+\Delta_{|\mathcal{G}|}}}{\sqrt{1-\Delta_{|\mathcal{G}|}}}, \tag{11}$$

where $\Delta_t \geq \frac{\tau_{\delta,\varepsilon}}{\sqrt{t}}$ *for* $t = |\mathcal{B}|$ *and* $|\mathcal{G}|$*, then with probability at least* $(1-\delta)^2$ *the interval*

$$\left[\frac{\|\mathbf{d}\|_2}{\sqrt{|\mathcal{G}|(1-\Delta_{|\mathcal{G}|})}-\sqrt{|\mathcal{B}|(1+\Delta_{|\mathcal{B}|})}}, \frac{(\alpha\sqrt{(1-\Delta_{|\mathcal{G}|})}-1)\|\mathbf{d}\|_2}{|\mathcal{B}|(\sqrt{(1+\Delta_{|\mathcal{B}|})}+\sqrt{(1+\Delta_{|\mathcal{G}|})})}\right] \tag{12}$$

is not empty and the system of inequalities

$$\begin{cases} \mu > \|\boldsymbol{\lambda}_{\mathcal{G}}\|_\infty \\ \alpha\|\mathbf{d}\|_2 > \|\boldsymbol{\lambda}_{\mathcal{G}}\|_1 + \mu|\mathcal{B}| \end{cases}$$

can be satisfied with $\boldsymbol{\lambda}_{\mathcal{G}} = -\mathbf{J}_{\mathcal{G}}^{+}(\mathbf{J}_{\mathcal{B}}\boldsymbol{\lambda}_{\mathcal{B}} + \mathbf{d})$ *for every* μ *in the interval* (12)*.*

Proof The proof is provided in Appendix C ■

Combining Theorems 1, 6 and 14 leads to the bound given below.

Corollary 15 *Consider the penalized SDP problem, together with the notations δ, ε and $\tau_{\delta,\varepsilon}$ introduced in Theorem 14. Suppose that $\bar{\mathbf{J}}$ is sampled from the normal standard Gaussian distribution. If*

$$\sqrt{|\mathcal{G}|} > \sqrt{|\mathcal{B}|}\frac{\sqrt{1+\Delta_{|\mathcal{B}|}}}{\sqrt{1-\Delta_{|\mathcal{G}|}}} + \frac{|\mathcal{B}|}{\alpha_{SDP}\sqrt{1-\Delta_{|\mathcal{G}|}}-1}\frac{\sqrt{1+\Delta_{|\mathcal{B}|}}+\sqrt{1+\Delta_{|\mathcal{G}|}}}{\sqrt{1-\Delta_{|\mathcal{G}|}}},$$

then with probability at least $(1-\delta)^2$ there exists a constant μ for which $(\mathbf{x}\mathbf{x}^T, \boldsymbol{\eta})$ is the unique solution of the penalized SDP problem.

Corollary 16 *Consider the penalized SOCP problem, together with the notations δ, ε and $\tau_{\delta,\varepsilon}$ introduced in Theorem 14. Suppose that $\tilde{\mathbf{J}}$ is sampled from the normal standard Gaussian distribution. If*

$$\sqrt{|\mathcal{G}|} > \sqrt{|\mathcal{B}|}\frac{\sqrt{1+\Delta_{|\mathcal{B}|}}}{\sqrt{1-\Delta_{|\mathcal{G}|}}} + \frac{|\mathcal{B}|}{\alpha_{SOCP}\sqrt{1-\Delta_{|\mathcal{G}|}}-1}\frac{\sqrt{1+\Delta_{|\mathcal{B}|}}+\sqrt{1+\Delta_{|\mathcal{G}|}}}{\sqrt{1-\Delta_{|\mathcal{G}|}}},$$

then with probability at least $(1-\delta)^2$ there exists a constant μ for which $(\mathbf{x}\mathbf{x}^T, \boldsymbol{\eta})$ is the unique solution of the penalized SOCP problem.

Unlike the results of the previous subsection, the stochastic bounds given above are established for a "random case scenario" when the adversary is oblivious and selects the indexes of nonzero components of the error vector $\boldsymbol{\eta}$ on random with a uniform distribution.

5. Analysis and Illustrations

The deterministic conditions from Section 4.1 may be hard to verify *a-priori*. One could approximately use $\hat{\mathbf{x}}$ instead of $\mathbf{x}$ in the definition of $\mathbf{J}$ to obtain estimations of $\|\mathbf{d}\|_2$ and α that are useful for the analysis of the developed deterministic bound. In contrast, the stochastic bound from Section 4.2 is significantly easier to analyze. In general, it can be verified that to satisfy the condition (11), the number of good measurements $|\mathcal{G}|$ must grow quadratically on the number of bad measurements $|\mathcal{B}|$ for both the penalized SDP and the penalized SOCP relaxations. Nevertheless, in the case when $\alpha \to \infty$, this condition on $|\mathcal{G}|$ and $|\mathcal{B}|$ can be reduced to the simple inequality

$$|\mathcal{G}|(1 - \frac{\tau_{\delta,\varepsilon}}{\sqrt{|\mathcal{G}|}}) > |\mathcal{B}|(1 + \frac{\tau_{\delta,\varepsilon}}{\sqrt{|\mathcal{B}|}})$$

or equivalently

$$\begin{cases} |\mathcal{G}| > \tau_{\delta,\varepsilon}^2 \\ |\mathcal{B}| < |\mathcal{G}| + \tau_{\delta,\varepsilon}^2 - 2\tau_{\delta,\varepsilon}\sqrt{|\mathcal{G}|} \end{cases} \tag{13}$$

The above inequalities imply that the number of good measurements $|\mathcal{G}|$ can grow linearly with the number of bad measurements $|\mathcal{B}|$ if α^{-1} is negligible. For example, this occurs when α has the potential to increase as $|\mathcal{B}|$ grows or when the initial guess $\hat{\mathbf{x}}$ is close to the true solution $\mathbf{x}$.

Numerical studies show that the function $\tau_{\delta,\varepsilon}$ is expected to be fairly flat with respect to ε for practically important values of the parameters. For illustration purposes, consider $\varepsilon = \varepsilon^* = 0.05514$ that is the minimum of $\tau_{\delta,\varepsilon}$ for $l = 100$ and $\delta = 0.05$. Therefore,

$$\tau_{\delta,\varepsilon^*} \simeq 1.123\sqrt{708.7l + 177.3\log\frac{2}{\delta}}$$

or equivalently

$$\tau^2_{\delta,\varepsilon^*} \simeq 893.7l + 223.6\log\frac{2}{\delta}$$

which demonstrates the asymptotic behaviour of the function. Since $l_{SDP} = n$ but $l_{SOCP} = n^2 - n$, it can be concluded that the guarantee for the SDP approach works whenever the number of measurements is on order of the size of the problem, while the guarantee for the SOCP approach requires a higher number of measurements. This gives rise to a salient difference between the penalized SDP and the penalized SOCP: the SDP approach offers a higher performance over the SOCP approach but is computationally more expensive to solve. Another important difference between the SDP and SOCP approaches — coming from the nature of the problem itself — is rooted in the definition of the coefficient α. The same amount of prior information (e.g. the values of $\mathbf{x}$ and $\hat{\mathbf{x}}$) may result in significantly different values of the functions κ and χ. The same argument holds for $\|\mathbf{M}_r\|_2$ and $\mathrm{tr}(\mathbf{M}_r^{ij})$.

It is important to discuss when $\mathbf{J}$ becomes a Gaussian matrix in order to use the stochastic bounds. The easiest scenario corresponds to the case where the true solution $\mathbf{x}$ is a deterministic vector while $\mathbf{M}_r$'s are stochastic matrices. For example, $[\mathbf{M}_r]_{ij}$ with the distribution $\mathcal{N}(0, \frac{1}{nx_j^2})$ makes $[\mathbf{M}_r\mathbf{x}]$ distributed as a standard normal vector, independent of any other column vector in the matrix $\mathbf{J}$ in the SDP case. Likewise, an example of the data distribution for the SOCP case corresponds to $[\mathbf{M}_r]_{ii} \sim \mathcal{N}(0, n-1)$ and $[\mathbf{M}_r]_{ij} \sim \mathcal{N}(0, (\frac{x_i}{x_j})^2)$ when $i \neq j$. Indeed, $R_{ij} \sim \mathcal{N}(0,1)$ when $i \neq j$ will make $\mathbf{J}$ a normal Gaussian matrix in this case.

In this paper, we simplified the deterministic bounds in a Gaussian setting, but the developed techniques could be used to study other distributions as well. As will be shown in empirical studies in Section 7, the constant μ could be just chosen to be a small positive number and does not need to obey the derived conservative bounds. The developed theoretical results do not limit applications of the proposed methods, but they provide intuitions into how large the range of parameters could be to provide guarantees despite the natural hardness of the problem.

