Convex Relaxation for Optimal Distributed Control Problem—Part II: Lyapunov Formulation and Case Studies

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***Abstract*—This two-part paper is concerned with the optimal distributed control (ODC) problem. In Part I, the finite-horizon ODC problem was investigated for deterministic systems. In this part, we first study the infinite-horizon ODC problem (for deterministic systems) and then generalize the results to a stochastic ODC problem (for stochastic systems). By adopting a Lyapunov approach, we show that each of these non-convex controller design problems admits a rank-constrained formulation, which can be relaxed to a semidefinite program (SDP). The notion of treewidth is then utilized to prove that the SDP relaxation has a matrix solution with rank at most 3. If the SDP relaxation has a rank-1 solution, a globally optimal solution can be recovered from it; otherwise, a near-optimal controller together with a bound on its optimality degree may be attained. Since the proposed SDP relaxation is not computationally attractive, a computationally-cheap SDP relaxation is also developed. It is shown that this relaxation works as well as Riccati equations in the extreme case of designing a centralized controller. The superiority of the proposed technique is demonstrated on several thousand simulations for two physical systems (mass spring and electrical power network) and random systems.**

I. Introduction

Real-world systems mostly consist of many interconnected subsystems, and designing an optimal controller for them pose several challenges to the field of control. The area of *distributed control* is created to address the challenges arising in the control of these systems. The objective is to design a constrained controller whose structure is specified by a set of permissible interactions between the local controllers with the aim of reducing the computation or communication complexity of the overall controller. If the local controllers are not allowed to exchange information, the problem is often called *decentralized controller* design. It has been long known that the design of an optimal distributed (decentralized) controller is a daunting task because it amounts to an NP-hard optimization problem in general [1], [2]. Great effort has been devoted to investigating this highly complex problem for special types of systems, including spatially distributed systems [3], [4], [5], [6], [7], dynamically decoupled systems [8], [9], weakly coupled systems [10], and strongly connected systems [11].

There is no surprise that the decentralized control problem is computationally hard to solve. This is a consequence of the fact that several classes of optimization problems, including polynomial optimization and quadratically-constrained quadratic program (QCQP) as a special case, are NP-hard in the worst case. Due to the complexity of such problems, various convex relaxation methods based on linear matrix inequality (LMI), semidefinite programming (SDP), and second-order cone programming (SOCP) have gained popularity [12], [13]. These techniques enlarge the possibly non-convex feasible set into a convex set characterizable via convex functions, and then provide the exact or a lower bound on the optimal objective value. The SDP relaxation usually converts an optimization with a vector variable to a convex optimization with a matrix variable, via a lifting technique. The exactness of the relaxation can then be interpreted as the existence of a low-rank (e.g., rank-1) solution for the SDP relaxation. Several papers have studied the existence of a low-rank solution to matrix optimizations with linear or nonlinear (e.g., LMI) constraints. For instance, the papers [14], [15], [16] provide an upper bound on the lowest rank among all solutions of a feasible LMI problem. A rank-1 matrix decomposition technique is developed in [17] to find a rank-1 solution whenever the number of constraints is small. We have shown in [18] and [19] that the SDP relaxation is able to solve a large class of non-convex energy-related optimization problems performed over power networks. We related the success of the relaxation to the hidden structure of those optimizations induced by the physics of a power grid. Inspired by this positive result, we developed the notion of "nonlinear optimization over graph" in [20] and [21]. Our technique maps the structure of an abstract nonlinear optimization into a graph from which the exactness of the SDP relaxation may be concluded. By adopting the graph technique developed in [20] and [21], the objective of the present work is to study the potential of the SDP relaxation for the optimal distributed control problem.

In Part I of the paper, the problem of finite-horizon optimal distributed control (ODC) was investigated. In this part, two problems of infinite-horizon ODC (for deterministic systems) and stochastic ODC (for stochastic systems) will be studied. Following the technique developed in Part I, our approach rests on formulating each of these problems as a rank-constrained optimization from which an SDP relaxation can be derived. With no loss of generality, this part focuses on the design of a static controller. As the first contribution of this part, we show that infinite-horizon ODC and stochastic ODC both admit sparse SDP relaxations with solutions of rank at most 3. Since a rank-1 SDP matrix can be mapped back into a globally-optimal controller, the rank-3 solution may be deployed to retrieve a near-global controller.

Since the proposed relaxations are computationally expensive, we propose two computationally-cheap SDP relaxations associated with infinite-horizon ODC and stochastic ODC. Af-

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terwards, we develop effective heuristic methods to recover a near-optimal controller from the low-rank SDP solution. Note that the computationally-cheap SDP relaxations associated with infinite-horizon ODC and stochastic ODC are both exact for the classical (centralized) LQR and H_2 problems. This implies that the relaxations indirectly solve Riccati equations in the extreme case where the controller under design is unstructured. In this work, we conduct thousands of simulations on a mass-spring system, an electrical power network, and 100 random systems to elucidate the efficacy of the proposed relaxations. In particular, the design of numerous near-optimal structured controllers with global optimality degrees above 99% will be demonstrated.

This paper is organized as follows. The infinite-horizon ODC problem is studied in Section II. The results are generalized to a stochastic ODC problem in Section III. Various experiments and simulations for two case studies are provided in Section IV-B. Concluding remarks are drawn in Section V.

Notations: $\mathbb{R}$ and $\mathbb{S}^n$ denote the sets of real numbers and $n\times n$ symmetric matrices, respectively. $\operatorname{rank}\{W\}$ and $\operatorname{trace}\{W\}$ denote the rank and trace of a matrix W. The notation $W \succeq 0$ means that W is symmetric and positive semidefinite. Given a matrix W, its (l,m) entry is denoted as W_{lm}. Given a block matrix $\mathbf{W}$, its (l,m) block is shown as $\mathbf{W}_{lm}$. The superscript $(\cdot)^{\text{opt}}$ is used to show the globally optimal value of an optimization parameter. The symbols $(\cdot)^T$ and $\|\cdot\|$ denote the transpose and 2-norm operators, respectively. The notation $|x|$ shows the size of a vector x. The expected value of a random variable x is shown as $\mathcal{E}\{x\}$.

II. Deterministic Control Systems

We study the optimal distributed control problem for deterministic systems in this section and then generalize our results to stochastic systems in the next section. Consider the discrete-time system

$$\begin{cases} x[\tau+1] = Ax[\tau] + Bu[\tau] \\ \quad\; y[\tau] = Cx[\tau] \end{cases} \qquad \tau = 0,1,2,... \tag{1}$$

with the known matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{r\times n}$, and $x[0] \in \mathbb{R}^n$. With no loss of generality, assume that C has full row rank. The goal is to design a distributed controller minimizing a quadratic cost function. Similar to Part I, we focus on the static case where the objective is to design a static controller of the form $u[\tau] = Ky[\tau]$ under the constraint that the controller gain K must belong to a given linear subspace $\mathcal{K} \subseteq \mathbb{R}^{m\times r}$. The set $\mathcal{K}$ captures the sparsity structure of the unknown constrained controller $u[\tau] = Ky[\tau]$ and, more specifically, it contains all $m \times r$ real-valued matrices with forced zeros in certain entries. This problem will be formalized below.

Optimal Distributed Control (ODC) problem: Design a stabilizing static controller $u[\tau] = Ky[\tau]$ to minimize the cost function

$$\sum_{\tau=0}^{p} \left(x[\tau]^T Q x[\tau] + u[\tau]^T R u[\tau] \right) + \alpha \operatorname{trace}\{KK^T\} \tag{2}$$

subject to the system dynamics (1) and the controller requirement $K \in \mathcal{K}$, for a terminal time p, a nonnegative scalar α, and positive-definite matrices Q and R.

Remark 1. *The third term in the objective function of the ODC problem is a soft penalty term aimed at avoiding a high-gain controller. Instead of this soft penalty, we could impose a hard constraint* $\operatorname{trace}\{KK^T\} \leq \beta$*, for a given number* β*. The method to be developed later can readily be adopted for the modified case.*

Part I of the paper tackled the finite-horizon ODC problem, where p was a finite number. In this section, we deal with the **infinite-horizon ODC** problem, corresponding to the case $p = +\infty$. This problem will be studied based on the following steps:

- First, the infinite-horizon ODC problem is cast as an optimization with linear matrix inequality constraints as well as quadratic constraints.
- Second, the resulting non-convex problem is formulated as a rank-constrained optimization.
- Third, an SDP relaxation of the problem is derived by dropping the non-convex rank constraint.
- Last, the rank of the minimum-rank solution of the SDP relaxation is analyzed.

A. Lyapunov Formulation

The finite-horizon ODC has been investigated in Part I of the paper through a time-domain formulation. However, to deal with the infinite dimension of the infinite-horizon ODC and its hard stability constraint, a Lyapunov approach will be taken below.

