Transformation of Optimal Centralized Controllers Into Near-Global Static Distributed Controllers

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***Abstract*—This paper is concerned with the optimal decentralized control problem for linear discrete-time deterministic and stochastic systems. The objective is to design a stabilizing static distributed controller with a given structure, whose performance is close to that of the optimal centralized controller. To this end, we derive a necessary and sufficient condition under which there exists a distributed controller that generates the same input and state trajectories as the optimal centralized one. This condition is then translated into a convex optimization problem. Subsequently, a regularization term is incorporated into the objective of the proposed optimization problem to indirectly account for the stability of the distributed control system. The designed optimization has a closed-form solution (explicit formula), which depends on the optimal centralized controller as well as the controller structure. If the optimal objective value of the proposed optimization is small enough at the explicit solution, the resulting controller is stabilizing and has a high performance. The derived formula may help partially answer some open problems, such as finding the minimum number of free elements required in the distributed controller to achieve a performance close to the optimal centralized one. The proposed approach is tested on a power network and several random systems.**

I. Introduction

The area of decentralized control has been created to address computation and communication challenges in the control of large-scale real-world systems. The main objective is to design a controller with a prescribed structure, as opposed to the traditional centralized controller, for an interconnected system consisting of an arbitrary number of interacting local subsystems. The structurally constrained controller is composed of a set of local controllers, associated with different subsystems, which are allowed to interact with one another according to the given control structure. The names “decentralized” and “distributed” are interchangeably used in the literature to refer to the underlying controller (the latter term is often used for geographically distributed systems). It has been known that solving the long-standing optimal decentralized control problem is a daunting task due to its NP-hardness [1], [2]. Great efforts have been made to solve this difficult problem for special structures, such as spatially distributed systems [3]–[6], dynamically decoupled systems [7], [8], strongly connected systems [9], and optimal static distributed systems [10], [11].

Due to the evolving role of convex optimization in solving complex problems, more recent approaches for the optimal decentralized control problem have shifted towards a convex reformulation of the problem [12]–[19]. Using the graph-theoretic analysis developed in [20], [21], we have shown in [22]–[24] that a semidefinite programming (SDP) relaxation for the decentralized control problem has a low-rank solution for finite- and infinite-time cost functions in both deterministic and stochastic settings. The low-rank SDP solution may be used to find a near-global distributed controller, but SDPs are often computationally expensive.

Consider the gap between the optimal costs of the centralized and decentralized control problems. This gap could be arbitrarily large in practice (as there may not exist a stabilizing controller with the prescribed structure). This paper is focused on systems for which this gap is relatively small. The main problem to be addressed is the following: given an optimal centralized controller, is it possible to design a stabilizing static distributed controller with a given structure whose performance is close to that of the best centralized one? The main objective of this paper is to propose a candidate distributed controller via an explicit formula, which is indeed the closed-form solution of an optimization problem.

In this work, we first derive a necessary and sufficient condition under which the states and inputs produced by a candidate distributed controller and the optimal centralized controller are the same for a given initial state. We translate the condition into an optimization problem, where the closeness of the optimal centralized and distributed control systems are captured by the smallness of the optimal objective value of this optimization. We then add a regularization term to the objective to account for the stability of the closed-loop system. This problem has a closed-form solution, which depends on the given sparsity pattern of the to-be-designed controller. The main advantage of the proposed technique is that it explicitly shows how the sparsity pattern affects the performance of the distributed control system. As a by-product, if the number of free elements in each row of the unknown static distributed control is higher than the rank of some Lyapunov matrix, then there exists a controller for which the above gap is zero (without considering the stability requirement). However, this controller may not be stabilizing and therefore having a higher number of free elements in each row increases the likelihood of finding both a stabilizing and a high-performance controller. We demonstrate the efficacy of our technique on a power network as well as random systems.

The rest of this paper is organized as follows. Deterministic systems are studied in Section II, followed by an extension to stochastic systems in Section III. Numerical examples are provided in Section IV. Finally, concluding remarks are drawn in Section V.

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II. Distributed Controller Design: Deterministic Systems

In this section, we study the design of static distributed controllers for deterministic systems. To this end, we will first formulate the problem and then develop our main results.

A. Problem Formulation

Consider the discrete-time system

$$x[\tau+1] = Ax[\tau] + Bu[\tau], \qquad \tau = 0, 1, ... \tag{1}$$

with the known matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $x[0] \in \mathbb{R}^n$. The main objective is to design a static controller $u[\tau] = Kx[\tau]$ to satisfy certain optimality and structural constraints to be specified later. Associated with the system (1) under an arbitrary controller $u[\tau] = Kx[\tau]$, we define the following cost function for the closed-loop system:

$$J(K) = \sum_{\tau=0}^{\infty} \left(x[\tau]^T Q x[\tau] + u[\tau]^T R u[\tau] \right) \tag{2}$$

where Q and R are known positive semidefinite matrices of appropriate dimensions. Let $K = K_c$ denote an optimal solution of the optimization problem

$$\min_{K\in\mathbb{R}^{m\times n}} J(K) \tag{3}$$

Note that K_c is the optimal centralized controller gain, which can be obtained from the Riccati equation.