6. Robust Least-Squares Regression

To build an iterative algorithm for solution of Problem (1), consider the optimization

$$\begin{array}{ll} \underset{\mathbf{W}\in\mathbb{S}^n,\ \boldsymbol{\nu}\in\mathbb{R}^m}{\text{minimize}} & \dfrac{1}{2}\displaystyle\sum_{r=1}^{m}\left(\langle\mathbf{W},\mathbf{M}_r\rangle + \nu_r - y_r\right)^2 \\ \text{subject to} & \mathbf{W} \succeq_{\mathcal{C}} 0 \\ & \|\boldsymbol{\nu}\|_0 \leq k \end{array} \tag{14}$$

where k is a parameter. This problem is nonconvex due to a cardinality constraint.

Definition 17 *Define* $HT_k(\mathbf{y}) : \mathbb{R}^m \to \mathbb{R}^m$ *as a hard thresholding operator such that*

$$[HT_k(\mathbf{z})]_i = \begin{cases} z_i & \text{if } |z_i| \text{ is among the } k \text{ largest-in-magnitude entries of } \mathbf{z} \\ 0 & \text{otherwise,} \end{cases}$$

where $[HT_k(\mathbf{z})]_i$ denotes the i^{th} entry of $HT_k(\mathbf{z})$.

Consider the function

$$f(\boldsymbol{\nu}) := \min_{\mathbf{W} \succeq_{\mathcal{C}} 0} \frac{1}{2} \sum_{r=1}^{m} \left(\langle \mathbf{W}, \mathbf{M}_r \rangle - (y_r - \nu_r) \right)^2$$

and let $\hat{\mathbf{W}}(\boldsymbol{\nu})$ denote a solution to this problem. We propose a Hard Thresholding method for solving the quadratic regression problem, which consists of the iterative scheme

$$\boldsymbol{\nu}^{t+1} = HT_k(\boldsymbol{\nu}^t - \mathbf{d}(\boldsymbol{\nu}^t))$$

where

$$\mathbf{d}(\boldsymbol{\nu}) = \frac{1}{2} \nabla_{\boldsymbol{\nu}} \left(\sum_{r=1}^{m} \left(\langle \mathbf{W}, \mathbf{M}_r \rangle - (y_r - \nu_r) \right)^2 \right) \Bigg|_{\mathbf{W} = \hat{\mathbf{W}}(\boldsymbol{\nu})}$$

(the symbol $\nabla_{\boldsymbol{\nu}}$ stands for the gradient with respect to $\boldsymbol{\nu}$). By Lemma 3.3.1 in Bertsekas (1995), if $\hat{\mathbf{W}}(\boldsymbol{\nu})$ is a continuously differentiable mapping, then $\nabla f(\boldsymbol{\nu}) = \mathbf{d}(\boldsymbol{\nu})$. Inspired by this fact, one may informally regard $\mathbf{d}(\boldsymbol{\nu})$ as the gradient of the optimal value of the optimization problem (14) without its cardinality constraint. Define $\mathbf{w} = \text{vec}(\mathbf{W})$, $\hat{\mathbf{w}}(\boldsymbol{\nu}) = \text{vec}(\hat{\mathbf{W}}(\boldsymbol{\nu}))$, $\mathbf{a}_r = \text{vec}(\mathbf{M}_r)$ for $r = 1, ..., m$, and $\mathbf{A} = [\mathbf{a}_1 \ \dots \ \mathbf{a}_m]^T$. It can be verified that

$$\mathbf{d}(\boldsymbol{\nu}) = \mathbf{A}\hat{\mathbf{w}}(\boldsymbol{\nu}) - \mathbf{y} + \boldsymbol{\nu}$$

which implies that

$$HT_k(\boldsymbol{\nu} - \mathbf{d}(\boldsymbol{\nu})) = HT_k(\mathbf{y} - \mathbf{A} \cdot \text{vec}(\hat{\mathbf{W}}(\boldsymbol{\nu})))$$

Based on this formula, we propose a conic hard thresholding method in Algorithm 1.

Algorithm 1 Conic Hard Thresholding

Input: Covariates $\mathbf{A}$, responses $\mathbf{y}$, corruption index k, tolerance ε, and cone $\mathcal{C}$

Initialization :

$\boldsymbol{\nu}^0 \leftarrow \mathbf{0}$, $t \leftarrow 0$;

LOOP Process

while $\|\boldsymbol{\nu}^t - \boldsymbol{\nu}^{t-1}\| > \varepsilon$ **do**

  $\hat{\mathbf{W}}^t = \arg \min_{\mathbf{W} \succeq_{\mathcal{C}} \mathbf{0}} \sum_{r=1}^{m} \left(\langle \mathbf{W}, \mathbf{M}_r \rangle - (y_r - \nu_r^t) \right)^2$;

  $\boldsymbol{\nu}^{t+1} = HT_k(\mathbf{y} - \mathbf{A} \cdot \text{vec}(\hat{\mathbf{W}}^t))$;

  $t \leftarrow t + 1$;

end while

return $\hat{\mathbf{W}}^{t+1}$

Unlike the the penalized SDP and SOCP methods, Algorithm 1 solves a sequence of conic programs to identify the set of bad measurements through a thresholding technique.

In the regime where $m \geq \frac{n(n+1)}{2}$, this algorithm with a high computational complexity can be further relaxed by letting the cone $\mathcal{C}$ be the set of symmetric matrices. We refer to this as **Algorithm 2**, where the condition $\mathbf{W} \succeq_{\mathcal{C}} \mathbf{0}$ is reduced to $\mathbf{W} = \mathbf{W}^T$. Note that Algorithm 2 is not effective if $m < n(n+1)/2$ because the number of measurements becomes less than the number of scalar variables in $\mathbf{W}$. On the other hand, as m grows, the feasibility constraint $\mathbf{W} \succeq_{\mathcal{C}} \mathbf{0}$ would more likely be satisfied for free (since the feasible set shrinks) and Algorithm 1 would perform similarly to Algorithm 2. Inspired by this property, we analyze the asymptotic behavior of Algorithm 2 for Gaussian systems below.

Lemma 18 *Suppose that* $|\mathcal{B}| < \frac{m}{20000}$, $m \geq n^2$, *and* $\mathbf{M}_r$ *is a random normal Gaussian matrix for* $r = 1, ..., m$. *For every* $\epsilon > 0$, *Algorithm 2 recovers a matrix* $\mathbf{W}$ *such that* $\|\mathbf{W} - \mathbf{x}\mathbf{x}^T\|_2 \leq \epsilon$ *within* $\mathcal{O}(\log(\frac{\|\boldsymbol{\eta}\|_2}{\epsilon}) + \log(\frac{2m}{n^2+n}))$ *iterations.*

Proof Follows from Theorem 4 of Bhatia et al. (2017). ■

Let $\mathbf{W}^*$ be any solution obtained by Algorithm 2. Then, one can use its eigenvalue decomposition to find a vector $\mathbf{u}$ such that $\mathbf{u} = \arg\min_{\mathbf{v} \in \mathbb{C}^n} \|\mathbf{v}\mathbf{v}^T - \mathbf{W}\|_2$. Therefore,

$$\begin{aligned} \|\mathbf{u}\mathbf{u}^T - \mathbf{x}\mathbf{x}^T\|_2 &= \|(\mathbf{u}\mathbf{u}^T - \mathbf{W}^*) - (\mathbf{x}\mathbf{x}^T - \mathbf{W}^*)\|_2 \\ &\leq \|\mathbf{u}\mathbf{u}^T - \mathbf{W}^*\|_2 + \|\mathbf{x}\mathbf{x}^T - \mathbf{W}^*\|_2 \leq 2\varepsilon \end{aligned} \tag{15}$$

This means that Algorithm 2 can be used to find an approximate solution $\mathbf{u}$ with any arbitrary precision for the robust regression problem for Gaussian systems with a large number of measurements and yet it allows up to a constant fraction of measurements to be completely wrong. Comparing this with the guarantee $O(|\mathcal{B}|) = O(|\mathcal{G}|^{\frac{1}{2}})$ for the penalized conic methods, given by Corollaries 15 and 16, it can be concluded that Algorithm 1 (or 2) is more robust to outliers than the penalized conic program since it solves a sequence of optimization problems iteratively as opposed to a single one. This leads to a tradeoff between the complexity of an estimation method and its robustness level.

The theoretical analyses of this work were all on a regression model subject to a sparse error vector. However, the results can be slightly modified to account for modest noise values in addition to sparse errors. The bounds derived in this work remain the same, but the solutions found by the penalized conic relaxation and Algorithm 1 would no longer match the true regression solution being sought (as expected, due to a corruption in all equations). The mismatch error is a function of the modest noise values. The details are omitted for brevity; however, the this scenario will later be analyzed in numerical examples.

7. Experiments

In this section, we study the numerical properties of the penalized conic methods and the conic hard thresholding Algorithm 1.

7.1 Synthetic Data

Following Madani et al. (2017a), we define the sparsity pattern of an arbitrary matrix $\mathbf{X} \in \mathbb{S}^n$ to be a binary matrix $\mathbf{N} \in \mathbb{S}^n$ whose (i, j)-entry is equal to 1 if and only if $X_{ij} \neq 0$.