Theorem 1. *The infinite-horizon ODC problem is equivalent to finding a controller* $K \in \mathcal{K}$*, a symmetric Lyapunov matrix* $P \in \mathbb{S}^n$*, an auxiliary symmetric matrix* $G \in \mathbb{S}^n$ *and an auxiliary matrix* $L \in \mathbb{R}^{n\times r}$ *to satisfy the following optimization problem:*

$$\min_{K,L,P,G} \; x[0]^T P x[0] + \alpha \operatorname{trace}\{KK^T\} \tag{3a}$$

subject to:

$$\begin{bmatrix} G & G & (AG+BL)^T & L^T \\ G & Q^{-1} & 0 & 0 \\ AG+BL & 0 & G & 0 \\ L & 0 & 0 & R^{-1} \end{bmatrix} \succeq 0, \tag{3b}$$

$$\begin{bmatrix} P & I \\ I & G \end{bmatrix} \succeq 0, \tag{3c}$$

$$L = KCG \tag{3d}$$

Proof. Given an arbitrary control gain K, consider the system (1) under the controller $u[\tau] = Ky[\tau]$. It is evident that

$$x[\tau] = (A+BKC)^\tau x[0], \qquad \tau = 0,1,...,\infty \tag{4}$$

Hence, the cost function (2) can be written as:

$$\sum_{\tau=0}^{\infty} \left(x[\tau]^T Q x[\tau] + u[\tau]^T R u[\tau] \right) + \alpha \operatorname{trace}\{KK^T\} = \\ = x[0]^T P x[0] + \alpha \operatorname{trace}\{KK^T\} \tag{5}$$

where

$$P=\sum_{\tau=0}^{\infty}((A+BKC)^{\tau})^{T}(Q+C^{T}K^{T}RKC)(A+BKC)^{\tau} \tag{6}$$

or equivalently

$$(A+BKC)^{T}P(A+BKC)-P+Q+(KC)^{T}R(KC)=0 \tag{7a}$$

$$P \succeq 0 \tag{7b}$$

On the other hand, it is well-known that replacing the equality sign "=" in (7a) with the inequality sign "$\preceq$" does not affect the solution of the optimization problem [13]. After pre- and post-multiplying the Lyapunov inequality obtained from (7a) with P^{-1} and using the Schur complement formula, the constraints (7a) and (7b) can be combined as

$$\begin{bmatrix} P^{-1} & P^{-1} & S^{T} & P^{-1}(KC)^{T} \\ P^{-1} & Q^{-1} & 0 & 0 \\ S & 0 & P^{-1} & 0 \\ (KC)P^{-1} & 0 & 0 & R^{-1} \end{bmatrix} \succeq 0 \tag{8}$$

where $S=(A+BKC)P^{-1}$ and 0's in the above matrix are zero matrices of appropriate dimensions. By replacing P^{-1} with a new variable G in the above matrix and defining L as KCG, the constraints (3b) and (3d) will be obtained. The minimization of $x[0]^{T}Px[0]$ subject to the constraint (3c) ensures that $P=G^{-1}$ is satisfied for at least one optimal solution of the optimization problem. ☐

Theorem 2. *Consider the special case where $C=I$, $\alpha=0$ and $\mathcal{K}$ contains the set of all unstructured controllers. Then, the infinite-horizon ODC problem has the same solution as the convex optimization problem obtained from the nonlinear optimization* (3) *by removing its non-convex constraint* (3d).

Proof. It is easy to verify that a solution ($K^{\text{opt}}, P^{\text{opt}}, G^{\text{opt}}$, L^{opt}) of the convex problem stated in the theorem can be mapped to the solution $(L^{\text{opt}}(G^{\text{opt}})^{-1}, P^{\text{opt}}, G^{\text{opt}}, L^{\text{opt}})$ of the non-convex problem (3) and vice versa (recall that $C=I$ by assumption). This completes the proof. ☐

B. SDP Relaxation

Theorem 2 states that a classical optimal control problem can be precisely solved via a convex relaxation of the nonlinear optimization (3) by eliminating its constraint (3d). However, this simple convex relaxation does not work satisfactorily for a general control structure $\mathcal{K}$. To design a better relaxation, define

$$w:=\begin{bmatrix} 1 & h^{T} & \text{vec}\{CG\}^{T} \end{bmatrix}^{T} \tag{9}$$

where h is a column vector containing the variables (free parameters) of K, and vec$\{CG\}$ is a column vector containing all scalar entries of CG. It is possible to write every entry of the bilinear matrix term KCG as a linear function of the entries of the parametric matrix ww^{T}. Hence, by introducing a new matrix variable W playing the role of ww^{T}, the nonlinear constraint (3d) can be rewritten as a linear constraint in term of W. In addition, the term α trace$\{KK^{T}\}$ in the objective function of the ODC problem is also linear in W. Now, one can relax the non-convex mapping constraint $W=ww^{T}$ to $W\succeq 0$ and another constraint stating that the first column of W is equal to w. This convex problem is referred to as **SDP relaxation of ODC** in this work. In the case where the relaxation has the same solution as ODC, the relaxation is said to be exact.

Theorem 3. *Consider the case where $\mathcal{K}$ contains only diagonal matrices. The following statements hold regarding the SDP relaxation of the infinite-horizon ODC problem:*

i) *The relaxation is exact if it has a solution (K^{opt}, P^{opt}, $G^{opt}, L^{opt}, W^{opt}$) such that* rank$\{W^{opt}\}=1$.
ii) *The relaxation always has a solution ($K^{opt}, P^{opt}, G^{opt}$, L^{opt}, W^{opt}) such that* rank$\{W^{opt}\}\leq 3$.

Proof. To study the SDP relaxation of the aforementioned control problem, we need to define a sparsity graph $\mathcal{G}$. Let η denote the number of rows of W. The graph $\mathcal{G}$ has η vertices with the property that two arbitrary disparate vertices $i,j\in\{1,2,...,\eta\}$ are connected in the graph if W_{ij} appears in at least one of the constraints of the SDP relaxation excluding the global constraint $W\succeq 0$. For example, vertex 1 is connected to all remaining vertices of the graph. The graph $\mathcal{G}$ with its vertex 1 removed is depicted in Figure 1. This graph is acyclic and therefore the treewidth of the graph $\mathcal{G}$ is at most 2. Hence, It follows from Theorem 1 provided in Part I of the paper that the SDP relaxation has a matrix solution with rank at most 2+1. ☐

Theorem 3 states that the SDP relaxation of the infinite-horizon ODC problem has a low-rank solution. However, it does not imply that every solution of the relaxation is low-rank. Theorem 1 developed in Part I provides a procedure for converting a high-rank solution of the SDP relaxation into a matrix solution with rank at most 3. The above theorem will be generalized below.

Proposition 1. *The infinite-horizon ODC problem has a convex relaxation with the property that its exactness amounts to the existence of a rank-1 matrix solution W^{opt}. Moreover, it is always guaranteed that this relaxation has a solution such that* rank$\{W^{opt}\}\leq 3$.

Proof. The procedure of designing an SDP relaxation with a guaranteed low-rank solution is spelled out for the time domain formulation in Part I of the paper. The idea will be only sketched here. As explained in the Part I paper, there are two binary matrices Φ_1 and Φ_2 such that $K=\Phi_1\text{diag}\{k\}\Phi_2$ for every $K\in\mathcal{K}$, where diag$\{k\}$ denotes a diagonal matrix whose diagonal contains the free (variable) entries of K. Hence, the design of a structured control gain K for the system (A,B,C) amounts to the design of a diagonal control gain diag$\{k\}$ for the system $(A,B\Phi_1,\Phi_2C)$ (after updating the matrices Q and R accordingly). It follows from Theorem 3 that the SDP relaxation of the ODC problem equivalently formulated for the new system satisfies the properties of this theorem. ☐

In this section, it has been shown that the infinite-horizon ODC problem has an SDP relaxation with a low-rank solution. Nevertheless, there are many SDP relaxations with this

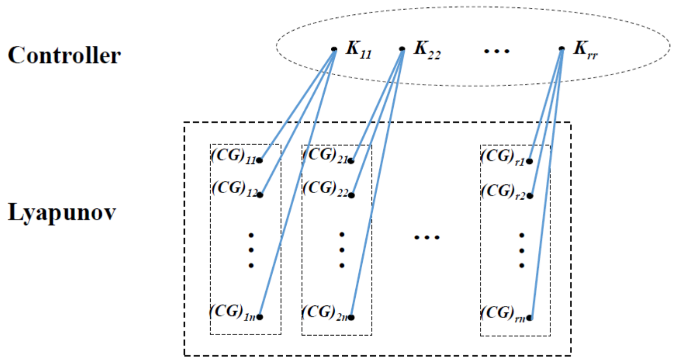


Fig. 1: The sparsity graph for the infinite-horizon ODC problem in the case where $\mathcal{K}$ consists of diagonal matrices (the central vertex 1 is removed for simplicity).

property and it is desirable to find the one offering the highest lower bound on the optimal solution of the ODC problem. To this end, the abovementioned SDP relaxation should be reformulated in such a way that the diagonal entries of the matrix W are incorporated into as many constraints of the problem as possible in order to indirectly penalize the rank of the matrix W. This idea will be flourished next, but for a computationally-cheap relaxation of the ODC problem.

C. Computationally-Cheap SDP Relaxation

The aforementioned SDP relaxation has a high dimension for a large-scale system, which makes it less interesting for computational purposes. Moreover, the quality of its optimal objective value can be improved using some indirect penalty technique. The objective of this subsection is to offer a computationally-cheap SDP relaxation for the ODC problem, whose solution outperforms that of the previous SDP relaxation. For this purpose, Consider an invertible matrix Φ such that

$$C\Phi = \begin{bmatrix} I & 0 \end{bmatrix} \tag{10}$$

where I the is identity matrix and "0" is an $r \times (n-r)$ zero matrix. Define also

$$\mathcal{K}^2 = \{KK^T \mid K \in \mathcal{K}\} \tag{11}$$

Indeed, $\mathcal{K}^2$ captures the sparsity pattern of the matrix KK^T. For example, if $\mathcal{K}$ consists of block-diagonal (rectangular) matrix, $\mathcal{K}^2$ will also include block-diagonal (square) matrices. Let $\mu \in \mathbb{R}$ be a positive number such that

$$Q \succ \mu \times \Phi^{-T}\Phi^{-1} \tag{12}$$

where Φ^{-T} denotes the transpose of the inverse of Φ. Define $\widehat{Q} := Q - \mu \times \Phi^{-T}\Phi^{-1}$.