Definition 1. *Define $\mathcal{K} \subseteq \mathbb{R}^{m\times n}$ as a linear subspace consisting of all distributed feedback gains K with a pre-specified sparsity pattern (forced zeros in certain entries).*

Definition 2. *Given a matrix $K_d \in \mathcal{K}$ and a percentage number $\mu \in [0, 100]$, it is said that the distributed controller $u[\tau] = K_d x[\tau]$ has the global optimality guarantee of μ if*

$$\frac{J(K_c)}{J(K_d)} \times 100 \geq \mu \tag{4}$$

To understand Definition 2, if μ is equal to 90% for instance, it means that the distributed controller $u[\tau] = K_d x[\tau]$ is at most 10% worse than the best centralized controller with respect to the cost function (2). The main objective of this part is to study the following problem.

Problem 1: Distributed Controller Design. Given a percentage number $\mu \in [0, 100]$, find a distributed controller $u[\tau] = K_d x[\tau]$ meeting three requirements:

i) $K_d \in \mathcal{K}$ is the solution of an explicit formula with respect to K_c without having to solve any optimization problem.
ii) The controller $u[\tau] = K_d x[\tau]$ has the global optimality guarantee of μ.
iii) The system (1) is stable under the controller $u[\tau] = K_d x[\tau]$.

Note that Requirement (i) in Problem 1 demands a low-complex design approach, which does not allow solving any optimization problem.

B. Performance Criterion

Consider the optimal centralized controller $u[\tau] = K_c x[\tau]$ and an arbitrary distributed controller $u[\tau] = K_d x[\tau]$. Let $x_c[\tau]$ and $u_c[\tau]$ denote the state and input of the system (1) under the centralized controller. Likewise, define $x_d[\tau]$ and $u_d[\tau]$ as the state and input of the system (1) under the distributed controller. The next lemma derives a necessary and sufficient condition under which the centralized and distributed controllers generate the same state trajectory for the system (1).

Lemma 1. *Given the optimal centralized gain K_c, an arbitrary distributed control gain $K_d \in \mathcal{K}$, and the initial state $x[0]$, the relation*

$$x_c[\tau] = x_d[\tau], \qquad \tau = 0, 1, 2, ... \tag{5}$$

holds if and only if

$$B(K_c - K_d)(A + BK_c)^\tau x[0] = 0, \qquad \tau = 0, 1, 2, ... \tag{6}$$

Proof. The proof is based on a simple induction on the time instance τ. The details are omitted due to its similarity to the proof of Lemma 2. □

Lemma 1 investigates the equivalence of the centralized and distributed controllers from the states' perspective. The next lemma studies the analogy of the input trajectories for the centralized and distributed control systems.

Lemma 2. *Given the optimal centralized gain K_c, an arbitrary distributed control gain $K_d \in \mathcal{K}$, and the initial state $x[0]$, the relation*

$$u_c[\tau] = u_d[\tau], \qquad \tau = 0, 1, 2, ... \tag{7}$$

holds if and only if

$$(K_c - K_d)(A + BK_c)^\tau x[0] = 0, \qquad \tau = 0, 1, 2, ... \tag{8}$$

Proof. First, we prove that (7) implies (8). To this end, assume that the equation (7) is satisfied. Since $x_c[0] = x_d[0] = x[0]$, we have $x_c[\tau] = x_d[\tau]$ for every nonnegative integer τ (note that the system (1) generates identical state signals under two identical input signals $u_c[\tau]$ and $u_d[\tau]$). Now, one can write

$$u_c[\tau] = K_c x_c[\tau] = K_c(A + BK_c)^\tau x[0] \tag{9a}$$
$$u_d[\tau] = K_d x_d[\tau] = K_d(A + BK_d)^\tau x[0] \tag{9b}$$

On the other hand, the relation $x_c[\tau] = x_d[\tau]$ can be expressed as

$$(A + BK_c)^\tau x[0] = (A + BK_d)^\tau x[0] \tag{10}$$

Combining (9) and (10) leads to (8).

To prove that (8) implies (7), suppose that the equation (8) is satisfied. By pre-multiply the left side of (8) with B, it follows from Lemma 1 that $x_c[\tau] = x_d[\tau]$. Now, one can write:

$$\begin{aligned} u_c[\tau] - u_d[\tau] &= K_c x_c[\tau] - K_d x_d[\tau] \\ &= K_c x_c[\tau] - K_d x_c[\tau] \\ &= (K_c - K_d)(A + BK_c)^\tau x[0] \\ &= 0 \end{aligned} \tag{11}$$

This yields the relation (7), and completes the proof. □

Theorem 1. *Given K_c, an arbitrary gain $K_d \in \mathcal{K}$, and the initial state $x[0]$, the relations*

$$u_c[\tau] = u_d[\tau], \qquad \tau = 0, 1, 2, ... \tag{12a}$$
$$x_c[\tau] = x_d[\tau], \qquad \tau = 0, 1, 2, ... \tag{12b}$$

hold if and only if

$$(K_c - K_d)(A + BK_c)^{\tau} x[0] = 0, \qquad \tau = 0, 1, 2, ... \tag{13}$$

Proof. This theorem is an immediate consequence of Lemmas 1 and 2. □

Theorem 1 derives a necessary and sufficient condition in order for a distributed control system to perform identically to its centralized counterpart. To flourish this condition, we introduce an optimization problem below.

