Uniqueness of Power Flow Solutions Using Graph-theoretic Notions

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***Abstract*—This paper extends the uniqueness theory in [1] and establishes general necessary and sufficient conditions for the uniqueness of P-Θ power flow solutions in an AC power system using some properties of the monotone regime and the power network topology. We show that the necessary and sufficient conditions can lead to tighter sufficient conditions for the uniqueness in several special cases. Our results are based on the existing notion of maximal girth and our new notion of maximal eye. Moreover, we develop a series-parallel reduction method and search-based algorithms for computing the maximal eye and maximal girth, which are necessary for the uniqueness analysis. Reduction to a single line using the proposed reduction method is guaranteed for 2-vertex-connected Series-Parallel graphs. The relations between the parameters of the network before and after reduction are obtained. It is verified on real-world networks that the computation of the maximal eye can be reduced to the analysis of a much smaller power network, while the maximal girth is computed during the reduction process.**

I. INTRODUCTION

The AC power flow problem plays a crucial role in various aspects of power systems, e.g., the daily operations in contingency analysis and security-constrained dispatch of electricity markets. black Hence, unexpected operating points may appear for some system conditions and can jeopardize the normal operations of power systems. Conditions that ensure the existence of a unique "physically realizable" power flow solution are important but not fully understood.

For a special case of the AC power flow problem, the uniqueness property of the P-Θ power flow problem [2] has been studied in [1]. In the P-Θ power flow problem, the magnitude of the complex voltage at each node is given and the objective is to find a set of voltage phases such that the power flow equations are satisfied. The "physically realizable" constraint requires that the angular difference across every line lies within the stability limit of $\pi/2$ for lossless networks. Sufficient conditions (on the angular differences) that depend on the topological properties of the power network are established in [1]. Specifically, the authors proposed the notion of monotone regime and an upper bound on the angular differences based on the power network topology, which together can ensure the uniqueness of solutions. However, due to the nonlinear property of sinusoidal functions and the low-rank structure of angular differences, it is unclear to what extent the sufficient conditions given in [1] are necessary.

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The authors affiliated with UC Berkeley were in part supported by grants from ARO, ONR, AFOSR and NSF. Ross Baldick was in part supported by the University of Texas Faculty Development Program.

The goal of this paper is to provide more general necessary and sufficient conditions for the uniqueness, using the notion of maximal eye defined in Section III and the notion of maximal girth introduced in [1]. The paper also designs algorithms to compute these graph-theoretic parameters.

A. Main results

In this paper, we extend the uniqueness theory of P-Θ power flow problem proposed in [1]. We focus on the uniqueness of the power flow problem in a stronger sense and derive general necessary and sufficient conditions that *depend only on the choice of the monotone regime and network topology*. Under certain circumstances, the general conditions can be simplified to obtain tighter sufficient conditions. In addition, some algorithms for computing the maximal eye and the maximal girth of undirected graphs are proposed. A reduction method is designed to reduce the size of graphs and accelerate the computation process. More specifically, the contributions of this paper are three-fold: black In summary, this paper constitutes a substantial generalization of the uniqueness theory in [1]. A stronger notion of uniqueness is proposed and general necessary and sufficient conditions are proposed. These two combined provides a tool for analyzing large-scale power networks and enables a deeper understanding of the uniqueness of the P-Θ power flow problem.

B. Related work

The study of solutions to the power flow problem has a long history dating back to [3], which gave an example showing the general non-uniqueness of solutions for the power flow problem. Then, the number of solutions of the power flow problem was estimated in [4], which also characterized the stability region for the power flow problem. However, these early works only considered lossless transmission networks consisting of PV buses.

black Under the assumption that resistive losses are negligible, conditions for the existence and uniqueness of both real power-phase (P-Θ) problem, and reactive power-voltage (Q-V) problem were derived in [5], [2].

In another line of work, the topology structure of the power network was also considered to derive stronger conditions for the uniqueness. The number of solutions was estimated for radial networks in [6], [7], and later for general networks. Moreover, a more recent work [8] gave several algorithms to compute the unique high-voltage solution. black In this paper, we consider the P-Θ problem [2] for general lossy power networks and utilize the topology information. We refer to [1] for a more detailed review of the existing literature.