Computationally-Cheap SDP Relaxation of ODC: This optimization problem is defined as the minimization of

$$\text{trace}\{x[0]^T P x[0] + \alpha \mathbf{W}_{33}\} \tag{13}$$

subject to the constraints

$$\begin{bmatrix} G-\mu\mathbf{W}_{22} & G & (AG+BL)^T & L^T \\ G & \widehat{Q}^{-1} & 0 & 0 \\ AG+BL & 0 & G & 0 \\ L & 0 & 0 & R^{-1} \end{bmatrix} \succeq 0, \tag{14a}$$

$$\begin{bmatrix} P & I \\ I & G \end{bmatrix} \succeq 0, \tag{14b}$$

$$\mathbf{W} := \left[\begin{array}{c|c|c} I_n & \Phi^{-1}G & \begin{bmatrix} K^T \\ 0 \end{bmatrix} \\ \hline G\Phi^{-T} & \mathbf{W}_{22} & L^T \\ \hline \begin{bmatrix} K & 0 \end{bmatrix} & L & \mathbf{W}_{33} \end{array}\right] \succeq 0, \tag{14c}$$

$$K \in \mathcal{K}, \tag{14d}$$

$$\mathbf{W}_{33} \in \mathcal{K}^2, \tag{14e}$$

with the parameter set $\{K, L, G, P, \mathbf{W}\}$, where the dependent variables $\mathbf{W}_{22}$ and $\mathbf{W}_{33}$ represent two blocks of $\mathbf{W}$.

The following remarks can be made regarding the computationally-cheap SDP relaxation:

- The constraint (14a) corresponds to the Lyapunov inequality associated with (7a), where $\mathbf{W}_{22}$ in its first block aims to play the role of $P^{-1}\Phi^{-T}\Phi^{-1}P^{-1}$.
- The constraint (14b) ensures that the relation $P = G^{-1}$ occurs at optimality (at least for one of the solution of the problem).
- The constraint (14c) is a surrogate for the only complicating constraint of the ODC problem, i.e., $L = KCG$.
- Since no non-convex rank constraint is imposed on the problem to maintain the convexity of the relaxation, the rank constraint is compensated in various ways. More precisely, the entries of $\mathbf{W}$ are constrained in the objective function (13) through the term trace$\{\alpha\mathbf{W}_{33}\}$, in the first block of the constraint (14a) through the term $G - \mu\mathbf{W}_{22}$, and also via the constraints (14d) and (14e). These terms aim to automatically penalize the rank of $\mathbf{W}$ indirectly.
- The proposed relaxation takes advantage of the sparsity of not only K, but also KK^T (through the constraint (14e)).

Theorem 4. *The computationally-cheap SDP relaxation is a convex relaxation of the infinite-horizon ODC problem. Furthermore, the relaxation is exact if and only if it possesses a solution $(K^{opt}, L^{opt}, P^{opt}, G^{opt}, \mathbf{W}^{opt})$ such that rank$\{\mathbf{W}^{opt}\} = n$.*

Proof. The objective function and constraints of the computationally-cheap SDP relaxation are all linear functions of the tuple $(K, L, P, G, \mathbf{W})$. Hence, this relaxation is indeed convex. To study the relationship between this optimization problem and the infinite-horizon ODC, consider a feasible point (K, L, P, G) of the ODC formulation (3). It can be deduced from the relation $L = KCG$ that $(K, L, P, G, \mathbf{W})$ is a feasible solution of the computationally-cheap SDP relaxation if the free blocks of $\mathbf{W}$ are considered as

$$\mathbf{W}_{22} = G\Phi^{-T}\Phi^{-1}G, \qquad \mathbf{W}_{33} = KK^T \tag{15}$$

(note that (3b) and (14a) are equivalent for this choice of

$\mathbf{W}$). This implies that computationally-cheap SDP problem is a convex relaxation of the infinite-horizon ODC problem.

Consider now a solution $(K^{\text{opt}}, L^{\text{opt}}, P^{\text{opt}}, G^{\text{opt}}, W^{\text{opt}})$ of the computationally-cheap SDP relaxation such that $\text{rank}\{\mathbf{W}^{\text{opt}}\} = n$. Since the rank of the first block of $\mathbf{W}^{\text{opt}}$ (i.e., I_n) is already n, a Schur complement argument on the blocks $(1,1)$, $(1,3)$, $(2,1)$ and $(2,3)$ of $\mathbf{W}^{\text{opt}}$ yields that

$$0 = L^{\text{opt}} - \begin{bmatrix} K^{\text{opt}} & 0 \end{bmatrix} (I_n)^{-1} \Phi^{-1} G^{\text{opt}} \tag{16}$$

or equivalently $L^{\text{opt}} = K^{\text{opt}} C G^{\text{opt}}$, which is tantamount to the constraint (3d). This implies that $(K^{\text{opt}}, L^{\text{opt}}, P^{\text{opt}}, G^{\text{opt}})$ is a solution of the ODC problem and hence the relaxation is exact. So far, we have shown that the existence of a rank-n solution $\mathbf{W}^{\text{opt}}$ guarantees the exactness of the relaxation. The converse of this statement can also be proved similarly. $\square$

The matrix variable W in the first SDP relaxation of the infinite-horizon ODC problem had $O(n^2)$ rows. In contrast, this number reduces to $O(n)$ for the matrix $\mathbf{W}$ in the computationally-cheap SDP relaxation, which significantly reduces the computation time of the relaxation.

Corollary 1. *Consider the special case where $C = I$, $\alpha = 0$ and $\mathcal{K}$ contains the set of all unstructured controllers. Then, the computationally-cheap SDP relaxation is exact for the infinite-horizon ODC problem.*

Proof. The proof follows from that of Theorem 2. $\square$

D. Controller Recovery

Once the computationally-cheap SDP relaxation is solved, a controller K must be recovered. This can be achieved in two ways as explained below.

Direct Recovery Method for ODC: A near-optimal controller $\hat{K}$ for the infinite-horizon ODC problem is chosen to be equal to the optimal matrix K^{opt} obtained from the computationally-cheap SDP relaxation.

Indirect Recovery Method for ODC: Let $(K^{\text{opt}}, L^{\text{opt}}, P^{\text{opt}}, G^{\text{opt}}, \mathbf{W}^{\text{opt}})$ denote a solution of the computationally-cheap SDP relaxation. A near-optimal controller $\hat{K}$ for the infinite-horizon ODC problem is recovered by solving a convex program with the variables $K \in \mathcal{K}$ and $\gamma \in \mathbb{R}$ to minimize the cost function

$$\varepsilon \times \gamma + \alpha \,\text{trace}\{KK^T\} \tag{17}$$

subject to the constraint

$$\begin{bmatrix} (G^{\text{opt}})^{-1} - Q + \gamma I_n & (A+BKC)^T & (KC)^T \\ (A+BKC) & G^{\text{opt}} & 0 \\ (KC) & 0 & R^{-1} \end{bmatrix} \succ 0 \tag{18}$$

where ε is a pre-specified nonnegative number.

The direct recovery method assumes that the controller K^{opt} obtained from the computationally-cheap SDP relaxation is near-optimal, whereas the indirect method assumes that the controller K^{opt} might be unacceptably imprecise while the inverse of the Lyapunov matrix is near-optimal. The indirect method is built on the SDP relaxation by fixing G at its optimal value and then perturbing Q as $Q - \gamma I_n$ to facilitate the recovery of a stabilizing controller. It may rarely happen that a stabilizing controller can be recovered from a solution G^{opt} if γ is set to zero. In other words, since the solution of the computationally-cheap SDP relaxation is not exact in general, there may not exist any controller $\hat{K}$ satisfying the Lyapunov equation jointly with G^{opt}. Nonetheless, perturbing the diagonal entries of Q with γ boosts the degree of the freedom of the problem and helps with the existence of a controller $\hat{K}$. Although none of the proposed recovery methods is universally better than the other one, we have verified in numerous simulations that the indirect recovery method significantly outperforms the direct recovery method with a high probability.

III. Stochastic Control Systems

The ODC problem was investigated for a deterministic system in the preceding section. The objective of this section is to generalize the results derived earlier to stochastic systems. To this end, consider the discrete-time system

$$\begin{cases} x[\tau+1] = Ax[\tau] + Bu[\tau] + Ed[\tau] \\ \quad y[\tau] = Cx[\tau] + Fv[\tau] \end{cases} \qquad \tau = 0, 1, 2, \ldots \tag{19}$$

with the known matrices A, B, C, E, and F, where

- $x[\tau] \in \mathbb{R}^n$, $u[\tau] \in \mathbb{R}^m$ and $y[\tau] \in \mathbb{R}^r$ denote the state, input and output of the system.
- $d[\tau]$ and $v[\tau]$ denote the input disturbance and measurement noise, which are assumed to be zero-mean white-noise random processes.

The goal is to design an optimal distributed controller. In order to simplify the presentation, we focus on the static case where the objective is to design a static controller of the form $u[\tau] = Ky[\tau]$ under the structural constraint $K \in \mathcal{K}$. This section of this paper is mainly concerned with the following problem.

Stochastic Optimal Distributed Control (SODC) problem: Design a stabilizing static controller $u[\tau] = Ky[\tau]$ to minimize the cost function

$$\lim_{\tau \to +\infty} \mathcal{E}\left(x[\tau]^T Q x[\tau] + u[\tau]^T R u[\tau] \right) + \alpha \,\text{trace}\{KK^T\} \tag{20}$$

subject to the system dynamics (19) and the controller requirement $K \in \mathcal{K}$, for a nonnegative scalar α and positive-definite matrices Q and R.

Define two covariance matrices as below:

$$\Sigma_d = \mathcal{E}\{Ed[0]d[0]^T E^T\}, \quad \Sigma_v = \mathcal{E}\{Fv[0]v[0]^T F^T\} \tag{21}$$

In what follows, the SODC problem will be formulated as a nonlinear optimization program.