Optimization A. This problem is defined as

$$\min_{K_d} \ \text{trace}\left\{(K_c - K_d)P(K_c - K_d)^T\right\} \tag{14a}$$
$$\text{s.t.} \ \ K_d \in \mathcal{K} \tag{14b}$$

where the symmetric positive-semidefinite matrix $P \in \mathbb{R}^{n\times n}$ is the unique solution of the Lyapunov equation

$$(A + BK_c)P(A + BK_c)^T - P + x[0]x[0]^T = 0 \tag{15}$$

Since P is positive semidefinite and the feasible set $\mathcal{K}$ is linear, Optimization A is convex. The next theorem explains how this optimization can be used to study the analogy of the centralized and distributed control systems.

Theorem 2. *Given K_c, an arbitrary gain $K_d \in \mathcal{K}$, and the initial state $x[0]$, the relations*

$$u_c[\tau] = u_d[\tau], \qquad \tau = 0, 1, 2, ... \tag{16a}$$
$$x_c[\tau] = x_d[\tau], \qquad \tau = 0, 1, 2, ... \tag{16b}$$

hold if and only if the optimal objective value of Optimization A is zero and K_d is a minimizer of this problem.

Proof. In light of Theorem 1, we need to show that the condition (13) is equivalent to the optimal objective value of Optimization A being equal to 0. To this end, define the semi-infinite matrix

$$X = \begin{bmatrix} x[0] & (A + BK_c)x[0] & (A + BK_c)^2 x[0] & \cdots \end{bmatrix} \tag{17}$$

Now, observe that (13) is satisfied if and only if the Frobenius norm of $(K_c - K_d)X$ is equal to 0 or equivalently

$$\text{trace}\{(K_c - K_d)XX^T(K_c - K_d)^T\} = 0 \tag{18}$$

On the other hand, if P is defined as XX^T, it will satisfy (15). This completes the proof. □

Theorem 2 states that if the optimal cost of the convex Optimization A is 0, then there exists a distributed controller $u_d[\tau] = K_d x_d[\tau]$ with the structure induced by $\mathcal{K}$ whose global optimality guarantee is 100%. Roughly speaking, a small optimal value for Optimization A implies that the centralized and distributed control systems can become very close to each other.

C. Stability Criterion

In the preceding subsection, we studied conditions under which the centralized and distributed control systems had the same input and state trajectories, for a given initial state. However, this condition does not guarantee the stability of the distributed closed-loop system. To elaborate on this statement, assume that condition (13) is satisfied, implying that $x_c[\tau] = x_d[\tau]$ and $u_c[\tau] = u_d[\tau]$ for every nonnegative integer τ. Assume also that $A + BK_d$ is diagonalizable as $A + BK_d = VDV^{-1}$, where V is a matrix consisting of the eigenvectors of $A + BK_d$ and D is a diagonal matrix containing the eigenvalues of $A + BK_d$. One can write

$$x_d[\tau] = (A + BK_d)^{\tau} x[0] = VD^{\tau}V^{-1}x[0] \tag{19}$$

Also, due to the stability of the centralized closed-loop system, we have

$$0 = \lim_{\tau\to\infty} \|x_c[\tau]\| = \lim_{\tau\to\infty} \|x_d[\tau]\| = \lim_{\tau\to\infty} \|VD^{\tau}V^{-1}x[0]\|$$

The above equation does not imply that all diagonal entries of D have norms less than 1 (i.e., stability). Instead, it implies that $x[0]$ is orthogonal to every eigenvector whose corresponding eigenvalue is unstable.

It follows from the above discussion that whenever the centralized and distributed control systems have the same input and state trajectories, $x[0]$ belongs to the stable manifold of the system $x[\tau + 1] = (A + BK_d)x[\tau]$, but the closed-loop system might be unstable. To address this issue, we introduce an optimization problem below.

Optimization B. This problem is defined as

$$\min_{K_d} \ \text{trace}\left\{(K_c - K_d)^T B^T B(K_c - K_d)\right\} \tag{20a}$$
$$\text{s.t.} \ \ K_d \in \mathcal{K} \tag{20b}$$

Lemma 3. *There exists a strictly positive number μ_s such that every distributed controller $u[\tau] = K_d x[\tau]$ with a gain $K_d \in \mathcal{K}$ stabilizes the system* (1) *if the objective value of Optimization B at the point K_d is less than μ_s.*

Proof. Notice that $A + BK_d$ could be interpreted as a structured additive perturbation of the closed-loop system matrix corresponding to the centralized controller K_c, i.e.,

$$A + BK_d = A + BK_c + (B(K_d - K_c)) \tag{21}$$

The proof follows from the above equation. □

Note that there are several techniques in matrix perturbation and robust control to maximize or find a sub-optimal value for μ_s [25]. Note also that the stability criterion (20a) is conservative, and can be improved by exploiting any possible structure in the matrices A and B together with the set $\mathcal{K}$.