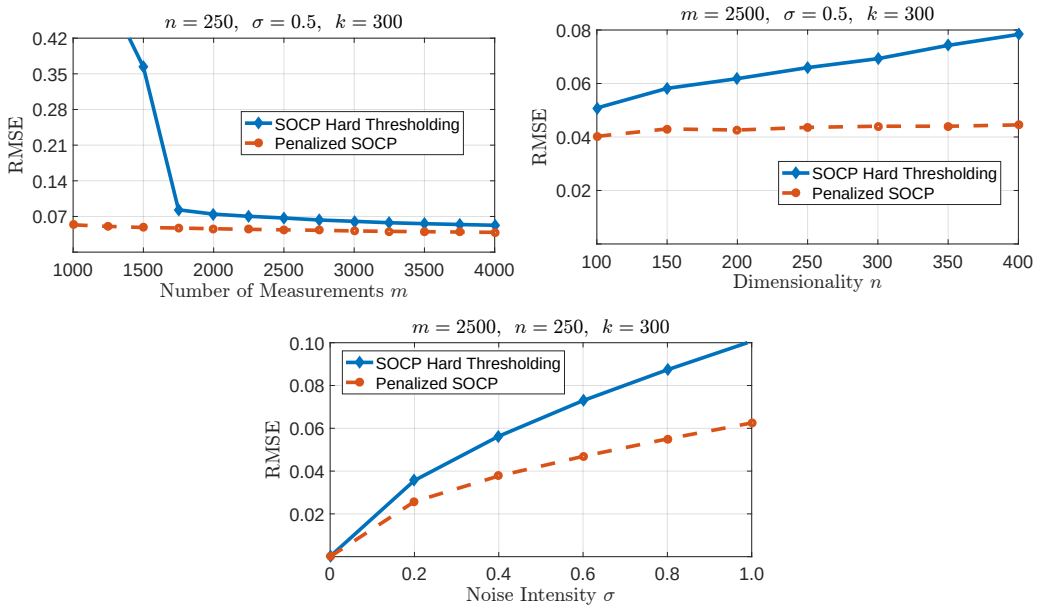

Figure 1: Estimation error as a function of: (a) the number of data points m, (b) the dimensionality n, (c) the additive white noise standard deviation σ.

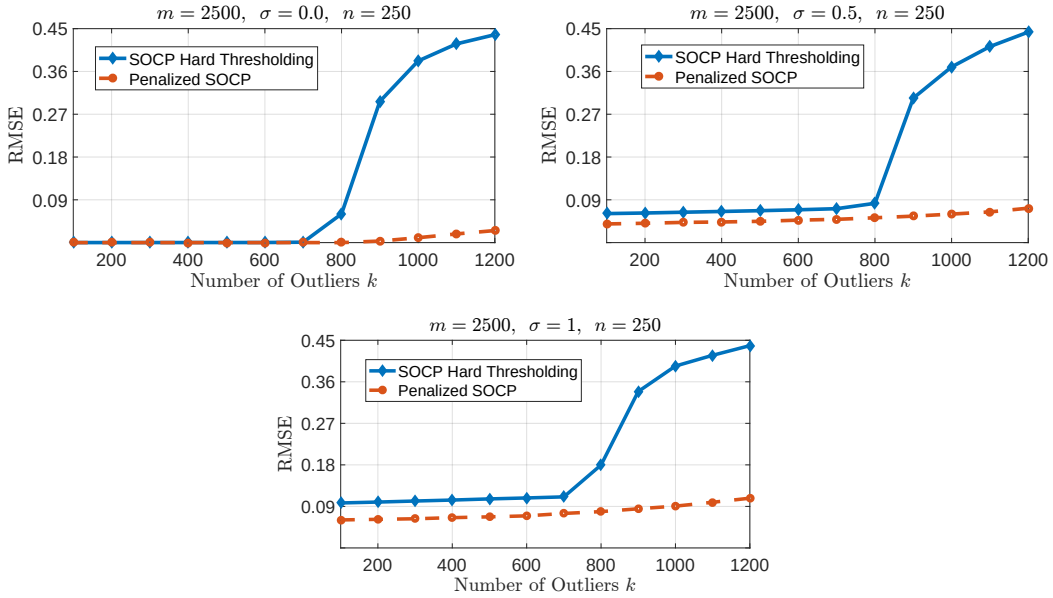

Figure 2: Estimation error as a function of the number of bad measurements k for different magnitudes of additive dense Gaussian noise.

Define the set

$$\mathcal{S}(\mathbf{N}) \triangleq \{\mathbf{X} \in \mathbb{S}^n | \mathbf{X} \circ \mathbf{N} = \mathbf{X}\}$$

We conduct experiments on synthetically generated quadratic regression datasets with corruptions. The true model vector $\mathbf{x}$ is chosen to be a random unit-norm vector, while the input matrices $\mathbf{M}_r$'s are chosen from $\mathcal{S}(\mathbf{N})$ according to a common random sparsity pattern $\mathbf{N}$. The nonzero entries of $\mathbf{M}_r$'s are generated from a normal standard distribution. The matrix $\mathbf{N}$ has all diagonal elements and $3n$ off-diagonal elements nonzero. The off-diagonal positions are selected uniformly. The measurements to be corrupted are chosen uniformly at random and the value of each corruption is generated uniformly from the interval $[10, 20]$. The measurements are then generated as $y_r = \mathbf{x}^*\mathbf{M}_r\mathbf{x} + \eta_r + \omega_r$, where in addition to the sparse error vector $\boldsymbol{\eta}$ there is a random dense noise vector $\boldsymbol{\omega}$ whose entries are Gaussian with zero mean and standard deviation σ. All reported results are averaged over 10 random trials.

By assuming that no prior information about the solution $\mathbf{x}$ is available, we set the matrix $\mathbf{M}$ to be equal to $\mathbf{I}_n$ in the penalized conic methods with the parameter μ chosen as 10^{-2}. Regarding Algorithm 1, the parameter k is selected as the true number of corrupted measurements, the tolerance ε is set to 10^{-3}, and the algorithm is terminated early if the number of conic iterations exceeds 50. In both of the methods, $\mathcal{C}$ is considered to be the SO cone. Hence, we refer to these methods as penalized SOCP and SOCP hard thresholding. Due to the sparsity in the data, the SOCP formulation can be simplified by only imposing those 2×2 constraints in (2) that correspond to the members of $\{(i, j) \mid N_{ij} = 1\}$.

We measure the performance of each algorithm using the root-mean-squared error (RMSE) defined as $\frac{\|\hat{\mathbf{x}}-\mathbf{x}\|_2}{\sqrt{n}}$. Figure 1 shows the RSME in three different plots as a function of the number of data points m, the dimensionality n, and the additive white noise standard deviation σ. Figure 2 depicts the RSME as a function of the number of bad measurements k for different magnitudes of additive dense Gaussian noise. It can be observed that both the penalized conic relaxation and the conic hard thresholding algorithm exhibit an exact recovery property for systems with up to 700 randomly corrupted measurements out of 2500 measurements in the absence of dense Gaussian noise. The same behavior is observed in the presence of dense Gaussian noise of different magnitudes: the error of the penalized SOCP solution grows gradually, while the error of the the hard thresholding algorithm has a jump at around 800 bad measurements. These simulations support the statement that up to a constant fraction of measurements could be completely wrong, and yet the unknown regression solution is found precisely.

Although the theoretical analyses provided in this paper favor Algorithm 1 over the penalized conic relaxation, our empirical analysis shows that the penalized SOCP method has a better performance than the hard thresholding algorithm uniformly in the number of measurements, dimensionality, noise magnitude and the number of outliers. To explain this observation, note that the derived theoretical bounds correspond to the worst-case scenario and are more conservative for an average scenario. Moreover, the implementation of Algorithm 1 in this section has limited the number of iterations to 50, while Theorem 18 requires the number of iterations to grow with respect to the amount of corruption.

The results of this part are produced using the standard MOSEK v7. SOCP-solving procedure, run in MATLAB on a 12-core 2.2GHz machine with 256GB RAM. The CPU

time for each round of solving SOCP ranges from 3s (for $n = 250$, $m = 2500$) to 30s (for $n = 400$, $m = 2500$).

7.2 State Estimation for Power Systems

In this subsection, we present empirical results for the penalized conic relaxation with a PSD cone $\mathcal{C}$ tested on the real data for the power flow state estimation with outliers. As discussed in Madani et al. (2017b), this problem can be formulated as robust quadratic regression. The experiment is run on the PEGASE 1354-bus European system borrowed from the MATPOWER package Fliscounakis et al. (2013); Josz et al. (2016). This system has 1354 nodes and the objective is to estimate the nodal voltages based on voltage magnitude and power measurements of the form $y_r = \mathbf{x}^* \mathbf{M}_r \mathbf{x} + \eta_r + \omega_r$, where $\boldsymbol{\omega}$ is a dense additive noise whose r^{th} entry is Gaussian with mean zero and the standard deviation equal to σ times the true value of the corresponding voltage/power parameter. The dimension of the complex vector $\mathbf{x}$ is 1354, which leads to 2708 real variables in the problem. In this model, the measurements are voltage magnitude squares, active and reactive nodal power injections, and active and reactive power flows from both sides of every line of the power system. This amounts to $3n + 4t = 12026$ measurements, where $t = 1991$ denotes the number of lines in the system. Note that the quadratic regression problem is complex-valued in this case.