The fixed-point technique is often used for proving the existence and uniqueness of equations. For the power flow problem, the fixed-point technique was first utilized in [9] and was further developed by several works [10], [11], [12], [13], [14], [15]. Another more recently applied approach is to treat the P-Θ power flow problem as a rank-1 matrix sensing problem and solve its convex relaxation counterpart [16], [17]. The work [18] also considered the domain of voltages over which the power flow operator is monotone. However, the relation between the rank-1-constrained problem and its convexification is not clear for general power networks.

black In [1], it was also shown that the solution of P-Θ problem is unique under the assumption that angle differences across the lines are bounded by some limit related to the maximal girth of the network, which is defined in [19]. black

The existing algorithms in the literature cannot be directly used to compute maximal eye (introduced in Section III) or maximal girth. A related problem is computing the maximal chordless cycle as an upper bound to these parameters. The computation of maximal chordless cycles was proved to be $\mathcal{NP}$-complete in [20]. Efficient algorithms for enumerating chordless cycles were proposed in [21], [22] and both take linear time to enumerate a single chordless cycle. The algorithms for enumerating maximal chordless cycles can be easily modified to compute the minimal chordless cycle containing a given edge. Series-parallel reduction method was introduced as an alternative definition of Generalized Series-Parallel (GSP) graphs in [23]. Under the assumption that the slack bus is the last bus to be reduced, all GSP graphs can be reduced to a single line [1]. However, whether the series-parallel reduction method can still reduce GSP graphs without the assumption on the slack bus is not known. In this paper, we show that 2-vertex-connected[1] Series-Parallel graphs can be reduced to a single line without the assumption.

C. Notations

We start with some mathematical notations. We use $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$ to denote the set of all natural numbers, integers, real numbers and complex numbers, respectively. We denote $[n] := \{1, \ldots, n\}$ for any $n \in \mathbb{N}$. The symbol $\mathbf{j}$ denotes the unit imaginary number. The notations $(\cdot)^T$ and $(\cdot)^H$ denote the transpose and Hermitian transpose of a matrix, respectively. For a complex number z, $|z|$ denotes its magnitude and for a set X, the symbol $|X|$ denotes its cardinality. $\Re(\cdot)$ denotes the real part of a given scalar or matrix.

For an undirected graph, the set of vertices and the set of edges are denoted as $\mathbb{V}$ and $\mathbb{E}$, respectively. black For a directed graph $(\mathbb{V}, \mathbb{E}, A)$, the matrix $A \in \mathbb{R}^{|\mathbb{V}| \times |\mathbb{V}|}$ gives the orientation of each line, black The undirected edge connecting two vertices k and ℓ is denoted by a set notation $\{k, \ell\}$, whereas (k, ℓ) denotes a directed edge coming out of vertex k and going into ℓ. For parallel edges, we use $\{k, \ell, i\}$ to represent different edges connecting k and ℓ, where $i \in \mathbb{Z}_+$ is the index of each parallel edge.

A power network $\mathbb{G} = (\mathbb{V}, \mathbb{E}, Y)$ consists of two parts: the underlying undirected graph $(\mathbb{V}, \mathbb{E})$ and the complex admittance matrix $Y \in \mathbb{C}^{n \times n}$, where n is the number of vertices in the underlying graph. The underlying graph is assumed to be a simple and connected graph. The set of vertices $\mathbb{V}$ and the set of edges $\mathbb{E}$ correspond to the set of buses and the set of lines of the power network. The series element of the equivalent Π-model of each line $\{k, \ell\}$ is modeled by admittance $Y_{k\ell} = G_{k\ell} - \mathbf{j} B_{k\ell}$, where $G_{k\ell}, B_{k\ell} \geq 0$.

We denote $\mathbf{v} \in \mathbb{C}^n$ as the vector of complex bus voltages. The complex voltage at bus k can be written in the polar form using its magnitude and phase angle $v_k = |v_k| e^{\mathbf{j}\Theta_k}$ for all $k \in [n]$, where $|v_k| \in \mathbb{R}$ and $\Theta_k \in \mathbb{R}$ denote the voltage magnitude and phase angle, respectively. We denote $\Theta_{k\ell} := \Theta_k - \Theta_\ell \in [-\pi, \pi)$ as the phase difference modulus by 2π for all $\{k, \ell\} \in \mathbb{E}$. In the rest of the paper, we use the corresponding values in $[-\pi, \pi)$ for phase differences.

D. Paper organization

The remainder of this paper is organized as follows. Section II gives the necessary background knowledge about the P-Θ power flow problem and the existing uniqueness theory for the P-Θ problem. The notions of strong uniqueness and weak uniqueness are also introduced. In Section III, we propose the general analysis framework of the uniqueness theory that only depends on the monotone regime and the topological structure. We show that necessary and sufficient conditions can be fully characterized by a feasibility problem, which has fewer variables than the P-Θ problem. Sufficient conditions for uniqueness are derived and it is shown that the uniqueness conditions in [1] follow as a natural corollary. Then, we consider three special cases in Section IV by assuming specific topological structures for the underlying graph or a specific monotone regime. In these special cases, the necessary and sufficient conditions are simplified and the intricate sinusoidal functions are avoided in the verification of those conditions. Furthermore, the sufficient conditions proposed in Section III are proved to be tight when no information beyond the monotone regime and the topological structure is available. Finally, a reduction method and search-based algorithms for computing the maximal girth and maximal eye are given in Section V. black Proofs are delineated in the technical report [24].