D. Candidate Distributed Controller

Optimization A and Optimization B were introduced earlier to guarantee a high performance and closed-loop stability for a to-be-designed controller K_d. We merge these two optimization problems below.

Optimization C. Given a constant number $\alpha \in [0, 1]$, this problem is defined as the minimization of the function

$$\begin{aligned}\alpha \times \text{trace}\left\{(K_c - K_d)P(K_c - K_d)^T\right\} + \\ (1-\alpha) \times \text{trace}\left\{(K_c - K_d)^T B^T B (K_c - K_d)\right\}\end{aligned} \tag{22}$$

with respect to the matrix variable $K_d \in \mathcal{K}$.

Assume matrix K_d has l free entries to be designed. Denote these parameters as $h_1, h_2, ...h_l$. The space of permissible controllers can be characterized as

$$\mathcal{K} \triangleq \left\{ \sum_{i=1}^{l} h_i M_i \;\middle|\; \mathbf{h} \in \mathbb{R}^l \right\} \tag{23}$$

for some (fixed) 0-1 matrices $M_1, ..., M_l \in \mathbb{R}^{m \times n}$ (note that h_i's are the entries of $\mathbf{h}$).

Theorem 3. *Consider two matrices $X \in \mathbb{R}^{l \times l}$ and $Y \in \mathbb{R}^l$ with the entries*

$$X_{ij} = \alpha \ \text{trace}\{M_i P M_j^T\} + (1-\alpha) \ \text{trace}\{M_i^T B^T B M_j\} \tag{24a}$$

$$Y_i = \alpha \ \text{trace}\{M_i P K_c^T\} + (1-\alpha) \ \text{trace}\{M_i^T B^T K_c\} \tag{24b}$$

for every $i, j \in \{1, 2, ..., l\}$. The optimal solution of Optimization C can be expressed as $K_d = \sum_{i=1}^{l} h_i M_i$, where $\mathbf{h} = X^{-1} Y$.

To illustrate Theorem 3, assume that K_c is a square matrix and $\mathcal{K}$ consists of diagonal matrices. Then, one naive strategy to design K_d is to simply remove the off-diagonal entries of K_c but keep its diagonal. However, Optimization C proposes a distributed controller K_d whose $(i,i)^{\text{th}}$ entry is a weighted sum of the elements of the i^{th} row and i^{th} column of K_c, where the weights come from the Lyapunov matrix P.

We will numerically demonstrate in Section IV that the explicit controller K_d proposed in Theorem 3 is stabilizing and has a high optimality guarantee for a class of systems.

Remark 1. It can be inferred from Theorem 2 and Lemma 3 that there exists a positive number μ' such that if the optimal cost of Optimization C is less than μ', then its solution K_d solves Problem 1. Note that μ' is indeed a function of μ, i.e., the prescribed global optimality guarantee. Although it is straightforward to find conservative values for μ' (e.g., by choosing it sufficiently small), finding a non-conservative value for μ' is left for future work. Note that μ' does not depend on $\mathcal{K}$. By plugging the optimal controller K_d obtained from (24) into (22), the optimal cost of Optimization C will be obtained as a function of the sparsity pattern $\mathcal{K}$. Then, one can evaluate how sparse the to-be-designed K_d should be so that the optimal cost of Optimization C becomes less than μ' in order to guarantee the existence of a solution to Problem 1.

E. Sparsity Pattern

Since $x[0]x[0]^T$ has rank-1 in (15), the Lyapunov matrix P tends to be low-rank. In the extreme case, if the closed-loop matrix $A + BK_c$ is 0 (the most stable discrete system), the matrix P becomes rank 1. To illustrate this property, we will later show in Example 3 that only 10% of the eigenvalues of P are dominant for random highly-unstable systems. On the other hand, Theorem 2 states that there exists a distributed controller with the global optimality guarantee of 100% if the optimal objective value of Optimization A is zero. In what follows, we will show that this optimal value becomes zero if the number of free elements of K_d is higher than a threshold that depends on the rank of P.

Given a natural number r, let $\hat{P}$ denote a rank-r approximation of P. Define an approximate version of Optimization A below.

Approximate Optimization A. This problem is defined as

$$\min_{K_d \in \mathcal{K}} \ \text{trace}\left\{(K_c - K_d)\hat{P}(K_c - K_d)^T\right\} \tag{25}$$

Let $V \in \mathbf{R}^{n \times r}$ be a matrix whose columns are those eigenvectors of $\hat{P}$ associated with the r nonzero eigenvalues of this matrix. To simplify the statement of the next theorem, we make the assumption that the spark of V^T is $r+1$, implying that every r rows of V are linearly independent. Note that this condition is satisfied generically.