The penalty parameter μ of the penalized conic relaxation is set to 10^2 and the matrix $\mathbf{M}$ is chosen as $-\mathbf{Y} + \gamma \mathbf{I}$, where $\mathbf{Y}$ is the susceptance matrix of the system and γ is the smallest positive number that makes $\mathbf{M}$ positive semidefinite. Since the penalized SDP relaxation is large-scale, we employ a tree decomposition technique to leverage the sparsity of the problem to solve it more efficiently (Madani et al. (2016)). The width of the tree decomposition used to reduce the complexity is equal to 12. We do not report any results on Algorithm 1 because it requires solving large-scale SDPs successively and this could be time-consuming. Moreover, the number of measurements is not high enough to use Algorithm 2, and, therefore, we will not test this method either.

The numerical results are reported in Figure 3. Remarkably, if the dense Gaussian noise is non-existent, the conic relaxation recovers the solution precisely as long as the number of bad measurements is less than 150 (note that $\sqrt{m} \simeq 109$). Note that power systems are sparse and their models are far from Gaussian, but the results of Corollary 15 are still valid in this numerical example.

7.3 Linear System Identification

Following Fattahi and Sojoudi (2018), this case study is concerned with the problem of identifying the parameters of a linear dynamical system given limited observation and non-uniform snapshots of the state vector. Consider a discrete-time linear system described by the equations

$$\mathbf{x}[\tau + 1] = \mathbf{A}\mathbf{x}[\tau] + \mathbf{B}\mathbf{u}[\tau] \qquad \tau = 1, 2, \ldots, T - 1, \tag{16a}$$

$$\mathbf{y}[\tau] = \mathbf{C}\mathbf{x}[\tau] + \mathbf{w}[\tau] \qquad \tau = 1, 2, \ldots, T, \tag{16b}$$

where

- $\{\mathbf{x}[\tau] \in \mathbb{R}^n\}_{\tau=1}^T$ are the state vectors that are known at times $\tau \in \{\tau_1, \ldots, \tau_o\}$,

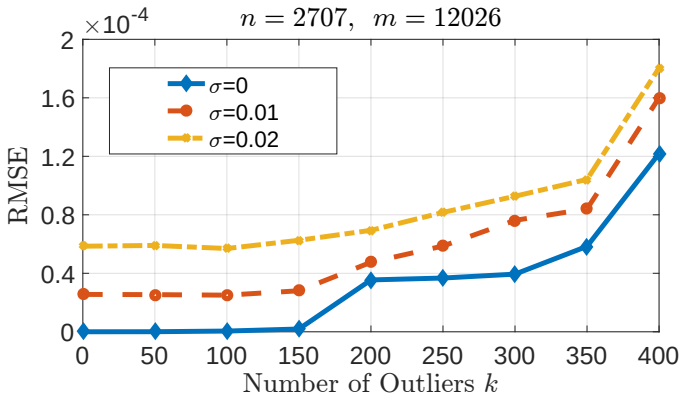

Figure 3: This plot shows the RMSE with respect to the number of corrupted measurements k for the PEGASE 1354-bus system.

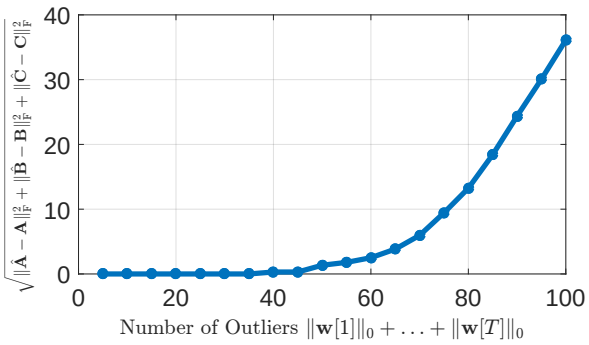

Figure 4: This plot shows the average estimation error of 15 random ground truth realizations with respect to the number of corrupted observations.

- $\{\mathbf{u}[\tau] \in \mathbb{R}^m\}_{\tau=1}^T$ and $\{\mathbf{y}[\tau] \in \mathbb{R}^k\}_{\tau=1}^T$ are the known control and observation vectors, respectively,
- $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$ and $\mathbf{C} \in \mathbb{R}^{k\times n}$ are fixed unknown matrices, and
- $\{\mathbf{w}[\tau] \in \mathbb{R}^k\}_{\tau=1}^T$ are the vectors of sparsely occurring observation errors that are unknown.

We propose to determine the triplet $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ by solving the following system of quadratic equations:

$$0 = \hat{e} \times \hat{\mathbf{x}}[\tau+1] - (\hat{e} \times \hat{\mathbf{B}})\mathbf{u}[\tau] - \hat{\mathbf{A}}\hat{\mathbf{x}}[\tau] \qquad \tau = 1, 2, \ldots, T-1, \tag{17a}$$

$$\mathbf{y}[\tau] = \hat{\mathbf{C}}\hat{\mathbf{x}}[\tau] + \hat{\mathbf{w}}[\tau] \qquad \tau = 1, 2, \ldots, T, \tag{17b}$$

$$\mathbf{x}[\tau] = \hat{e} \times \hat{\mathbf{x}}[\tau] \qquad \tau = \tau_1, \tau_2, \ldots, \tau_o, \tag{17c}$$

$$1 = \hat{e}^2, \tag{17d}$$

with the unknown vector

$$\hat{\mathbf{z}} \triangleq [\hat{e},\ \hat{\mathbf{x}}[1]^\top,\ \hat{\mathbf{x}}[2]^\top,\ \ldots,\ \hat{\mathbf{x}}[T]^\top, \mathrm{vec}\{\hat{\mathbf{A}}\}^\top, \mathrm{vec}\{\hat{\mathbf{B}}\}^\top, \mathrm{vec}\{\hat{\mathbf{C}}\}^\top]^\top \tag{18}$$

and the noise estimation vectors $\{\hat{\mathbf{w}}[\tau] \in \mathbb{R}^k\}_{\tau=1}^T$. The auxiliary variable $\hat{e}$ is added to make the system of equations homogeneous, similar to the canonical quadratic regression problem (1). In order to solve the system of equations (17), we formulate the penalized SDP relaxation (2) by introducing the matrix variable $\hat{\mathbf{Z}}$ accounting for $\hat{\mathbf{z}}\hat{\mathbf{z}}^\top$. In this experiment, we use the objective function

$$\langle \mathbf{M}, \hat{\mathbf{Z}} \rangle + \eta \times \sum_{\tau=1}^{T} \|\hat{\mathbf{w}}[\tau]\|_1 \tag{19}$$

where $\mathbf{M} = \mathrm{diag}\{[0,\ 0.001 \times \mathbf{1}_{1\times nT},\ \mathbf{1}_{1\times n^2},\ \mathbf{0}_{1\times nm},\ \mathbf{1}_{1\times nk}]\}$ and $\eta = 0.1$.

We consider system identification problems with $n = 5$, $m = 2$, $k = 3$, and $T = 50$ time epochs. We assume that, for every $\tau \in \{5, 10, 15, 20, 25, 30, 35, 40, 45, 50\}$, the state vector $\mathbf{x}[\tau]$ is unknown. The elements of the ground truth matrices $\mathbf{A} \in \mathbb{R}^{5\times 5}$, $\mathbf{B} \in \mathbb{R}^{5\times 2}$, $\mathbf{C} \in \mathbb{R}^{3\times 5}$ and the control vectors $\{\mathbf{u}[\tau]\}_{\tau=1}^T$, as well as the initial state $\mathbf{x}[1]$ have independent Gaussian distribution with zero mean and variance $\frac{1}{3}$. Unstable ground truth matrices $\mathbf{A}$ with an eigenvalue outside of the unit circle are excluded. For various values of ρ, we randomly choose ρ elements of $\{\mathbf{y}[\tau]\}_{\tau=1}^T$ and corrupt them by adding observation errors chosen uniformly from the interval $[10, 20]$. Figure 4, demonstrates the average estimation error for 15 trials. As shown in Figure 4, with up to 35 corrupted observations, the triplet $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ can be recovered with zero error. By exploiting the sparsity of the resulting problems (see Nakata et al. (2003)), each round of penalized SDP has been solved within 5 minutes.

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Appendix A.

The following lemma studies Slater's condition for the dual problem (3).