Theorem 5. *The SODC problem is equivalent to finding a controller $K \in \mathcal{K}$, a symmetric Lyapunov matrix $P \in \mathbb{S}^n$, and auxiliary matrices $G \in \mathbb{S}^n$, $L \in \mathbb{R}^{n \times r}$ and $M \in \mathbb{S}^r$ to minimize the objective function*

$$\text{trace}\{P\Sigma_d + M\Sigma_v + K^T RK\Sigma_v\} + \alpha \,\text{trace}\{KK^T\} \tag{22}$$

subject to the constraints

$$\begin{bmatrix} G & G & (AG+BL)^T & L^T \\ G & Q^{-1} & 0 & 0 \\ AG+BL & 0 & G & 0 \\ L & 0 & 0 & R^{-1} \end{bmatrix} \succeq 0, \tag{23a}$$

$$\begin{bmatrix} P & I \\ I & G \end{bmatrix} \succeq 0, \tag{23b}$$

$$\begin{bmatrix} M & (BK)^T \\ BK & G \end{bmatrix} \succeq 0, \tag{23c}$$

$$L = KCG \tag{23d}$$

Proof. It is straightforward to verify that

$$\begin{aligned} x[\tau] = &(A+BKC)^\tau x[0] \\ &+ \sum_{t=0}^{\tau-1} (A+BKC)^t Ed[\tau-t-1] \\ &+ \sum_{t=0}^{\tau-1} (A+BKC)^t BKFv[\tau-t-1] \end{aligned} \tag{24}$$

for $\tau = 1, 2, \ldots$. On the other hand, since the controller under design must be stabilizing, $(A+BKC)^\tau$ approaches zero as τ goes to $+\infty$. In light of the above equation, it can be verified that

$$\begin{aligned} &\mathcal{E}\left\{ \lim_{\tau\to+\infty} \Big(x[\tau]^T Q x[\tau] + u[\tau]^T R u[\tau] \Big) + \alpha \operatorname{trace}\{KK^T\} \right\} = \\ &= \mathcal{E}\left\{ \lim_{\tau\to+\infty} x[\tau]^T \Big(Q + C^T K^T R K C \Big) x[\tau] \right\} \\ &+ \mathcal{E}\left\{ \lim_{\tau\to+\infty} v[\tau]^T F^T K^T R K F v[\tau] \right\} + \alpha \operatorname{trace}\{KK^T\} \\ &= \operatorname{trace}\{P\Sigma_d + (BK)^T P (BK)\Sigma_v + K^T R K \Sigma_v + \alpha K K^T\} \end{aligned} \tag{25}$$

where

$$P = \sum_{t=0}^{\infty} \big((A+BKC)^t\big)^T (Q + C^T K^T R K C)(A+BKC)^t \tag{26}$$

Similar to the proof of Theorem 1, the above infinite series can be replaced by the following expanded Lyapunov inequality:

$$\begin{bmatrix} P^{-1} & P^{-1} & S^T & P^{-1}(KC)^T \\ P^{-1} & Q^{-1} & 0 & 0 \\ S & 0 & P^{-1} & 0 \\ (KC)P^{-1} & 0 & 0 & R^{-1} \end{bmatrix} \succeq 0 \tag{27}$$

where $S = (A + BKC)P^{-1}$. After replacing P^{-1} and KCP^{-1} with new variables G and L, it can be concluded that:

- The condition (27) is identical to the set of constraints (23a) and (23d).
- The cost function (25) can be expressed as
$$\operatorname{trace}\{P\Sigma_d + (BK)^T G^{-1} (BK)\Sigma_v + K^T R K \Sigma_v + \alpha K K^T\} \tag{28}$$
- Since P appears only once in the constraints of the optimization problem (22)-(23) (i.e., the condition (23b)) and the objective function of this optimization includes the term trace$\{P\Sigma_d\}$, the optimal value of P is equal to G^{-1}.
- Similarly, the optimal value of M is equal to $(BK)^T G^{-1}(BK)$.

The proof follows from the above observations. □

The SODC problem is cast as a (deterministic) nonlinear program in Theorem 5. This optimization problem is non-convex due only to the complicating constraint (23d) . More precisely, the removal of this nonlinear constraint makes the optimization problem a semidefinite program (note that the term $K^T RK$ in the objective function is convex due to the assumption $R \succ 0$).

The traditional H_2 optimal control problem (i.e., in the centralized case) can be solved using Riccati equations. It will be shown in the next proposition that the abovementioned semidefinite program correctly solves the centralized H_2 optimal control problem.

Proposition 2. *Consider the special case where $C = I$, $\alpha = 0$, $\Sigma_v = 0$, and $\mathcal{K}$ contains the set of all unstructured controllers. Then, the SODC problem has the same solution as the convex optimization problem obtained from the nonlinear optimization* (22)-(23) *by removing its non-convex constraint* (23d).

Proof. It is similar to the proof of Theorem 2. □

Proposition 2 states that a classical optimal control problem can be precisely solved via a convex relaxation of the nonlinear optimization (22)-(23) by eliminating its constraint (23d). However, this simple convex relaxation does not work satisfactorily for a general control structure $\mathcal{K}$. To design a better relaxation, consider the vector w defined in (9). Similar to infinite-horizon ODC, the bilinear matrix term KCG can be represented as a linear function of the entries of the parametric matrix $\mathbf{W}$ defined as ww^T. Now, relaxing the constraint $\mathbf{W} = ww^T$ to $\mathbf{W} \succeq 0$ and adding another constraint stating that the first column of $\mathbf{W}$ is equal to w leads to an SDP relaxation. This convex problem is referred to as **SDP relaxation of SODC**. In the case where the relaxation has the same solution as SODC, the relaxation is said to be exact.

Proposition 3. *Consider the case where $\mathcal{K}$ contains only diagonal matrices. The following statements hold regarding the SDP relaxation of the SODC problem:*

i) *The relaxation is exact if it has a solution* $(K^{opt}, P^{opt}, G^{opt}, L^{opt}, M^{opt}, W^{opt})$ *such that* rank$\{W^{opt}\} = 1$.
ii) *The relaxation always has a solution* $(K^{opt}, P^{opt}, G^{opt}, L^{opt}, M^{opt}, W^{opt})$ *such that* rank$\{W^{opt}\} \leq 3$.

Proof. The proof is omitted (see Theorems 3 and 5). □

As before, it can be deduced from Proposition 3 that the SODC problem has a convex relaxation with the property that its exactness amounts to the existence of a rank-1 matrix solution W^{opt}. Moreover, it is always guaranteed that this relaxation has a solution such that rank$\{W^{\text{opt}}\} \leq 3$.

A computationally-cheap SDP relaxation will be derived below. Let μ_1 and μ_2 be two nonnegative numbers such that

$$Q \succ \mu_1 \times \Phi^{-T}\Phi^{-1}, \quad \Sigma_v \succeq \mu_2 \times I \tag{29}$$

Define $\widehat{Q} := Q - \mu_1 \times \Phi^{-T}\Phi^{-1}$ and $\widehat{\Sigma}_v := \Sigma_v - \mu_2 \times I$.

Computationally-Cheap SDP Relaxation of SODC: This optimization problem is defined as the minimization of

$$\text{trace}\{P\Sigma_d + M\Sigma_v + \mu_2 R\mathbf{W}_{33} + \alpha\mathbf{W}_{33} + K^T RK\widehat{\Sigma}_v\} \quad (30)$$

subject to the constraints

$$\begin{bmatrix} G-\mu_1\mathbf{W}_{22} & G & (AG+BL)^T & L^T \\ G & \widehat{Q}^{-1} & 0 & 0 \\ AG+BL & 0 & G & 0 \\ L & 0 & 0 & R^{-1} \end{bmatrix} \succeq 0, \quad (31a)$$

$$\begin{bmatrix} P & I \\ I & G \end{bmatrix} \succeq 0, \quad (31b)$$

$$\begin{bmatrix} M & (BK)^T \\ BK & G \end{bmatrix} \succeq 0, \quad (31c)$$

$$\mathbf{W} := \left[\begin{array}{c:c:c} I_n & \Phi^{-1}G & \begin{bmatrix} K^T \\ 0 \end{bmatrix} \\ \hdashline G\Phi^{-T} & \mathbf{W}_{22} & L^T \\ \hdashline \begin{bmatrix} K & 0 \end{bmatrix} & L & \mathbf{W}_{33} \end{array}\right] \succeq 0, \quad (31d)$$

$$K \in \mathcal{K}, \quad (31e)$$

$$\mathbf{W}_{33} \in \mathcal{K}^2, \quad (31f)$$

with the parameter set $\{K, L, G, P, M, \mathbf{W}\}$.

It should be noted that the constraint (31c) ensures that the relation $M = (BK)^T G^{-1}(BK)$ occurs at optimality.

Theorem 6. *The computationally-cheap SDP relaxation is a convex relaxation of the SODC problem. Furthermore, the relaxation is exact if and only if possesses a solution* $(K^{opt}, L^{opt}, P^{opt}, G^{opt}, M^{opt}, \mathbf{W}^{opt})$ *such that rank*$\{\mathbf{W}^{opt}\} = n$.

Proof. Since the proof is similar to that of the infinite-horizon case presented earlier, it is omitted here. □

For the retrieval of a near-optimal controller, the Direct Recovery Method delineated for the infinite-horizon ODC problem can be readily deployed. However, the Indirect Recovery Method explained earlier should be modified.

Indirect Recovery Method for SODC: Let $(K^{\text{opt}}, L^{\text{opt}}, P^{\text{opt}}, G^{\text{opt}}, M^{\text{opt}}, \mathbf{W}^{\text{opt}})$ denote a solution of the computationally-cheap SDP relaxation of SODC. A near-optimal controller $\hat{K}$ for the SODC problem is recovered by solving a convex program with the variables $K \in \mathcal{K}$ and $\gamma \in \mathbb{R}$ to minimize the cost function

$$\varepsilon\times\gamma + \text{trace}\{(BK)^T(G^{\text{opt}})^{-1}(BK)\Sigma_v + K^T RK\Sigma_v + \alpha\; KK^T\} \quad (32)$$

subject to the constraint

$$\begin{bmatrix} (G^{\text{opt}})^{-1} - Q + \gamma I_n & (A+BKC)^T & (KC)^T \\ (A+BKC) & G^{\text{opt}} & 0 \\ (KC) & 0 & R^{-1} \end{bmatrix} \succ 0 \quad (33)$$

where ε is a pre-specified nonnegative number.

The above recovery method is obtained by assuming that G^{opt} is the optimal value of the inverse Lyapunov matrix for the ODC problem.