Theorem 4. *The optimal objective value of Approximate Optimization A is 0 if every row of K_d has at least r free elements.*

Proof. Approximate Optimization A has the optimal objective value 0 if $(K_c - K_d)V = 0$ or equivalently

$$K_d^j V = K_c^j V, \qquad j = 1, 2, ..., n \tag{26}$$

where K_c^j and K_d^j denote the j^{th} rows of K_c and K_d, respectively. Note that the rows of $K_d \in \mathcal{K}$ can be designed independently. On the other hand, (26) has a solution K_d^j with the right sparsity pattern because it has at least r free elements to be designed and the corresponding rows of V are linearly independent by assumption. This completes the proof. □

Corollary 1. *Given a natural number r, assume that the rank of P is r and that every row of the unknown controller K_d has at least r free elements. Then, there exists a controller $K_d \in \mathcal{K}$ whose global optimality degree is 100%.*

Proof. The proof follows from Theorems 2 and 4. □

Remark 2. The difference between the optimal solutions of Optimization A and Approximate Optimization A can be upper bounded by $\|P - \hat{P}\|$. On the other hand, Theorem 4 states that the objective of Approximate Optimization A is zero at optimality if every row of the to-be-designed distributed controller K_d has at least r free elements. Now, due to Theorem 2, there is a controller K_d whose global optimality degree is close to 100% if the number of free elements in each row of K_d is greater than or equal to the approximate rank of P (i.e., the number of clearly dominant eigenvalues). If the degree of freedom of K_d in each row is higher than r, then there are infinitely many distributed controllers with a high optimality degree, and then the chance of existence of a stabilizing controller among those candidates would be higher.

III. Distributed Controller Design: Stochastic Systems

In this section, we generalize the results developed earlier to stochastic systems. Consider the discrete-time system

$$\begin{cases} x[\tau+1] = Ax[\tau] + Bu[\tau] + Ed[\tau] \\ y[\tau] = x[\tau] + Fv[\tau] \end{cases} \qquad \tau = 0, 1, 2, ... \tag{27}$$

where A, B, E, F are known matrices, and $d[\tau]$ and $v[\tau]$ denote the input disturbance and measurement noise, respectively. Consider the cost functional

$$\lim_{\tau \to +\infty} \mathcal{E}\left(x[\tau]^T Q x[\tau] + u[\tau]^T R u[\tau]\right) \tag{28}$$

where $\mathcal{E}\{\cdot\}$ is the expectation operator. Assume that the input disturbance and measurement noise are zero-mean white-noise random processes. Define the covariance matrices

$$\Sigma_d = \mathcal{E}\{Ed[\tau]d[\tau]^T E^T\}, \quad \Sigma_v = \mathcal{E}\{Fv[\tau]v[\tau]^T F^T\} \tag{29}$$

for every $\tau \in \{0, 1, 2, ...\}$. Let K_c denote the gain of the optimal static centralized controller $u[\tau] = K_c y[\tau]$ minimizing (28) for the stochastic system (27). Note that if $F = 0$, K_c can be found using the Riccati equation. The goal is to design a stabilizing distributed controller $u[\tau] = K_d y[\tau]$ with a high global optimality degree such that $K_d \in \mathcal{K}$. Consider the counterpart of Optimization A for stochastic systems defined as the minimization problem

$$\min_{K_d \in \mathcal{K}} \quad \text{trace}\left\{(K_c - K_d)P_s(K_c - K_d)^T\right\} \tag{30}$$

where

$$\begin{aligned}(A + BK_c)P_s(A + BK_c)^T - P_s + x[0]x[0]^T \\ + \Sigma_d + (BK_c)\Sigma_v(BK_c)^T = 0\end{aligned} \tag{31}$$

Similarly to Theorem 2, it can be shown that there is a distributed controller $u[\tau] = K_d y[\tau]$ resulting in the same optimal cost (28) as the centralized controller $u[\tau] = K_c y[\tau]$ if the optimal objective value of (30) is zero. Since (30) is equivalent to Optimization A after replacing P_s with P, the results stated in the preceding section all hold for stochastic systems after changing P to P_s.

IV. Numerical Results

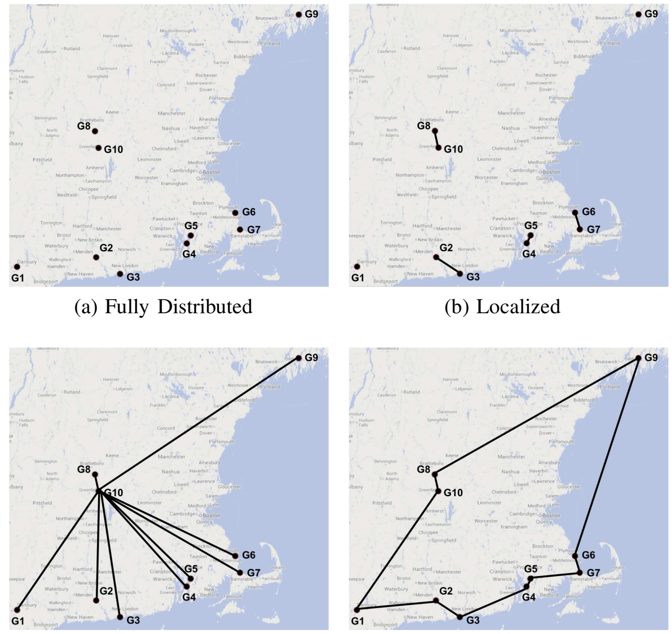
Three examples will be offered in this section to demonstrate the efficacy of the proposed controller design technique.