II. Preliminaries

A. P-Θ problem formulation

As mentioned in the introduction, we focus our attention to the P-Θ problem, which describes the relationship between the voltage phasor angles and the real power injections. We first make the following assumptions. black Recall that the following injection operator describes the P-Θ problem, where the shunt elements are assumed to be purely reactive.

Definition 1. *black define $\hat{P}_k : \{0\} \times \mathbb{R}^{n-1} \to \mathbb{R}$ as the map from the vector of phasor angles to the real power injection at bus k:*

$$\hat{P}_k(\Theta) := \Re\{(Yv)_k^H v_k\}, \quad \forall \Theta \in \{0\} \times \mathbb{R}^{n-1}.$$

Moreover, define the injection operator $\hat{P} : \{0\} \times \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ as

$$\hat{P}(\Theta) := [\hat{P}_2(\Theta), \ldots, \hat{P}_n(\Theta)].$$

[1]A graph is called 2-vertex-connected if it is connected after the deletion of any single vertex.

The goal of the P-Θ problem is, given $P \in \mathbb{R}^{n-1}$, to find the voltage phasor angles $\Theta \in \{0\} \times \mathbb{R}^{n-1}$ such that

$$\hat{P}(\Theta) = P. \tag{1}$$

B. Monotone regime and allowable sets

We are interested in the uniqueness property of the solution to problem (1). In general, the number of solutions to problem (1) is hard to estimate because of the periodic behavior of sinusoidal functions, especially when there is no symmetrical structure in the power network. Thus, we limit the phase angle vectors to the monotone regime, within which the real power flow from bus k to bus ℓ increases monotonically with respect to the phase difference $\Theta_{k\ell}$ for each line $\{k, \ell\} \in \mathbb{E}$. The monotone regime is defined in [1] as follows.

Definition 2. *The **monotone regime** of a power network $(\mathbb{V}, \mathbb{E}, Y)$ is the set*

$$\{\Theta \in \mathbb{R}^n \mid \Theta_1 = 0, \Theta_{k\ell} \in [-\gamma_{k\ell}, \gamma_{k\ell}], \forall \{k, \ell\} \in \mathbb{E}\},$$

where $\gamma_{k\ell} := \tan^{-1}(B_{k\ell}/G_{k\ell}) \in [0, \pi/2]$ for all $\{k, \ell\} \in \mathbb{E}$.

blackThe constraint that the angular difference across every line lies within the stability limit of $[-\gamma_{k\ell}, \gamma_{k\ell}]$ is equivalent to the steady-state stability limit if each line is considered individually. As shown in [1], the phase angle vectors of leaf buses except the slack bus are uniquely determined by the phase angle vectors of non-leaf buses in the monotone regime. Hence, we assume that all vertices in the underlying graph except vertex 1 have degree at least 2.

Assumption 1. *The graph $(\mathbb{V}, \mathbb{E})$ is connected. All vertices except vertex 1 in the graph $(\mathbb{V}, \mathbb{E})$ have degree at least 2.*

We focus on finding a neighborhood of a solution in which there is no other solution to the P-Θ problem. The neighborhood is defined as follows.

Definition 3. *The set of **allowable perturbations** is defined as*

$$\mathcal{W} := \{\omega_{k\ell} \geq 0 \mid \forall \{k, \ell\} \in \mathbb{E}\}.$$

*Suppose that Θ is a solution to the P-Θ problem in the monotone regime. Then, the set of **neighboring phases** is defined as*

$$\mathcal{N}(\mathbb{G}, \Theta, \mathcal{W}) := \{\tilde{\Theta} \in \mathbb{R}^n \mid \tilde{\Theta}_1 = 0,$$
$$\tilde{\Theta}_{k\ell} \in [-\gamma_{k\ell}, \gamma_{k\ell}] \cap [\Theta_{k\ell} - \omega_{k\ell}, \Theta_{k\ell} + \omega_{k\ell}], \forall \{k, \ell\} \in \mathbb{E}\}.$$

We note that $\tilde{\Theta}_{k\ell}$ refers to the value of $\tilde{\Theta}_k - \tilde{\Theta}_\ell$ modulo 2π.

black It is desirable to analyze the uniqueness of the solution in the neighborhood $\mathcal{N}(\mathbb{G}, \Theta, \mathcal{W})$. In [1], the authors considered the *set of allowable angles*, which is defined as

$$\{\tilde{\Theta} \in \mathbb{R}^n | \tilde{\Theta}_1 = 0, \tilde{\Theta}_{k\ell} \in [-\omega_{k\ell}/2, \omega_{k\ell}/2], \forall \{k, \ell\} \in \mathbb{E}\}.$$

Note that the set of allowable angles is a special case of the set of allowable perturbations, since any two phase vectors in the set of allowable angles are in the corresponding sets of neighboring phases of each other. In this paper, we use the *set of allowable perturbations* but the sufficient conditions we derive can be naturally applied to using the *set of allowable angles*.