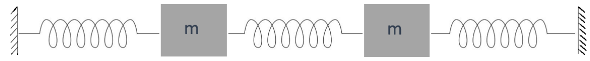


Fig. 2: Mass-spring system with two masses

IV. Case Studies

In this section, we elucidate the results of this two-part paper on a mass-spring system, an electrical power network, and 100 random system. We will solve thousands of SDP relaxations for these systems and evaluate their performance for different control topologies and a wide range of values for $(\alpha, \Sigma_d, \Sigma_v)$. Note that the computation time for each SDP relaxation is from a fraction of a second to 4 seconds on a desktop computer with an Intel Core i7 quad-core 3.4 GHz CPU and 16 GB RAM.

A. Case Study 1: Mass-Spring Systems

In this subsection, the aim is to evaluate the performance of the developed controller design techniques on the *Mass-Spring* system, as a classical physical system. Consider a mass-spring system consisting of N masses. This system is exemplified in Figure 2 for $N = 2$. The system can be modeled in the continuous-time domain as

$$\dot{x}_c(t) = A_c x_c(t) + B_c u_c(t) \quad (34)$$

where the state vector $x_c(t)$ can be partitioned as $[o_1(t)^T \;\; o_2(t)^T]$ with $o_1(t) \in \mathbb{R}^n$ equal to the vector of positions and $o_2(t) \in \mathbb{R}^n$ equal to the vector of velocities of the N masses. We assume that $N = 10$ and adopt the values of A_c and B_c from [22]. The goal is to design a static sampled-data controller with a pre-specified structure (i.e., the controller is composed of a sampler, a static discrete-time structured controller and a zero-order holder). Three ODC problems will be solved below.

Finite-Horizon ODC: In this experiment, we first discretize the system with the sampling time of 0.4 second and denote the obtained system as

$$x[\tau+1] = Ax[\tau] + Bu[\tau], \qquad \tau = 0, 1, \ldots \quad (35)$$

It is aimed to design a constrained controller $u[\tau] = Kx[\tau]$ to minimize the cost function

$$\sum_{\tau=0}^{p} \left(x[\tau]^T x[\tau] + u[\tau]^T u[\tau]\right) \quad (36)$$

with $x[0]$ equal to the vector of 1's. We solve an SDP relaxation for the six different control structures shown in Figure 3. The free parameters of each controller are colored in red in this figure. For example, Structure (c) corresponds to a fully decentralized controller, where each local controller has access to the position and velocity of its associated mass. In contrast, Structure (e) allows only five of the masses to be controlled. Similarly, Structure (a) implies limited communications between neighboring local controllers, whereas Structure (d) enables some communications between the local control of Mass 1 and the remaining local controllers. For each structure, the SDP relaxation of Problem D-2 is solved for four different terminal times $p = 5,\ 10,\ 15$ and 20 (please refer to

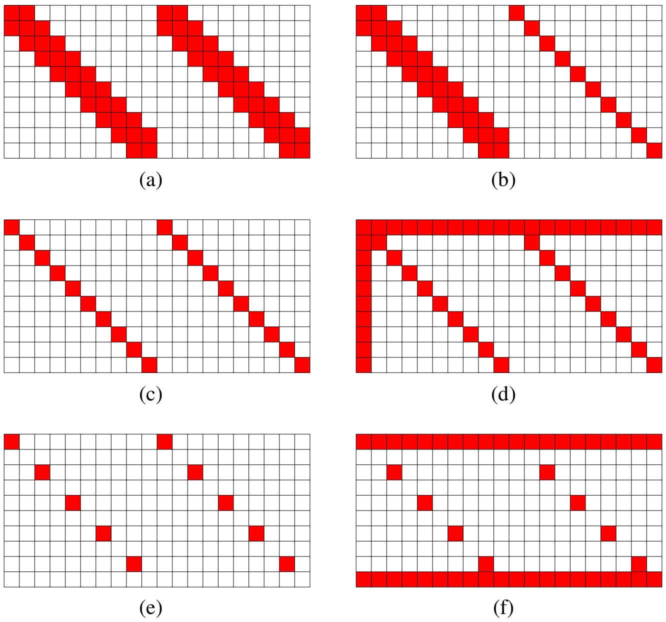


Fig. 3: Six different structures for the controller K: the free parameters are colored in red (uncolored entries are set to zero).

K	bounds	$p=5$	$p=10$	$p=15$	$p=30$
(a)	upper bound	126.752	140.105	140.681	140.691
	lower bound	126.713	140.080	140.660	140.690
	infinite-horizon perf.	∞	∞	∞	140.691
	stability	unstable	unstable	unstable	stable
(b)	upper bound	126.809	140.183	140.685	140.702
	lower bound	126.713	140.080	140.661	140.690
	infinite-horizon perf.	∞	∞	140.770	140.702
	stability	unstable	unstable	stable	stable
(c)	upper bound	127.916	140.762	140.792	140.795
	lower bound	126.713	140.080	140.660	140.690
	infinite-horizon perf.	150.972	140.992	140.796	140.795
	stability	stable	stable	stable	stable
(d)	upper bound	127.430	140.761	140.762	140.761
	lower bound	126.713	140.080	140.661	140.690
	infinite-horizon perf.	159.633	141.020	140.766	140.761
	stability	stable	stable	stable	stable
(e)	upper bound	175.560	235.240	240.189	242.973
	lower bound	167.220	215.202	222.793	226.797
	infinite-horizon perf.	277.690	282.580	271.675	267.333
	stability	stable	stable	stable	stable
(f)	upper bound	175.401	230.210	231.022	230.382
	lower bound	164.114	208.484	214.723	216.431
	infinite-horizon perf.	357.197	287.767	242.976	232.069
	stability	stable	unstable	stable	stable

TABLE I: The outcome of the SDP relaxation of Problem D-2 for the 6 different control structures given in Figure 3.

Part I of the paper for more details about the SDP relaxations of the finite-horizon ODC problem). The results are tabulated in Table I. Four metrics are reported for each structure and terminal time:

- *Lower bound:* This number is equal to the optimal objective value of the SDP relaxation, which serves as a lower bound on the minimum value of the cost function (36).
- *Upper bound:* This number corresponds to the cost function (36) at a near-optimal controller $\hat{K}$ retrieved using the Direct Recovery Method. This number serves as an upper bound on the minimum value of the cost function (36).
- *Infinite-horizon performance:* This is equal to the infinite sum $\sum_{\tau=0}^{\infty}\left(x[\tau]^T x[\tau] + u[\tau]^T u[\tau]\right)$ associated with the system (35) under the designed near-optimal controller.
- *Stability:* This indicates the stability or instability of the closed-loop system.

Note that since a stability constraint was not imposed on the aforementioned finite-horizon control problem, the stability was not guaranteed. However, it can be observed that the designed controller is always stabilizing for $p = 20$. As demonstrated in Table I, the upper and lower bounds are very close to each other in many scenarios, in which cases the recovered controllers are almost globally optimal. It can also be observed that there is a non-negligible gap between the lower and upper bounds for Structures (e) and (f), implying that the design of a controller with any of these structures may be computationally hard. Note that a powerful sparsity promoting technique is proposed in [22], which is able to design a controller of Structure (a) or (c) for $p = \infty$ but cannot handle the other structures or a finite terminal time.

Infinite-Horizon ODC: To study the effects of the initial state on the designed near-optimal controller, we generated 100 random initial states with entries drawn from a normal distribution. We then solved the computationally-cheap SDP relaxation combined with the Direct Recovery Method to design a controller of Structure (c) minimizing the cost function (36). In this experiment, the sampling time is considered as 0.1 second. The values of controllers' parameters are depicted in Figure 4, where the 20 points on the x-axis represent 20 different entries of the designed decentralized controller. As can be seen, the parameters of the controller vary over the 100 trials. This contrasts with the fact that the optimal controller associated with a centralized (classical) LQR problem is universally optimal and its parameters are independent of the initial state. Define a measure of near-global optimality as follows:

$$\text{Optimality degree (\%)} = 100 - \frac{\text{upper bound - lower bound}}{\text{upper bound}} \times 100$$

The optimality degrees of the controllers designed for these 100 random trials are depicted in Figure 5. As can be seen, the optimality degree is better than 95% for more than 98 trials. It should be mentioned that all of these controllers stabilize the system.

Stochastic ODC: In this experiment, two control structures of "decentralized" and "distributed" (shown in Figures 3(c) and (a)) will be studied for the matrix $K \in \mathbb{R}^{10\times 20}$. We assume that the system is subject to both input disturbance and measurement noise. Consider the case $\Sigma_d = I$ and $\Sigma_v = \sigma I$, where σ varies from 0 to 5. Using the computationally-cheap SDP relaxation in conjunction with the indirect recovery method, a near-optimal controller is designed for each of the aforementioned control structures under various noise levels. The results are reported in Figure 6. The structured controllers designed using the SDP relaxation are all stable

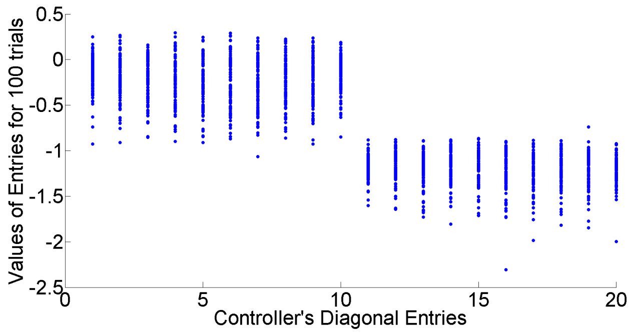


Fig. 4: The near-optimal values of the free parameters of the decentralized controller $\hat{K}$ for a mass-spring system under 100 random initial states. Corresponding to each free parameter $i \in \{1, 2, ..., 20\}$, the 100 values of this parameter (associated with different trials) are shown as 100 points on a vertical line.