A. Example 1: Power Networks

In this example, the objective is to design a distributed controller for the primary frequency control of a power network. The system under investigation is the IEEE 39-Bus New England test System [26]. The state-space model of this system, after linearizing the swing equations, can be described as

$$\dot{x}(\tau) = A_c x(\tau) + B_c u(\tau) \tag{32}$$

where $A_c \in \mathbb{R}^{20\times 20}$, $B_c \in \mathbb{R}^{20\times 10}$, and $x(\tau)$ contains the rotor angles and frequencies of the 10 generators in the system (see [23] for the details of this model). The input of the system is the mechanical power applied to each generator. The goal is to first discretize the system with the sampling time of 0.2 second, and then design a distributed controller to stabilize the system while achieving a high degree of optimality. We consider four different topologies for the structure of the controller: distributed, localized, star and ring. A visual illustration of these topologies is provided in Figure 1, where each node represents a generator and each line specifies what generators are allowed to communicate. In the fully distributed structure, the generators are not allowed to communicate with each other. In the localized structure, the generators can only communicate with neighboring generators. In the star topology, a single generator communicates with all generators. In the ring communication structure, the generators may communicate with two of their neighbors. We will study both deterministic and stochastic cases below, by choosing the entries of the initial state $x(0)$ uniformly from the interval $[0, 1]$.



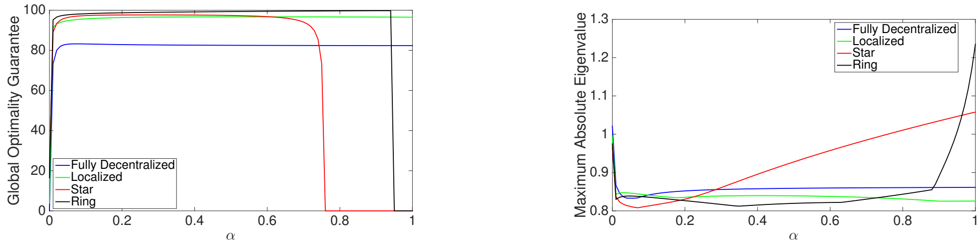
(a) Fully Distributed (b) Localized

(c) Star Topology (G10 in center) (d) Ring

Fig. 1: Communication structures studied in Example 1 for IEEE 39-Bus test System

Deterministic Case: In this experiment, we generate the weighting matrices Q and R in a random fashion as $Q = \tilde{Q}\tilde{Q}^T$ and $R = 0.1 \times \tilde{R}\tilde{R}^T$, where $\tilde{Q}$ and $\tilde{R}$ are normal random matrices. We then design distributed controllers using the explicit formula given in Theorem 3. Figure 2a shows the global optimality guarantee for the four aforementioned control topologies for different values of the parameter α. Figure 2b draws the maximum absolute eigenvalue of the closed-loop matrix $A + BK_d$. A number of observations can be made:

- For those values of α that make the system stable, the optimality guarantee is at least 80% for all topologies. Note that a distributed controller with a significantly higher optimality guarantee may not exist due to its low



(a) Optimality Guarantee

(b) Maximum Absolute Eigenvalues

Fig. 2: Global optimality guarantee and maximum absolute eigenvalue of $A + BK_d$ for four different topologies and different values of α (deterministic case).

degree of freedom.

- For those values of α making all topologies stabilizing, the ring structure has the best optimality guarantee (near 100%) while the fully distributed one has the lowest guarantee (near 80%).
- According to Figure 2b, the choice of α is critical for the ring and star structures. $0.92 \leq \alpha \leq 1$ and $0.77 \leq \alpha \leq 1$ make the system unstable for the ring and star topologies, respectively. Also, $[0.02, 1]$ is the best region for α, for the fully distributed and localized topologies.

Stochastic Case: Suppose that the power system is subject to input disturbance and measurement noise. The disturbance may be caused by certain non-dispatchable supplies and fluctuating loads. The measurement noise could arise from the inaccuracy of the rotor angle and frequency measurements. We assume that $\Sigma_d = I$ and $\Sigma_v = \sigma I$, with σ varying from 0 to 5. Q and R are set to I and $0.1 \times I$, respectively. Also, α is chosen as 0.4. The simulation results are provided in Figure 3. It can be observed that the designed controllers are all stabilizing with no exceptions. Moreover, the global optimality guarantees for the ring, star, localized, and fully distributed topologies are above 96.5%, 94.5%, 94%, and 90.8%, respectively.