Lemma 19 *If there exists an index $r \in \{1, \ldots, m\}$ such that $\hat{\mathbf{x}}^T \mathbf{M}_r \hat{\mathbf{x}} \neq 0$, then the interior of the feasible region of the problem (3) is not empty and strong duality holds for the penalized SDP.*

Proof Choose $c \in \{-1, +1\}$ such that $c\hat{\mathbf{x}}^T \mathbf{M}_r \hat{\mathbf{x}} > 0$. To construct a strictly feasible point for Problem (3), it is enough to consider $\boldsymbol{\lambda} = u\mathbf{e}_r$, where $u > 0$ is a constant that is smaller than μ and $\mathbf{M} + cu\mathbf{M}_{\hat{r}} \succ 0$. Such a constant exists due to Lemma 3.2.1 in Bertsekas (1999). ■

Proof of Theorem 1 Strong duality of the penalized SDP follows from Lemma 19. We aim to prove that under such a choice of $\hat{\boldsymbol{\lambda}}$, the matrix $\mathbf{M} + \sum \hat{\lambda}_r \mathbf{M}_r$ is a PSD matrix. The complementary slackness condition:

$$\langle \mathbf{x}\mathbf{x}^T, \mathbf{M} + \sum_{r=1}^{m} \lambda_r \mathbf{M}_r \rangle = 0$$

or equivalently

$$(\mathbf{M} + \sum_{r=1}^{m} \lambda_r \mathbf{M}_r)\mathbf{x} = \mathbf{0} \tag{20}$$

It is straightforward to verify that the condition (20) is satisfied for $\hat{\boldsymbol{\lambda}}$. Therefore, $\operatorname{rank}(\mathbf{M} + \sum_{r=1}^{m} \lambda_r \mathbf{M}_r) \leq n - 1$. In light of Corollary 4.3.39 in Horn (2013), that κ is a concave function. Now, it follows from condition (5b) that

$$\kappa(\mathbf{M} + \sum_{r=1}^{m} \hat{\lambda}_r \mathbf{M}_r) \geq \kappa(\mathbf{M}) + \sum_{r=1}^{m} \kappa(\hat{\lambda}_r \mathbf{M}_r) \geq \kappa(\mathbf{M}) - 2\sum_{r=1}^{m} |\hat{\lambda}_r| \ \|\mathbf{M}_r\|_2 \quad > 0$$

which yields that $\operatorname{rank}(\mathbf{M} + \sum_{r=1}^{m} \hat{\lambda}_r \mathbf{M}_r) \geq n - 1$. Dual feasibility for $\hat{\boldsymbol{\lambda}}_{\mathcal{G}}$ follows from condition (20), the above inequality and condition (5a). On the other hand, primal feasibility is satisfied for $(\mathbf{x}\mathbf{x}^T, \eta)$. Therefore, $(\mathbf{x}\mathbf{x}^T, \eta)$ and $\hat{\boldsymbol{\lambda}}$ is a primal-dual optimal pair for the problem. This completes the proof. ■

Appendix B.

Lemma 20 *The sequence $\{\mathbf{A}^{ij} \in \mathbb{S}^2\}_{i<j}$ is a decomposition of $\mathbf{A}$ if and only if:*

$$\begin{cases} \left[\mathbf{A}^{ij}\right]_{21} = \left[\mathbf{A}^{ij}\right]_{12} = A_{ij} = A_{ji} \\ \sum_{i=2}^{n} \sum_{j=1}^{i-1} \left[\mathbf{A}^{ji}\right]_{22} + \sum_{j=2}^{n} \sum_{i=1}^{j-1} \left[\mathbf{A}^{ij}\right]_{11} = \mathbf{A}_{ii} \end{cases}$$

Proof The proof is based on basis linear algebra and is omitted for brevity. ■

Definition 21 *Given the sequences* $\{\mathbf{A}^{ij} \in \mathbb{S}^2\}_{i<j}^{j\leq n}$ *and* $\{\mathbf{B}^{ij} \in \mathbb{S}^2\}_{i<j}^{j\leq n}$*, define the sum:*

$$\{\mathbf{A}^{ij}\}_{i<j}^{j\leq n} + \{\mathbf{B}^{ij}\}_{i<j}^{j\leq n} := \{\mathbf{A}^{ij} + \mathbf{B}^{ij}\}_{i<j}^{j\leq n}$$

Definition 22 *For a sequence* $\{\mathbf{A}^{ij} \in \mathbb{S}^2\}_{i<j}^{j\leq n}$ *and a scalar* $c \in \mathbb{R}$*, define the multiplication:*

$$c\{\mathbf{A}^{ij}\}_{i<j}^{j\leq n} := \{c\mathbf{A}^{ij}\}_{i<j}^{j\leq n}$$

Lemma 23 *If* $\{\mathbf{A}^{ij} \in \mathbb{S}^2\}_{i<j}^{j\leq n}$ *and* $\{\mathbf{B}^{ij} \in \mathbb{S}^2\}_{i<j}^{j\leq n}$ *are decompositions of* $\mathbf{A}$ *and* $\mathbf{B}$ *respectively, then* $\{\mathbf{A}^{ij} + \mathbf{B}^{ij}\}$ *is a decomposition of* $\mathbf{A} + \mathbf{B}$ *and* $c\{\mathbf{A}^{ij}\}$ *is a decomposition of* $c\mathbf{A}$*, for all* $c \in \mathbb{R}$*.*

Proof To prove the first part, one can write:

$$\sum_{i<j} [\mathbf{e}_i \;\; \mathbf{e}_j] \left(\mathbf{A}^{ij} + \mathbf{B}^{ij}\right) [\mathbf{e}_i \;\; \mathbf{e}_j]^T =$$
$$\sum_{i<j} [\mathbf{e}_i \;\; \mathbf{e}_j] \mathbf{A}^{ij} [\mathbf{e}_i \;\; \mathbf{e}_j]^T + \sum_{i<j} [\mathbf{e}_i \;\; \mathbf{e}_j] \mathbf{B}^{ij} [\mathbf{e}_i \;\; \mathbf{e}_j]^T = \mathbf{A} + \mathbf{B}$$

Moreover,

$$\sum_{i<j} [\mathbf{e}_i \;\; \mathbf{e}_j] c\mathbf{A}^{ij} [\mathbf{e}_i \;\; \mathbf{e}_j]^T = c \sum_{i<j} [\mathbf{e}_i \;\; \mathbf{e}_j] \mathbf{A}^{ij} [\mathbf{e}_i \;\; \mathbf{e}_j]^T = c\mathbf{A}$$

This proves the second part of the lemma. ■

Note that κ is a concave function, and an analogous property of χ will be stated below.

Lemma 24 *Given the sequences* $\{\mathbf{A}^{ij}\} = \{\mathbf{A}^{ij} \in \mathbb{S}^2\}_{i<j}^{j\leq n}$ *and* $\{\mathbf{B}^{ij}\} = \{\mathbf{B}^{ij} \in \mathbb{S}^2\}_{i<j}^{j\leq n}$ *as well as* $c \in \mathbb{R}$*, the following properties hold:*

$$\chi(\{\mathbf{A}^{ij}\} + \{\mathbf{B}^{ij}\}) \geq \chi(\{\mathbf{A}^{ij}\}) + \chi(\{\mathbf{B}^{ij}\})$$
$$\chi(c\{\mathbf{A}^{ij}\}) \geq -|c| \max_{i'<j'} \; |\mathrm{tr}(\mathbf{A}^{i'j'})|$$

Proof Introduce

$$(i', j') = \arg\min_{i<j} \; \mathrm{tr}(\mathbf{A}^{ij});$$
$$(i'', j'') = \arg\min_{i<j} \; \mathrm{tr}(\mathbf{B}^{ij});$$
$$(i^*, j^*) = \arg\min_{i<j} \; \mathrm{tr}(\mathbf{A}^{ij} + \mathbf{B}^{ij});$$

The proof of the first inequality follows from the following expression:

$$\chi(\{\mathbf{A}^{ij}\} + \{\mathbf{B}^{ij}\}) \geq \mathrm{tr}(\mathbf{A}^{i^*j^*} + \mathbf{B}^{i^*j^*}) = \mathrm{tr}(\mathbf{A}^{i^*j^*}) + \mathrm{tr}(\mathbf{B}^{i^*j^*}) \geq$$
$$\mathrm{tr}(\mathbf{A}^{i'j'}) + \mathrm{tr}(\mathbf{B}^{i''j''}) = \chi(\{\mathbf{A}^{ij}\}) + \chi(\{\mathbf{B}^{ij}\})$$

For the second inequality, one can write

$$\chi(c\{\mathbf{A}^{ij}\}) = \min_{i<j}\ \mathrm{tr}(c\mathbf{A}^{ij}) = \min_{i<j}\ c\ \mathrm{tr}(\mathbf{A}^{ij}) \geq -\max_{i<j}\ |c\ \mathrm{tr}(\mathbf{A}^{ij})| \geq -|c| \max_{i<j}\ |\mathrm{tr}(\mathbf{A}^{ij})|$$

This completes the proof. ■

Lemma 25 *By defining* $\mathbf{M}_r^{ij} := \begin{bmatrix} R_{ij}^r & M_{ij}^r \\ M_{ji}^r & R_{ji}^r \end{bmatrix}$, *the sequence* $\{\mathbf{M}_r^{ij}\}_{i<j}^{j\leq n}$ *becomes a decomposition of* $\mathbf{M}_r$.