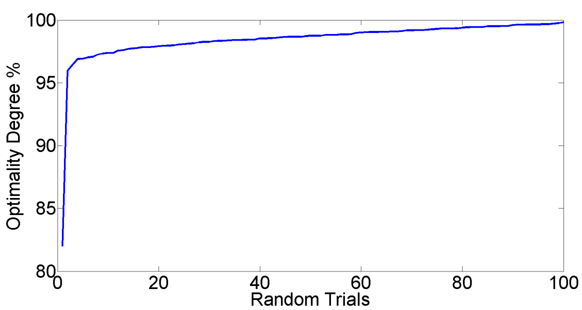


Fig. 5: Optimality degree (%) of the decentralized controller $\hat{K}$ for a mass-spring system under 100 random initial states.

with optimality degrees higher than 95% in the worst case and close to 99% in many cases.

B. Case Study II: Frequency Control in Power Systems

In this subsection, the performance of the computationally-cheap SDP relaxation combined with the indirect recovery method will be evaluated on the problem of designing an optimal distributed frequency control for IEEE 39-Bus New England Power System. The one-line diagram of this system is shown in Figure 7. The main objective of the unknown controller is to optimally adjust the mechanical power input to each generator as well as being structurally constrained by a user-defined communication topology. This pre-determined communication topology specifies which generators exchange their rotor angle and frequency measurements with one another.

In this example, we stick with a simple classical model of the power system. However, our result can be deployed for a complicated high-order model with nonlinear terms (our SDP relaxation may be revised to handle possible nonlinear terms in the dynamics). To derive a simple state-space model of the power system, we start with the widely-used per-unit swing equation

$$M_i \ddot{\theta}_i + D_i \dot{\theta}_i = P_{Mi} - P_{Ei} \tag{37}$$

where θ_i denotes the voltage (or rotor) angle at bus i (in rad), P_{Mi} is the mechanical power input to the generator at bus i (in per unit), P_{Ei} is the electrical active power injection at bus i (in per unit), M_i is the inertia coefficient of the generator at bus i (in pu-sec^2/rad), and D_i is the damping coefficient of the generator at bus i (in pu-sec/rad) [23]. The electrical real power P_{Ei} in (37) comes from the nonlinear AC power flow equation:

$$P_{Ei} = \sum_{j=1}^{n} |V_i||V_j| \; [\; G_{ij} \; cos(\theta_i - \theta_j) + B_{ij} \; sin(\theta_i - \theta_j) \;] \tag{38}$$

where n denotes the number of buses in the system, V_i is the voltage phasor at bus i, G_{ij} is the line conductance, and B_{ij} is the line susceptance. To simplify the formulation, a commonly-used technique is to approximate equation (38) by its corresponding DC power flow equation stated below:

$$P_{Ei} = \sum_{j=1}^{n} B_{ij}(\theta_i - \theta_j) \tag{39}$$

The approximation error is often small in practice due to the common practice of power engineering, which rests upon the following assumptions:

- For most networks, $G \ll B \longrightarrow G = 0$
- For most neighbouring buses, $|\theta_i - \theta_j| \leq (10^o \text{ to } 15^o)$
  $\longrightarrow sin(\theta_i - \theta_j) \approx \theta_i - \theta_j$
  $\longrightarrow cos(\theta_i - \theta_j) \approx 1$
- In per unit, $|V_i|$ is close to 1 (0.95 to 1.05)
  $\longrightarrow |V_i||V_j| \approx 1$

It is possible to rewrite (39) into the matrix format $P_E = L\theta$, where P_E and θ are the vectors of real power injections and voltage (or rotor) angles at only the generator buses (after removing the load buses and the intermediate zero buses). In this equation, L denotes the Laplacian matrix and can be found as follows [24]:

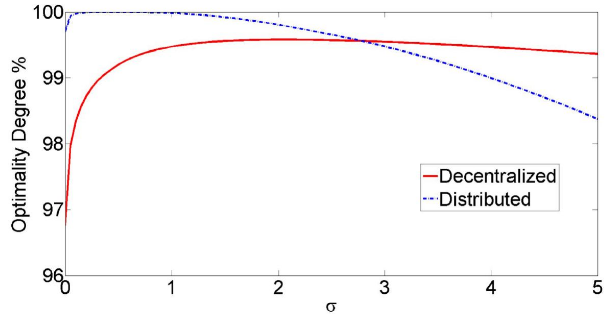
$$\begin{aligned} L_{ii} &= \sum_{j=1, j\neq i}^{\bar{n}} B_{ij}^{\text{Kron}} \quad \text{if } i = j \\ L_{ij} &= -B_{ij}^{\text{Kron}} \qquad\quad \text{if } i \neq j \end{aligned} \tag{40}$$

where B^{Kron} is the susceptance of the Kron reduced admittance matrix Y^{Kron} defined as

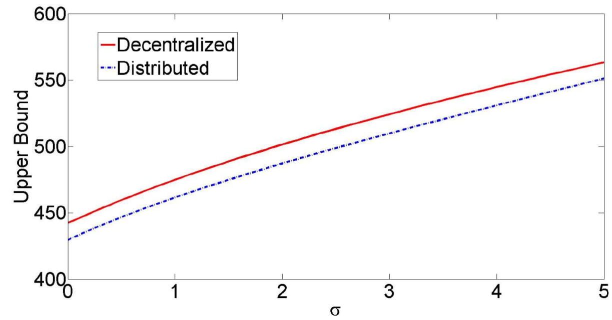
$$Y_{ij}^{\text{Kron}} = Y_{ij} - \frac{Y_{ik}Y_{kj}}{Y_{kk}} \quad (i, j = 1, 2, \ldots, n \text{ and } i, j \neq k) \tag{41}$$

where k is the index of the non-generator bus to be eliminated from the admittance matrix and $\bar{n}$ is the number of generator buses. Note that the Kron reduction method aims to eliminate the static buses of the network because the dynamics and interactions of only the generator buses are of interest [25].

By defining the rotor angle state vector as $\theta = [\theta_1, \ldots, \theta_{\bar{n}}]^T$ and the frequency state vector as $w = [w_1, \ldots, w_{\bar{n}}]^T$ and by substituting the matrix format of P_E into (37), the state space model of the swing equation used for frequency control in power systems could be written as



(a) Optimality degree of the near-optimal controller for a stochastic mass spring system.



(b) Cost of the near-optimal controller for a stochastic mass spring system.

Fig. 6: The optimality degree and the optimal cost of the near-optimal controller designed for the mass-spring system for two different control structures. The noise covariance matrix Σ_v is assumed to be equal to σI, where σ varies over a wide range.

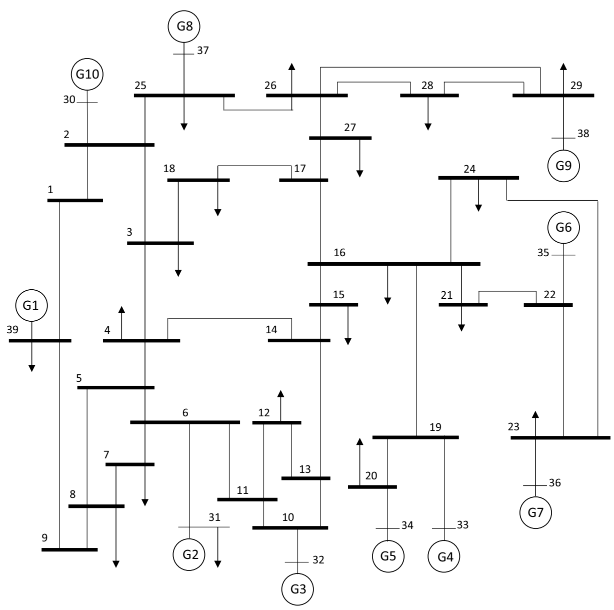


Fig. 7: Single line diagram of IEEE 39-Bus New England Power System.

$$\begin{bmatrix}\dot{\theta}\\ \dot{w}\end{bmatrix} = \begin{bmatrix}0_{\bar{n}\times\bar{n}} & I_{\bar{n}}\\ -M^{-1}L & -M^{-1}D\end{bmatrix}\begin{bmatrix}\theta\\ w\end{bmatrix} + \begin{bmatrix}0_{\bar{n}\times\bar{n}}\\ M^{-1}\end{bmatrix} P_M \tag{42}$$

$$y = \begin{bmatrix}\theta\\ w\end{bmatrix}$$

where $M = \text{diag}(M_1, \ldots, M_{\bar{n}})$ and $D = \text{diag}(D_1, \ldots, D_{\bar{n}})$. It is assumed that both rotor angle and frequency are available for measurement at each generator (implying that $C = I_{2\bar{n}}$). This is a reasonable assumption with the recent advances in Phasor Measurement Unit (PMU) technology [26].

By substituting the per-unit inertia (M) and damping (D) coefficients for the 10 generators of IEEE 39-Bus system [27] based on the data in Table II, the continuous-time state space model matrices A_c, B_c and C_c can be found. The system is then discretized to the discrete-time model matrices A, B and C with the sampling time of 0.2 second. The initial values

Bus	Gen	M	D	θ_0	w_0
30	G10	4	5	-0.0839	1.0
31	G2	3	4	0.0000	1.0
32	G3	2.5	4	0.0325	1.0
33	G4	4	6	0.0451	1.0
34	G5	2	3.5	0.0194	1.0
35	G6	3.5	3	-0.0073	1.0
36	G7	3	7.5	0.1304	1.0
37	G8	2.5	4	0.0211	1.0
38	G9	2	6.5	0.127	1.0
39	G1	6	5	-0.2074	1.0

TABLE II: The data and initial values of generators (in per unit) for IEEE 39-Bus New England Power System.