B. Example 2: Stable Random Systems

The objective is to demonstrate the performance of the proposed design technique on stable systems. Consider $n = m = 40$ and $Q = R = I$. We generate random continuous-time systems and then discretize them according to following rules:

- The entries of A_c are chosen randomly from a Gaussian distribution with the mean 0 and variance 25.
- The entries of B_c are chosen randomly from a normal distribution.
- After constructing A_c, the matrix is rescaled by a real number so that its maximum absolute eigenvalue becomes equal to 0.8.
- K is assumed to be diagonal (off-diagonal elements are forced to be zero).
- A and B are obtained by discretizing (A_c, B_c) using the zero-order hold method with the sampling time of 0.1 second.

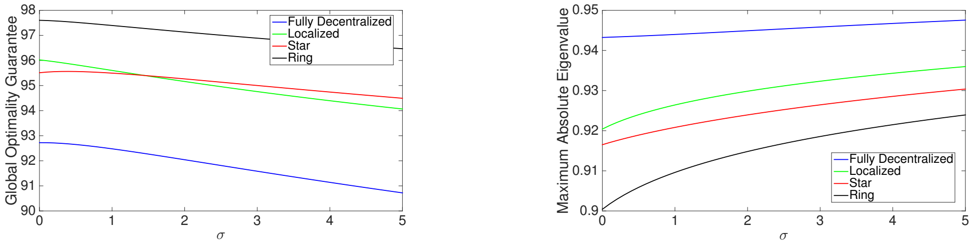
Notice that for $K = 0$ is a trivial stabilizing distributed controller for the above system. However, this choice of the controller may not result in a high optimality guarantee. We design a diagonal controller K_d using the explicit formula given in Theorem 3 for 100 random systems generated as above, with $\alpha = 0.98$. We arrange the resulting optimality guarantees in ascending order and label their corresponding trials as $1, 2, ..., 100$. Also, we calculate the optimality guarantee for the trivial controller $K = 0$ for each random system. The results of this experiment are provided in Figure 4. It can be observed that the designed diagonal controller always has a better optimality guarantee than the trivial solution $K = 0$. Note that our formula designs the $(i, i)^{\text{th}}$ entry of K_d using a linear combination of the entries in the i^{th} row and column of the optimal centralized controller K_c. Another approach is to simply use the diagonal of K_c as a candidate diagonal controller K_d. To assess this difference, we compute the cross-correlation between the controller designed using Theorem 3 and a truncated version of K_c for these 100 trials. The cross-correlation for these trials has the mean 0.4892 and standard deviation 0.1217. This shows that the designed controller is completely different from the naive method of truncating the centralized controller.

C. Example 3: Highly-Unstable Random Systems

In Example 2, it was shown that the proposed explicit formula could simply design high-performance distributed controllers for stable systems. This example investigates highly unstable systems for which the design of a stabilizing distributed controller is challenging itself without even imposing any performance criterion. Consider Example 2, but assume that the randomly generated matrices A_c's are not scalded down to make the maximum absolute eigenvalue to 0.8 and indeed their unstable eigenvalues are untouched. Suppose that each entry of K is forced to be zero with probability p. We consider three scenarios associated with p equal to 0.1, 0.2 and 0.3.

Note that the above class of random systems is highly unstable with the maximum absolute eigenvalue of A as high as 9. The optimality guarantees and maximum absolute eigenvalues of the design distributed controllers K_d are given in Figure 5 for a sample random system with a varying parameter α. The following observations can be made:

- As log as α is not very close to 1, the designed controllers are stabilizing. This implies that it is crucial to



(a) Optimality Guarantee

(b) Maximum Absolute Eigenvalues

Fig. 3: Global optimality guarantee and maximum absolute eigenvalue of $A + BK_d$ for four different topologies and different values of σ, under the assumption $\Sigma_d = I$ and $\Sigma_v = \sigma I$ (stochastic case).

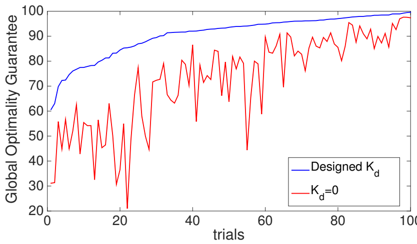


Fig. 4: Optimality guarantee of K_d for Example 2.

incorporate both stability and performance terms in the objective of Optimization C.

- As the probability of forced zeros in the controller increases, the optimality guarantee of the designed controller becomes more dependent on the value of α. Also, the optimality guarantee for the controller with $p = 0.1$ is higher than those of the other controllers, and it is almost 100% for a wide range of α. This is due to the fact that the controller has many free elements for design.