Proof It is straightforward to verify that

$$\sum_{i=2}^{n}\sum_{j=1}^{i-1}[\mathbf{M}_r^{ji}]_{22} + \sum_{j=2}^{n}\sum_{i=1}^{j-1}[\mathbf{M}_r^{ij}]_{11} = \sum_{j=1}^{n} R_{ij}^r = M_{ii}^r$$

The proof follows from Lemma 20. ■

Lemma 26 $\{\mathbf{M}^{ij}\}_{i<j} + \sum_{r=1}^{m} \lambda_r\{\mathbf{M}_r^{ij}\}_{i<j}$ *is a decomposition of* $\mathbf{M} + \sum_{r=1}^{m} \lambda_r\mathbf{M}_r$

Proof The proof follows immediately from Lemmas 23 and 25. ■

Lemma 27 *If the components of the solution are nonzero:* $x_i \neq 0$ *for all* $i \in \{1,\ldots,n\}$; *and there exists an index* $r \in \{1,\ldots,m\}$ *such that* $\hat{\mathbf{x}}^*\mathbf{M}_r\hat{\mathbf{x}} \neq 0$, *then the interior of the feasible region of Problem* (6) *is not empty, and strong duality holds for the penalized SOCP.*

Proof Recall that $\hat{\mathbf{x}}$ is an initial guess for the solution $\mathbf{x}$ and $\mathbf{M}$ a matrix in the objective function constructed based on $\hat{\mathbf{x}}$. We choose $c \in \{-1,+1\}$ such that $c\hat{\mathbf{x}}^T\mathbf{M}_r\hat{\mathbf{x}} > 0$, and select $\boldsymbol{\lambda} = uc\mathbf{e}_r$. It is desirable to show that if u is a sufficiently small positive number, then $\mathbf{M} + uc\mathbf{M}_k$ belongs to the interior of the $\mathcal{SDD}$ cone, i.e., it can be written as

$$\mathbf{M} + uc\mathbf{M}_k = \sum_{i,j}[\mathbf{e}_i\ \mathbf{e}_j]\mathbf{H}^{ij}[\mathbf{e}_i\ \mathbf{e}_j]^T,$$

where each $\mathbf{H}^{ij}$ is a 2×2 symmetric positive-definite matrix. The matrix $\mathbf{M}$ can be written as

$$\mathbf{M} = \sum_{i,j}[\mathbf{e}_i\ \mathbf{e}_j]\mathbf{M}^{ij}[\mathbf{e}_i\ \mathbf{e}_j]^T$$

where each $\mathbf{M}^{ij}$ is a 2×2 symmetric positive semidefinite matrix that has rank 1 and $[x_i, x_j]$ belongs to the null space of $\mathbf{M}^{ij}$. Now, we need to find a decomposition $\{\mathbf{B}^{ij}\}_{i<j}$ of $\mathbf{F} := c\mathbf{M}_k$ such that $\mathbf{M}^{ij} + u\mathbf{B}^{ij}$ becomes positive definite if u is small. Since the null space of $\mathbf{M}^{ij}$ is one dimensional, it suffices to show that $[x_i\ \ x_j]\mathbf{B}^{ij}[x_i\ \ x_j]^T > 0$ (due to Lemma 3.2.1 in Bertsekas (1999)). To this end, consider the following decomposition:

$$\begin{aligned}
\left[\mathbf{B}^{ij}\right]_{11} &= (d_i - d_j - F_{ij})\tfrac{x_j}{x_i} + \tfrac{s}{n-1} \\
\left[\mathbf{B}^{ij}\right]_{12} &= F_{ij} \\
\left[\mathbf{B}^{ij}\right]_{21} &= F_{ji} \\
\left[\mathbf{B}^{ij}\right]_{22} &= (d_j - d_i - F_{ji})\tfrac{x_i}{x_j} + \tfrac{s}{n-1}
\end{aligned}$$

where $d_i = \frac{\mathbf{x}^T\mathbf{F}\mathbf{e}_i - s x_i}{\mathbb{1}^T\mathbf{x}}$ for every $i \in \{1, \ldots, n\}$ and $s = \frac{\mathbf{x}^T\mathbf{F}\mathbf{x}}{\mathbf{x}^T\mathbf{x}}$. To complete the proof, it suffices to show that

$$\begin{cases} \mathbf{F} = \sum_{i,j} [\mathbf{e}_i \ \mathbf{e}_j] \mathbf{B}^{ij} [\mathbf{e}_i \ \mathbf{e}_j]^T \\ [x_i \ x_j] \mathbf{B}^{ij} [x_i \ x_j]^T > 0 \end{cases}$$

Which according to lemma 20 is equivalent to the following three conditions satisfied simultaneously:

$$B^{ij}_{12} = F_{ij}, \quad B^{ij}_{21} = F_{ji}, \quad \forall i < j \tag{21}$$

$$B^{ij}_{11} x_i^2 + B^{ij}_{22} x_j^2 > -(F_{ij} + F_{ji}) x_i x_j \quad \forall\ i < j \tag{22}$$

$$\sum_{j<i} B^{ji}_{22} + \sum_{j>i} B^{ij}_{11} = F_{ii}, \quad \forall i \tag{23}$$

Condition (21) is straightforward to verify. To verify (22), notice that

$$\begin{aligned}
B^{ij}_{11} x_i^2 + B^{ij}_{22} x_j^2 = \\
((d_i - d_j - F_{ij})\frac{x_j}{x_i} + \frac{s}{n-1}) x_i^2 + ((d_j - d_i - F_{ji})\frac{x_i}{x_j} + \frac{s}{n-1}) x_j^2 = \\
(-F_{ij} - F_{ji}) x_i x_j + s\frac{x_i^2 + x_j^2}{n-1} > -(F_{ij} + F_{ji}) x_i x_j
\end{aligned}$$

To analyze (23), one can write:

$$\begin{aligned}
\sum_{j<i} B^{ji}_{22} + \sum_{j>i} B^{ij}_{11} &= \\
\sum_{j<i} ((d_i - d_j - F_{ij})\tfrac{x_j}{x_i} + \tfrac{s}{n-1}) + \sum_{j>i} ((d_i - d_j - F_{ij})\tfrac{x_j}{x_i} + \tfrac{s}{n-1}) &= \\
\sum_{j \neq i} (d_i - d_j - F_{ij})\tfrac{x_j}{x_i} + s &= \\
\tfrac{d_i}{x_i}(\sum_j x_j - x_i) - \tfrac{1}{x_i}(\sum_j d_j x_j - d_i x_i) - \tfrac{1}{x_i}(\sum_j F_{ij} x_j - F_{ii} x_i) + s &= \\
\tfrac{d_i}{x_i} \sum_j x_j - \tfrac{1}{x_i} \sum_j d_j x_j - \tfrac{1}{x_i} \sum_j F_{ij} x_j + s + F_{ii} &= \\
\tfrac{d_i}{x_i} \mathbb{1}^T\mathbf{x} - \tfrac{1}{x_i} \sum_j d_j x_j - \tfrac{1}{x_i} \mathbf{e}_i^T \mathbf{F}\mathbf{x} + s + F_{ii} &= \\
\tfrac{d_i}{x_i} \mathbb{1}^T\mathbf{x} - \tfrac{1}{x_i \mathbb{1}^T\mathbf{x}} \sum_j [\mathbf{x}^T\mathbf{F}\mathbf{e}_j - x_j s] x_j - \tfrac{1}{x_i} \mathbf{e}_i^T \mathbf{F}\mathbf{x} + s + F_{ii} &= \\
\tfrac{d_i}{x_i} \mathbb{1}^T\mathbf{x} - \tfrac{1}{x_i \mathbb{1}^T\mathbf{x}} [\mathbf{x}^T\mathbf{F}\mathbf{x} - s\mathbf{x}^T\mathbf{x}] - \tfrac{1}{x_i} \mathbf{e}_i^T \mathbf{F}\mathbf{x} + s + F_{ii} &= \\
\tfrac{d_i}{x_i} \mathbb{1}^T\mathbf{x} - \tfrac{1}{x_i} \mathbf{e}_i^T \mathbf{F}\mathbf{x} + s + F_{ii} &= F_{ii}.
\end{aligned}$$

As a result, if u is small then $\|uc\mathbf{e}_r\|_\infty \leq \mu$ and $\mathbf{A}^{ij} + u\mathbf{B}^{ij}$ is positive definite, and therefore $\mathbf{M} + u\mathbf{F}$ belongs to the interior of the $\mathcal{SDD}$ cone. ■