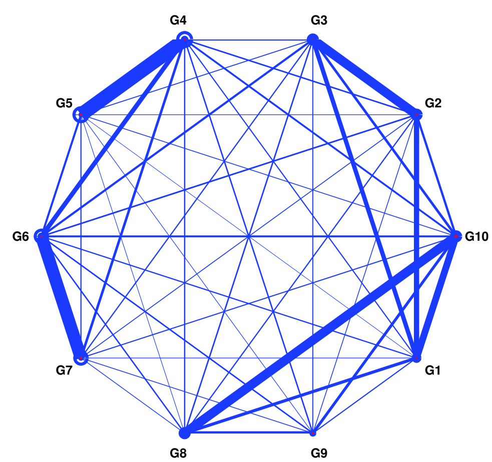
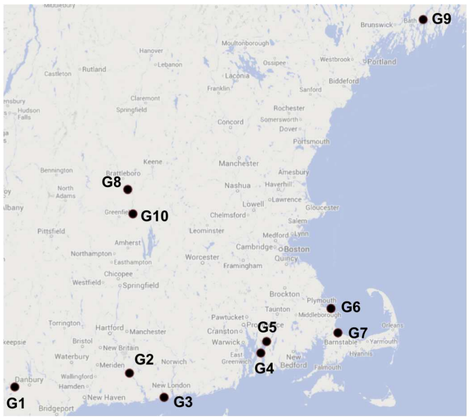


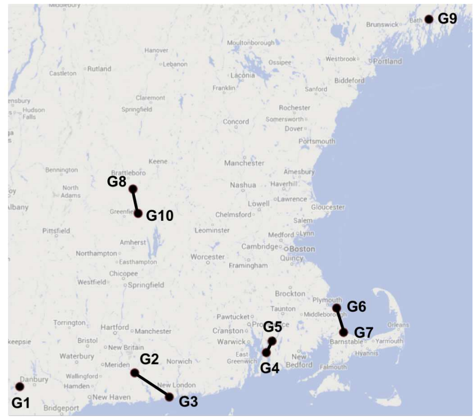
Fig. 8: Weighted graph of the Kron reduced network of IEEE 39-Bus New England Power System. Weights (thicknesses) of all edges are normalized to the minimum off-diagonal entry of the susceptance B^{Kron}.

of the rotor angle (θ_0) were calculated by solving power (or load) flow problem for the system using MATPOWER [28]. In practice, the rotor speed does not vary significantly from synchronous speed and thus the initial frequency (w_0) was assumed to be 1.0 per unit. Both θ_0 and w_0 are reported for each generator in Table II.

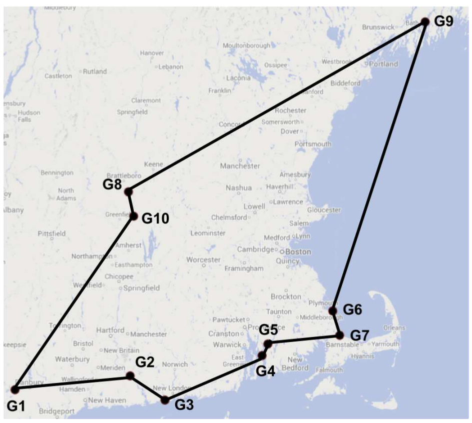
The 39-bus system has 10 generators, labeled as $G_1, G_2, \ldots, G_{10}$. Four communication topologies are considered in this work: decentralized, localized, star, and ring. In order to better understand how the interactions among the 10 generators in the



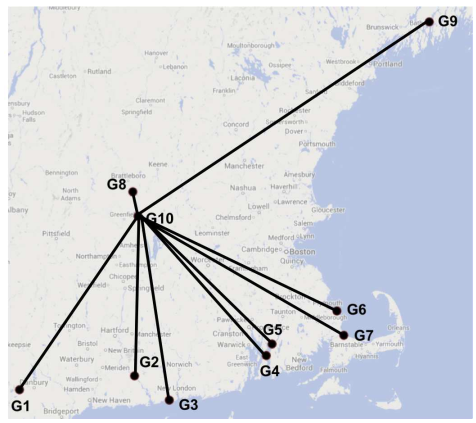
(a) Decentralized



(b) Localized



(c) Ring



(d) Star Topology (G_{10} in center)

Fig. 9: Four communication topologies studied for IEEE 39-bus system.

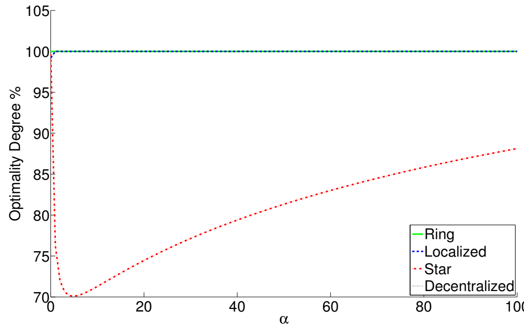
system are related to the communication structures, the Kron reduced network of the system is visualized by the weighted graph shown in Figure 8. In a fully decentralized structure, none of the generators communicate with each other. In a localized communication structure, the generators may only communicate with their close neighbors. In a star topology, a single generator is able to communicate with all other generators in the system. The ring communication structure—forming a closed path—aims to provide communications between neighbors. These topologies are visualized in Figure 9. The locations of the generators in the figure are based on the exact coordinates of the power plants named in [29]. Note that G_1 represents a group of generators, but it is considered as a single node near the border between New York and Connecticut in this map. G_4 and G_5 are very close in distance, but G_4 was somewhat shifted from its real coordinates to make the communication link between them visible in this map.

Finite-Horizon ODC: Assume that $Q = I$, $R = 0.1I$, and $p = 80$. Suppose also that α is a parameter between 0 and 100. The goal is to solve a finite-horizon ODC problem for each value of α and for each of the four aforementioned communication topologies. This will be achieved in two steps. First, a computationally-cheap SDP relaxation is solved. Second, a near-optimal controller $\hat{K}$ is designed by choosing the best solution of the direct and indirect recovery methods. The results are reported in Figures 10(a)-(c). The following observations can be made:

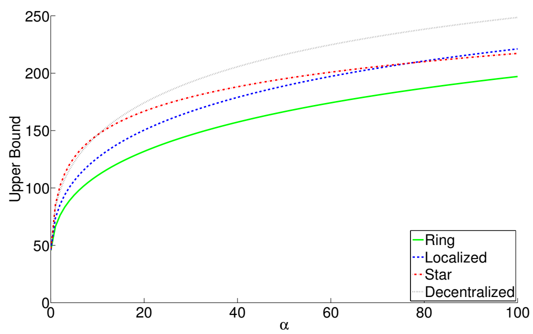
- The designed controllers are almost 100% optimal for three control topologies of decentralized, localized and ring, and this result holds for all possible values of α. The optimality degree for the star controller is above 70% and approaches 100% (even though slowly) as α grows.
- For every value of $\alpha \in [0, 100]$, the decentralized controller has the lowest performance while the ring controller offers the best performance.
- The closed-loop system is always stable for all 4 control topologies and all possible values of α.

Infinite-Horizon ODC: Consider the problem of solving an infinite-horizon ODC problem for each value of α in the interval $[0, 15]$ and each of the four aforementioned communication topologies. Similar to the previous experiment, stabilizing near-optimal controllers are designed for all these cases. The results are summarized in Figure 11.

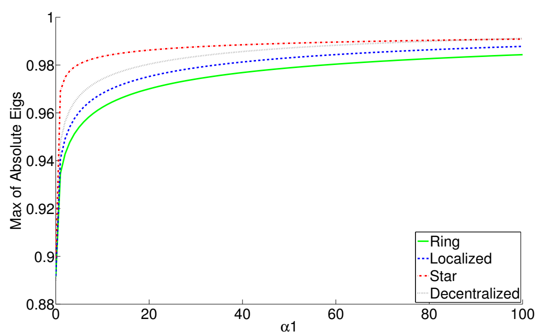
Stochastic ODC: Assume that the power system is under input disturbance and measurement noise. The disturbance can arise from non-dispatchable supplies (such as renewable energy) and fluctuating loads, among others. The measurement noise may account for the inaccuracy of the rotor angle and



(a) Optimality degree for finite-horizon ODC

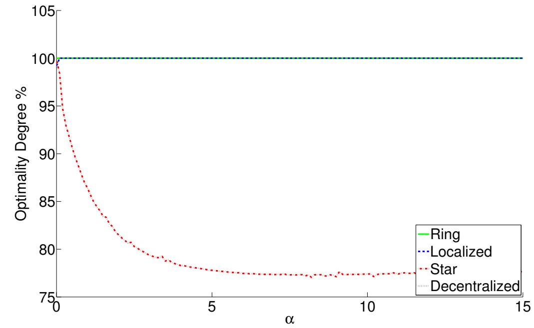


(b) Near-optimal cost for finite-horizon ODC

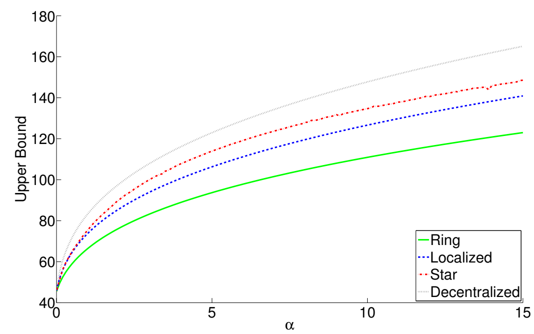


(c) Stability degree for finite-horizon ODC

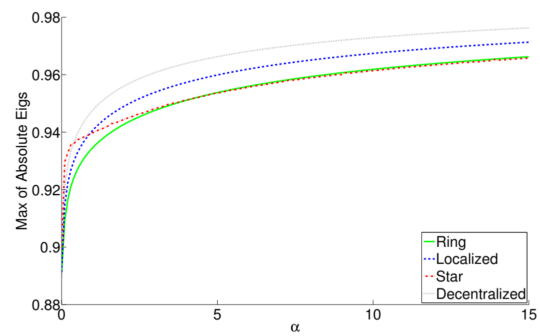
Fig. 10: A near-optimal controller $\hat{K}$ is designed to solve the finite-horizon ODC problem for every control topology given in Figure 9 and every α between 0 and 100: (a) optimality degree, (b) near-optimal cost, and (c) closed-loop stability (maximum of the absolute eigenvalues of the closed-loop system).