Now, consider 100 random systems generated according to the aforementioned rules and set $\alpha = 0.98$. We design distributed controllers for the three scenarios of p equal to 0.1, 0.2 and 0.3. We arrange the obtained maximum absolute eigenvalues in ascending order and subsequently label their corresponding trials as $1, 2, \dots, 100$. Figure 6 shows the optimality guarantees and maximum absolute eigenvalues of the designed distributed controllers. For $p = 0.1$, the proposed method always yields stabilizing controllers with optimality guarantees near to 100%. For $p = 0.2$, 99 control systems are stable with optimality guarantees near to 100%. For $p = 0.3$, 54 control systems are stable with very high optimality guarantees. Note that the designed controllers are very different from truncated versions of K_c (by simply discarding 10%-30% entries of K_c). More precisely, the cross-correlation between the controller designed using Theorem 3 and a truncated version of K_c for the above 100 trials with $p = 0.1$ has the mean 0.6245 and standard deviation 0.0677.

As mentioned earlier, the eigenvalues of P in the Lyapunov equation (15) may decay very rapidly. To support this statement, we compute the eigenvalues of P for previous 100 randomly generated unstable systems. Then, we arrange the absolute eigenvalues of P for each system in ascending order and label them as $\lambda_1, \lambda_2, \dots, \lambda_{40}$. For every $i \in \{1, 2, \dots, 40\}$, the mean of λ_i for these 100 independent random systems is drawn in Figure 7 (the variance is very low). It can be seen that only 10% of the eigenvalues are dominant and P can be well approximated by a low-rank matrix. Due to Corollary 1, there exist distributed controllers with global optimality degrees close to 100% even for a large value of p, e.g., $p = 0.9$. However, our stability term in Optimization C is not able to find a stabilizing controller in those cases. Designing a better convex compensation term for stability assurance is left as future work.

V. Conclusions

This paper studies the optimal distributed control problem for linear discrete-time systems. The goal is to design a stabilizing static distributed controller with a pre-defined structure, whose performance is close to that of the best centralized controller. To this end, we derive a necessary and sufficient condition under which there exists a distributed controller that produces the same input and state trajectories as the optimal centralized controller. We then convert this condition into a convex optimization problem. Roughly speaking, the smaller the optimal value of this optimization problem is, the closer the state and input trajectories of the centralized and distributed control systems can become. We also add a regularization term to the objective of the proposed optimization problem to account for the stability of the distributed control system indirectly. The designed optimization has a closed-form solution, which depends on the optimal centralized controller as well as the prescribed sparsity pattern for the unknown distributed controller. Hence, we provide a candidate distributed controller with the right structure through an explicit formula. If the optimal objective value of the proposed optimization is small enough at this solution, the resulting controller is stabilizing and has a high performance. The derived formula may help partially answer some open problems, such as finding the minimum number of free elements needed in the distributed controller to attain a performance close to the optimal centralized one. The proposed method is evaluated on a power network and several random systems.

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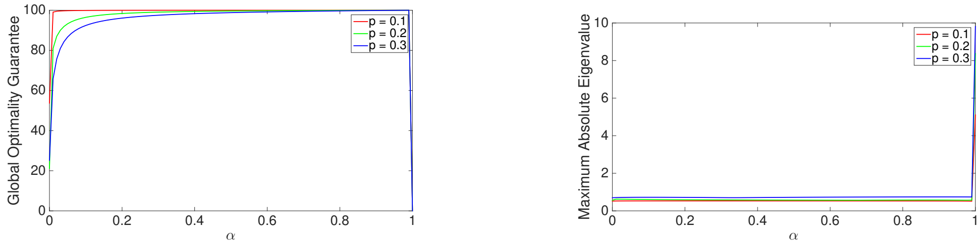
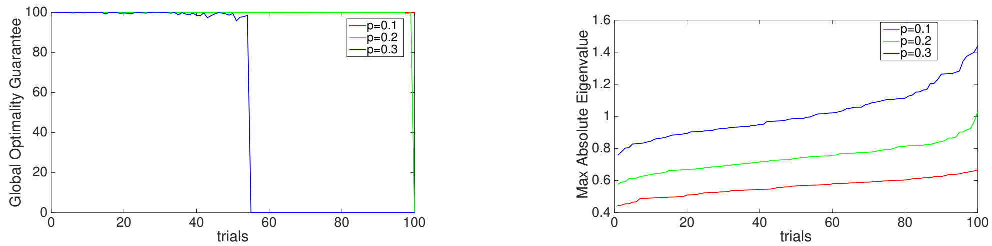


(a) Optimality Degree

(b) Maximum Absolute Eigenvalues

Fig. 5: Optimality guarantee and maximum absolute eigenvalue of $A + BK_d$ for a highly-unstable random system in Example 3.



(a) Optimality Degree

(b) Maximum Absolute Eigenvalues

Fig. 6: Optimality guarantee and maximum absolute eigenvalue of $A + BK_d$ for 100 highly-unstable random systems in Example 3.

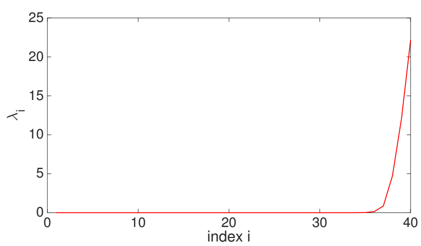


Fig. 7: Absolute eigenvalues of P for 100 random systems in Example 3.

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