Proof of Theorem 6 Strong duality of the penalized SOCP follows from Lemma 27. It is desirable to show that under such a choice of $\hat{\boldsymbol{\lambda}}$ each matrix $\mathbf{M}^{ij} + \sum \hat{\lambda}_r \mathbf{M}_r^{ij}$ is a PSD matrix. The complementary slackness condition:

$$\langle [x_i \;\; x_j][x_i \;\; x_j]^T, \mathbf{M}^{ij} + \sum \lambda_r \mathbf{M}_r^{ij} \rangle = 0$$

or equivalently

$$(\mathbf{M}^{ij} + \sum \lambda_r \mathbf{M}_r^{ij}) \begin{bmatrix} x_i \\ x_j \end{bmatrix} = \mathbf{0} \tag{24}$$

The condition (24) combined with $\chi(\{\mathbf{M}^{ij}\}_{i<j} + \sum_{r=1}^{m} \hat{\lambda}_r \{\mathbf{M}_r^{ij}\}_{i<j}) > 0$ yields $\mathbf{M}^{ij} + \sum_{r=1}^{m} \hat{\lambda}_r \mathbf{M}_r^{ij} \succeq 0$ for all $i, j \in \{1, \ldots, n\}$, and thus: $\mathbf{M} + \sum_{r=1}^{m} \hat{\lambda}_r \mathbf{M}_r \in \mathcal{SDD}$ (by lemma 23). To satisfy the condition (24), $\boldsymbol{\lambda}$ must be such that:

$$\sum_{r=1}^{m} \lambda_r \begin{bmatrix} R_{ij}^r & M_{ij}^r \\ M_{ji}^r & R_{ji}^r \end{bmatrix} \begin{bmatrix} x_i \\ x_j \end{bmatrix} = - \begin{bmatrix} G_{ij} \\ G_{ji} \end{bmatrix} \quad \forall \; i < j$$

or equivalently

$$\sum_{r \in \mathcal{G} \cup \mathcal{B}} \lambda_r R_{ij}^r = -\frac{x_j}{x_i} \sum_{r \in \mathcal{G} \cup \mathcal{B}} \lambda_r M_{ij}^r - G_{ij} \quad \forall \; i \neq j$$

Use the definitions given in (7) and rewrite

$$\tilde{\mathbf{J}}_{\mathcal{G}} \boldsymbol{\lambda}_{\mathcal{G}} = -(\tilde{\mathbf{J}}_{\mathcal{B}} \hat{\boldsymbol{\lambda}}_{\mathcal{B}} + \tilde{\mathbf{d}})$$

One solution to the above system is

$$\hat{\boldsymbol{\lambda}}_{\mathcal{G}} = -\tilde{\mathbf{J}}_{\mathcal{G}}^{+} (\tilde{\mathbf{J}}_{\mathcal{B}} \hat{\boldsymbol{\lambda}}_{\mathcal{B}} + \tilde{\mathbf{d}})$$

To conclude with dual feasibility, it is sufficient to show that

$$\chi(\{\mathbf{M}^{ij}\}_{i<j} + \sum_{r=1}^{m} \hat{\lambda}_r \{\mathbf{M}_r^{ij}\}_{i<j}) \quad > 0,$$

which is guaranteed by condition (8b) and Lemma 24, and $\|\hat{\boldsymbol{\lambda}}\|_\infty \leq \mu$ which is guaranteed by condition (8a). On the other hand, primal feasibility is satisfied for $(\mathbf{x}\mathbf{x}^*, \eta)$. Therefore, $(\mathbf{x}\mathbf{x}^*, \eta)$ and $(\hat{\boldsymbol{\lambda}}, \{\mathbf{M}^{ij}\} + \sum_{r=1}^{m} \hat{\lambda}_r \{\mathbf{M}_r^{ij}\})$ is a primal-dual optimal pair for the problem. This completes the proof. ■

Proof of Lemma 7 Consider strong duality:

$$\begin{aligned} \langle \mathbf{x}\mathbf{x}^T, \mathbf{M} \rangle + \mu \|\boldsymbol{\eta}\|_1 = -\mathbf{y}^T \hat{\boldsymbol{\lambda}} \iff \\ \mathbf{x}^T \mathbf{M} \mathbf{x} + \mu \|\boldsymbol{\eta}_{\mathcal{B}}\|_1 = -\sum_{r=1}^{m} \mathbf{x}^T \hat{\boldsymbol{\lambda}}_r \mathbf{M}_r \mathbf{x} - \boldsymbol{\eta}_{\mathcal{B}}^T \hat{\boldsymbol{\lambda}}_{\mathcal{B}} \iff \\ \mathbf{x}^T \left(\mathbf{M} + \sum_{r=1}^{m} \hat{\boldsymbol{\lambda}}_r \mathbf{M}_r \right) \mathbf{x} = -\left(\mu \|\boldsymbol{\eta}_{\mathcal{B}}\|_1 + \boldsymbol{\eta}_{\mathcal{B}}^T \hat{\boldsymbol{\lambda}}_{\mathcal{B}} \right) \end{aligned}$$

By complementary slackness condition, we have

$$\begin{aligned} \mathbf{x}^T(\mathbf{M} + \sum_{r=1}^{m} \hat{\lambda}_r \mathbf{M}_r)\mathbf{x} &= \\ \mathbf{x}^T \left\{ \sum_{i<j} [\mathbf{e}_i \;\; \mathbf{e}_j](\mathbf{M}^{ij} + \sum \hat{\lambda}_r \mathbf{M}_r^{ij})[\mathbf{e}_i \;\; \mathbf{e}_j]^T \right\} \mathbf{x} &= \\ \sum_{i<j} [x_i \;\; x_j](\mathbf{M}^{ij} + \sum \hat{\lambda}_r \mathbf{M}_r^{ij}) \begin{bmatrix} x_i \\ x_j \end{bmatrix} &= 0 \end{aligned}$$

Subject to the constraint $\|\hat{\boldsymbol{\lambda}}\|_\infty < \mu$, the only solution of $\mu\|\boldsymbol{\eta}_\mathcal{B}\|_1 + \boldsymbol{\eta}_\mathcal{B}^T \hat{\boldsymbol{\lambda}}_\mathcal{B} = 0$ is $\hat{\boldsymbol{\lambda}}_\mathcal{B} = -\mu \operatorname{sign}(\boldsymbol{\eta}_\mathcal{B})$ ■

Appendix C.

The next lemma will help prove some key results of the paper.

Lemma 28 *Let* $\mathbf{J}$ *be a matrix in* $\mathbb{R}^{l\times m}$, $\mathbf{d}$ *be a vector in* $\mathbb{R}^l$ *and* $\boldsymbol{\lambda}$ *be a vector in* $\mathbb{R}^m$ *such that* $\boldsymbol{\lambda}_\mathcal{B} = \mu \cdot \mathbf{s}$, *where* μ *is a scalar and* $\mathbf{s}$ *consists of* $+1$ *or* -1. *If*

$$\sigma_{min}(\mathbf{J}_\mathcal{G}) > \sigma_{max}(\mathbf{J}_\mathcal{B})$$

and

$$(\sigma_{min}(\mathbf{J}_\mathcal{G}) - \sigma_{max}(\mathbf{J}_\mathcal{B}))(\alpha\sigma_{min}(\mathbf{J}_\mathcal{G}) - \sqrt{|\mathcal{G}|}) > \sqrt{|\mathcal{B}|}\sigma_{max}(\mathbf{J}_\mathcal{B})\sqrt{|\mathcal{G}|} + |\mathcal{B}|\sigma_{min}(\mathbf{J}_\mathcal{G}) \tag{25}$$

then the interval

$$\left[\frac{\|\mathbf{d}\|_2}{\sigma_{min}(\mathbf{J}_\mathcal{G}) - \sigma_{max}(\mathbf{J}_\mathcal{B})}, \frac{(\alpha\sigma_{min}(\mathbf{J}_\mathcal{G}) - \sqrt{|\mathcal{G}|})\|\mathbf{d}\|_2}{\sqrt{|\mathcal{B}||\mathcal{G}|}\sigma_{max}(\mathbf{J}_\mathcal{B}) + |\mathcal{B}|\sigma_{min}(\mathbf{J}_\mathcal{G})} \right] \tag{26}$$

is not empty and the system of inequalities

$$\begin{cases} \mu > \|\boldsymbol{\lambda}_\mathcal{G}\|_\infty \\ \alpha\|\mathbf{d}\|_2 > \|\boldsymbol{\lambda}_\mathcal{G}\|_1 + \mu|\mathcal{B}| \end{cases} \tag{27}$$

can be satisfied by $\boldsymbol{\lambda}_\mathcal{G} = -\mathbf{J}_\mathcal{G}^+(\mathbf{J}_\mathcal{B}\boldsymbol{\lambda}_\mathcal{B} + \mathbf{b})$ *for every* μ *in the interval given in* (26).