(a) Optimality degree for infinite-horizon ODC



(b) Near-optimal cost for infinite-horizon ODC



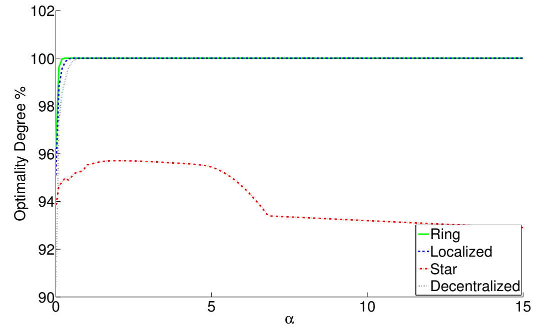
(c) Stability degree for infinite-horizon ODC

Fig. 11: A near-optimal controller $\hat{K}$ is designed to solve the infinite-horizon ODC problem for every control topology given in Figure 9 and every α between 0 and 15: (a) optimality degree, (b) near-optimal cost, and (c) closed-loop stability (maximum of the absolute eigenvalues of the closed-loop system).

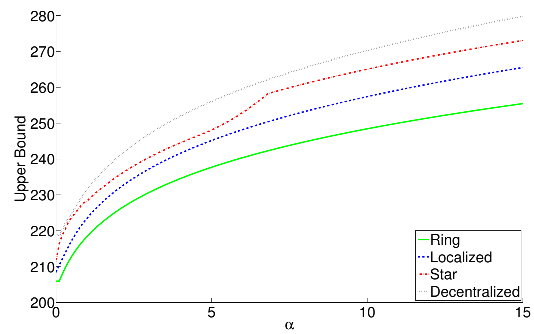
frequency measurements. Assume that Σ_d is equal to I. We consider two different scenarios:

i) Suppose that $\Sigma_v = 0$, while α varies from 0 to 15. For each SODC problem, we solve a computationally-cheap SDP relaxation, from which a near-optimal solution $\hat{K}$ is designed by choosing the best solution of the direct and indirect recovery methods. The outcome is plotted in Figure 12.
ii) Suppose that $\alpha = 0$, while Σ_v is equal to σI with σ varying between 0 and 15. As before, we design a near-optimal controller for each SODC problem. The results are reported in Figure 13.

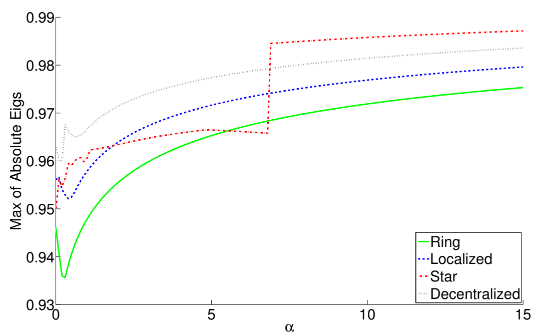
In the above experiments, we designed structured controllers to optimize a finite-horizon ODC, an infinite-horizon ODC or a stochastic ODC problem. This was achieved by solving their associated computationally-cheap SDP relaxations. Interestingly, the designed controllers were all stabilizing (with no exception), and their optimality degrees were close to 99% in case of decentralized, localized and ring structures. In case of the star structure, the optimality degree was higher than 70% in finite-horizon ODC, higher than 77% in infinite-horizon ODC and around 94% for various levels of σ and α in stochastic ODC.



(a) Optimality degree for stochastic ODC

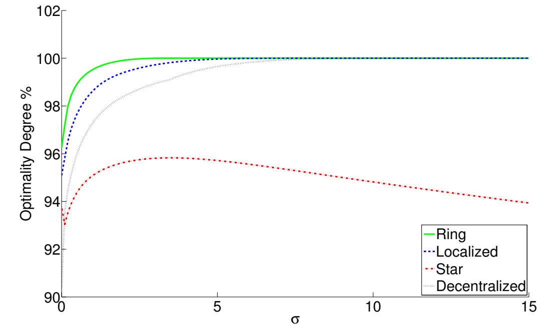


(b) Near-optimal cost for stochastic ODC

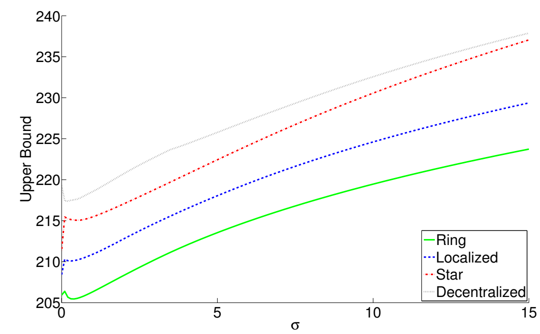


(c) Stability degree for stochastic ODC

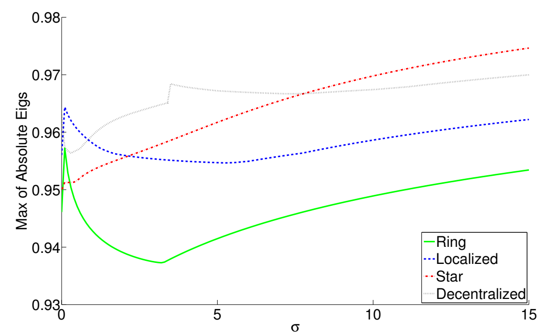
Fig. 12: A near-optimal controller $\hat{K}$ is designed to solve the stochastic ODC problem for every control topology given in Figure 9 and every α between 0 and 15 under the assumptions that $\Sigma_d = I$ and $\Sigma_v = 0$: (a) optimality degree, (b) near-optimal cost, and (c) closed-loop stability



(a) Optimality degree for stochastic ODC



(b) Near-optimal cost for stochastic ODC



(c) Stability degree for stochastic ODC

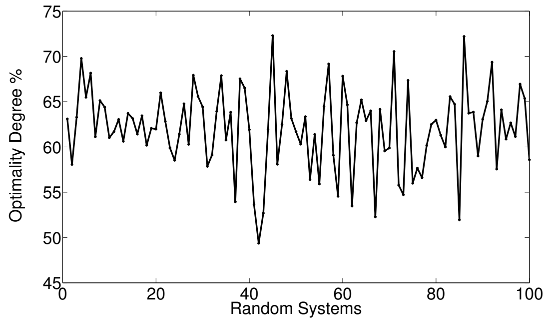
Fig. 13: A near-optimal controller $\hat{K}$ is designed to solve the stochastic ODC problem for every control topology given in Figure 9 and every σ between 0 and 15 under the assumptions that $\Sigma_d = I$, $\alpha = 0$ and $\Sigma_v = \sigma I$: (a) optimality degree, (b) near-optimal cost, and (c) closed-loop stability

C. Random Systems

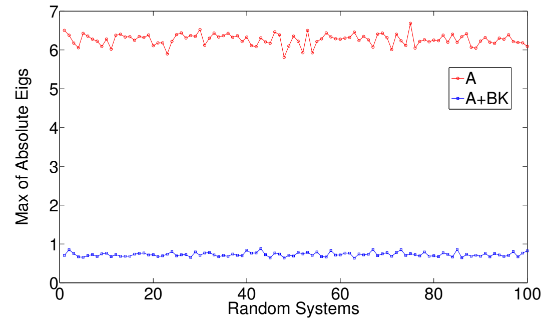
The goal of this example is to test the efficiency of the computationally-cheap SDP relaxation combined with the indirect recovery method on 100 highly-unstable random systems. Assume that $n = m = r = 25$, and that C, Q, R are identity matrices of appropriate dimensions. Suppose that $\Sigma_d = I$ and $\Sigma_v = 0$. To make the problem harder, assume that the controller under design must satisfy the hard constraint $\text{trace}\{KK^T\} \leq 2$ (to avoid a high gain K). We generated hundred random tuples $(A, B, \mathcal{K})$ according to the following rules:

- The entries of A were uniformly chosen from the interval $[0, 0.5]$ at random.
- The entries of B were uniformly chosen from the interval $[0, 1]$ at random.
- Each entry of the matrix K was enforced to be zero with the probability of 70%.

Note that although the matrices A and B are nonnegative, the matrix K under design can have both positive and negative entries. The randomly generated systems are highly unstable with the maximum absolute eigenvalue as high as 6 (instability for discrete-time systems requires a maximum magnitude less than 1). Although the control of such systems was not easy and the control structure was enforced to be 70% sparse with an enforced sparsity pattern, the proposed technique was always able to design a "stabilizing" near-optimal controller with an optimality degree between 50% and 75%. The results are reported in Figure 14.



(a) Optimality degree



(b) Stability level of open-loop and closed-loop systems

Fig. 14: The optimality degree and the stability level (maximum of the absolute eigenvalues) associated with 100 near-optimal sparse controllers designed for 100 highly-unstable random systems.

V. Conclusions

Part I of the paper was concerned with a finite-horizon optimal distributed control (ODC) problem. This part studies an infinite-horizon ODC problem as well as a stochastic ODC problem. The objective is to design a fixed-order distributed controller with a pre-determined structure to minimize a quadratic cost functional for either a deterministic or a stochastic system. For both infinite-horizon ODC and stochastic ODC, the problem is cast as a rank-constrained optimization with only one non-convex constraint requiring the rank of a variable matrix to be 1. This paper proposes a semidefinite program (SDP) as a convex relaxation, which is obtained by dropping the rank constraint. The notion of treewidth is exploited to study the rank of the minimum-rank solution of the SDP relaxation. This method is applied to the static distributed control case and it is shown that the SDP relaxation has a matrix solution with rank at most 3. Moreover, multiple recovery methods are proposed to round the rank-3 solution to rank 1, from which a near-global controller may be retrieved. Computationally-cheap SDP relaxations are also developed for infinite-horizon ODC and stochastic ODC. These relaxations are guaranteed to exactly solve the LQR and H_2 problems for the classical centralized control problem. The results of this two-part paper are tested on real-world and random systems through thousands of simulations.

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