Proof Set $\boldsymbol{\lambda}_\mathcal{G} = -\mathbf{J}_\mathcal{G}^+(\mathbf{J}_\mathcal{B}\boldsymbol{\lambda}_\mathcal{B} + \mathbf{d})$ and check the set of values of μ under which the system (27) is satisfied. It can be shown that $\|\boldsymbol{\lambda}_\mathcal{B}\|_\infty = \mu$; $\|\boldsymbol{\lambda}_\mathcal{B}\|_2 = \mu\sqrt{|\mathcal{B}|}$. Get several auxiliary inequalities:

1. $\|\mathbf{J}_\mathcal{G}^+\mathbf{d}\|_1 \leq \sqrt{|\mathcal{G}|}\|\mathbf{J}_\mathcal{G}^+\mathbf{d}\|_2 \leq \sqrt{|\mathcal{G}|}\|\mathbf{J}_\mathcal{G}^+\|_2\|\mathbf{d}\|_2$
2. $\|\mathbf{J}_\mathcal{G}^+\mathbf{d}\|_\infty \leq \|\mathbf{J}_\mathcal{G}^+\mathbf{d}\|_2 \leq \|\mathbf{J}_\mathcal{G}^+\|_2\|\mathbf{d}\|_2$
3. $\|\mathbf{J}_\mathcal{G}^+\mathbf{J}_\mathcal{B}\boldsymbol{\lambda}_\mathcal{B}\|_1 \leq \sqrt{|\mathcal{G}|}\|\mathbf{J}_\mathcal{G}^+\mathbf{J}_\mathcal{B}\boldsymbol{\lambda}_\mathcal{B}\|_2 \leq \sqrt{|\mathcal{G}|}\|\mathbf{J}_\mathcal{G}^+\|_2\|\mathbf{J}_\mathcal{B}\lambda_\mathcal{B}\|_2 \leq \mu\sqrt{|\mathcal{G}||\mathcal{B}|}\|\mathbf{J}_\mathcal{G}^+\|_2\|\mathbf{J}_\mathcal{B}\|_2$

4. $\|\mathbf{J}_{\mathcal{G}}^{+}\mathbf{J}_{\mathcal{B}}\boldsymbol{\lambda}_{\mathcal{B}}\|_\infty \leq \|\mathbf{J}_{\mathcal{G}}^{+}\mathbf{J}_{\mathcal{B}}\|_\infty\|\boldsymbol{\lambda}_{\mathcal{B}}\|_\infty \leq \mu\|\mathbf{J}_{\mathcal{G}}^{+}\mathbf{J}_{\mathcal{B}}\|_2 \leq \mu\|\mathbf{J}_{\mathcal{G}}^{+}\|_2\|\mathbf{J}_{\mathcal{B}}\|_2$

We want to satisfy:

$$\begin{cases} \mu > \|\mathbf{J}_{\mathcal{G}}^{+}(\mathbf{J}_{\mathcal{B}}\boldsymbol{\lambda}_{\mathcal{B}} + \mathbf{d})\|_\infty \\ \alpha\|\mathbf{b}\|_2 > \|\mathbf{J}_{\mathcal{G}}^{+}(\mathbf{J}_{\mathcal{B}}\boldsymbol{\lambda}_{\mathcal{B}} + \mathbf{d})\|_1 + \mu|\mathcal{B}| \end{cases} \tag{28}$$

One can use $\|\mathbf{J}_{\mathcal{G}}^{+}(\mathbf{J}_{\mathcal{B}}\boldsymbol{\lambda}_{\mathcal{B}} + \mathbf{d})\| \leq \|\mathbf{J}_{\mathcal{G}}^{+}\mathbf{d}\| + \|\mathbf{J}_{\mathcal{G}}^{+}\mathbf{J}_{\mathcal{B}}\boldsymbol{\lambda}_{\mathcal{B}}\|$ and relax the inequalities in (28) by applying the auxiliary inequalities:

$$\begin{cases} \mu > \|\mathbf{J}_{\mathcal{G}}^{+}\|_2\|\mathbf{d}\|_2 + \mu\|\mathbf{J}_{\mathcal{G}}^{+}\|_2\|\mathbf{J}_{\mathcal{B}}\|_2 \\ \alpha\|\mathbf{d}\|_2 > \sqrt{|\mathcal{G}|}\|\mathbf{J}_{\mathcal{G}}^{+}\|_2\|\mathbf{d}\|_2 + \mu\sqrt{|\mathcal{G}|}\|\mathbf{J}_{\mathcal{G}}^{+}\|_2\sqrt{|\mathcal{B}|}\|\mathbf{J}_{\mathcal{B}}\|_2 + \mu|\mathcal{B}| \end{cases}$$

Using $\|\mathbf{J}_B\|_2 = \sigma_{max}(\mathbf{J}_{\mathcal{B}})$ and $\|\mathbf{J}_{\mathcal{G}}^{+}\|_2 = \sigma_{min}(\mathbf{J}_{\mathcal{G}})^{-1}$, it yields that

$$\begin{cases} \mu(1 - \frac{\sigma_{max}(\mathbf{J}_{\mathcal{B}})}{\sigma_{min}(\mathbf{J}_{\mathcal{G}})}) > \frac{\|\mathbf{d}\|_2}{\sigma_{min}(\mathbf{J}_{\mathcal{G}})} \\ \alpha\|\mathbf{d}\|_2 > \frac{\sqrt{|\mathcal{G}|}}{\sigma_{min}(\mathbf{J}_{\mathcal{G}})}\|\mathbf{d}\|_2 + \mu\left(\frac{\sqrt{|\mathcal{G}|}}{\sigma_{min}(\mathbf{J}_{\mathcal{G}})}\sqrt{|\mathcal{B}|}\sigma_{max}(\mathbf{J}_{\mathcal{B}}) + |\mathcal{B}|\right) \end{cases}$$

One can express the bounds on μ as

$$\begin{cases} \mu > \frac{\|\mathbf{d}\|_2}{\sigma_{min}(\mathbf{J}_{\mathcal{G}}) - \sigma_{max}(\mathbf{J}_{\mathcal{B}})} \\ \mu < \frac{\alpha\sigma_{min}(\mathbf{J}_{\mathcal{G}})\|\mathbf{d}\|_2 - \sqrt{|\mathcal{G}|}\|\mathbf{b}\|_2}{\sqrt{|\mathcal{B}|}\sigma_{max}(\mathbf{J}_{\mathcal{B}})\sqrt{|\mathcal{G}|} + |\mathcal{B}|\sigma_{min}(\mathbf{J}_{\mathcal{G}})} \end{cases}$$

This gives rise to a condition to guarantee that the interval is not empty:

$$\frac{(\alpha\sigma_{min}(\mathbf{J}_{\mathcal{G}}) - \sqrt{|\mathcal{G}|})\|\mathbf{d}\|_2}{\sqrt{|\mathcal{B}|}\sigma_{max}(\mathbf{J}_{\mathcal{B}})\sqrt{|\mathcal{G}|} + |\mathcal{B}|\sigma_{min}(\mathbf{J}_{\mathcal{G}})} > \frac{\|\mathbf{d}\|_2}{\sigma_{min}(\mathbf{J}_{\mathcal{G}}) - \sigma_{max}(\mathbf{J}_{\mathcal{B}})}$$

The above inequality holds by (25). This concludes the proof. ■

Proof of Theorem 14 Note that the inequality (11) is stronger than

$$\sqrt{|\mathcal{G}|(1 - \Delta_{|\mathcal{G}|})} > \sqrt{|\mathcal{B}|(1 + \Delta_{|\mathcal{B}|})}$$

In light of Lemma 14 in Bhatia et al. (2015), any randomly sampled Gaussian matrix $\mathbf{X} \in \mathbb{R}^{l\times m}$ satisfies the inequalities

$$\lambda_{max}(\mathbf{X}\mathbf{X}^T) \leq m + (1 - 2\varepsilon)^{-1}\sqrt{cml + c'm\log\tfrac{2}{\delta}}$$
$$\lambda_{min}(\mathbf{X}\mathbf{X}^T) \geq m - (1 - 2\varepsilon)^{-1}\sqrt{cml + c'm\log\tfrac{2}{\delta}}$$

with probability at least $1 - \delta$ for every $\varepsilon > 0$, where $c = 24e^2\log\frac{3}{\varepsilon}$ and $c' = 24e^2$. This implies that the relations

$$\sigma_{min}(\mathbf{J}_{\mathcal{G}}) \in [\sqrt{|\mathcal{G}|(1 - \Delta_{|\mathcal{G}|})}, \sqrt{|\mathcal{G}|(1 + \Delta_{|\mathcal{G}|})}]$$

and

$$\sigma_{max}(\mathbf{J}_{\mathcal{B}}) \in [\sqrt{|\mathcal{B}|(1 - \Delta_{|\mathcal{B}|})}, \sqrt{|\mathcal{B}|(1 + \Delta_{|\mathcal{B}|})}]$$

are each satisfied with the probability $1 - \delta$, and both are met simultaneously with probability at least $(1-\delta)^2$. By tightening the bounds in Lemma 28 with these limits on singular values, it is straightforward to verify the statement of the theorem. ■

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