C. Notions of weak and strong uniqueness

Informally, we say that the P-Θ problem (1) has a unique solution Θ under the allowable perturbation set $\mathcal{W}$, if there exists at most one solution in the set $\mathcal{N}(\mathbb{G}, \Theta, \mathcal{W})$. We give two different definitions of uniqueness. Firstly, we introduce the uniqueness in the weak sense.

Definition 4. *We say that a solution Θ to the P-Θ problem (1) is **weakly unique** with the given set of allowable perturbations $\mathcal{W}$, if for any solution $\tilde{\Theta} \in \mathcal{N}(\mathbb{G}, \Theta, \mathcal{W})$, there exists a line $\{k, \ell\} \in \mathbb{E}$ such that $\Theta_{k\ell} = \tilde{\Theta}_{k\ell}$.*

In other words, two solutions are different according to Definition 4 if and only if they have different phase differences for every line. Next, we extend the definition of weak uniqueness to a stronger sense that is also more useful and usual.

Definition 5. *We say that a solution Θ to the P-Θ problem (1) is **strongly unique** with the given set of allowable perturbations $\mathcal{W}$, if for any solution $\tilde{\Theta} \in \mathcal{N}(\mathbb{G}, \Theta, \mathcal{W})$ and any $\{k, \ell\} \in \mathbb{E}$, we have $\Theta_{k\ell} = \tilde{\Theta}_{k\ell}$.*

In other words, two solutions are different according to Definition 5 if and only if the phase differences are different on at least one line.

III. Uniqueness Theory for General Graphs

In this section, we derive necessary and sufficient conditions on the set of allowable perturbations $\mathcal{W}$ such that the solution to problem (1) becomes strongly or weakly unique. In particular, we aim to analyze *the impact of the power system topology and the size of the monotone regime* on the uniqueness property. Namely, given the topological structure and the monotone regime, we aim to find conditions on $\mathcal{W}$ such that the uniqueness of solutions holds. To achieve this, we need to derive conditions under which all power networks with the same topological structure and monotone regime have unique solutions. To formalize the problem, we fix the underlying graph $(\mathbb{V}, \mathbb{E})$ and the angles specifying the monotone regime $\Gamma := \{\gamma_{k\ell} \in (0, \pi/2] \mid \{k, \ell\} \in \mathbb{E}\}$. We define the set of possible admittances with the same monotone regime as

$$\mathcal{S}(\gamma) := \{(C\cos(\gamma), C\sin(\gamma)) \mid C > 0\},, \quad \forall \gamma \in [0, \pi/2].$$

The set of complex admittance matrices with the same monotone regime is defined as

$$\mathcal{Y}(\mathbb{V}, \mathbb{E}, \Gamma) := \{Y \text{ is an admittance matrix } |$$
$$Y_{k\ell} = G_{k\ell} - \mathbf{j}B_{k\ell}, (G_{k\ell}, B_{k\ell}) \in \mathcal{S}(\gamma_{k\ell}), \{k, \ell\} \in \mathbb{E}\}.$$

Then, we define the set of power networks with the same topological structure and same monotone regime as

$$\mathcal{G}(\mathbb{V}, \mathbb{E}, \Gamma) := \{\mathbb{G} = (\mathbb{V}, \mathbb{E}, Y) \mid Y \in \mathcal{Y}(\mathbb{V}, \mathbb{E}, \Gamma)\},$$

or simply $\mathcal{G}$ if there is no confusion about $\mathbb{V}$, $\mathbb{E}$ and Γ. Hence, the problem under study in this paper can be stated as follows:

- What are the necessary conditions and sufficient conditions on the allowable perturbations $\mathcal{W}$ such that the solution to problem (1) is unique within the set of allowable perturbations for any power network $\mathbb{G} \in \mathcal{G}$?

The necessary conditions and the sufficient conditions provide two sides on the uniqueness theory. The sufficient conditions give a guarantee for the uniqueness of solutions for any single power network with the given topological structure and monotone regime, while the necessary conditions bound the optimal conditions we can derive only using the knowledge of topological structure and monotone regime. We first give an equivalent characterization of strong and weak uniqueness.

Lemma 1. (Necessary and Sufficient Conditions for Uniqueness) *Given the set of power networks $\mathcal{G}(\mathbb{V},\mathbb{E},\Gamma)$ and the set of allowable perturbations $\mathcal{W}$, the following two statements are equivalent:*

1) *For any power network $\mathbb{G}\in\mathcal{G}(\mathbb{V},\mathbb{E},\Gamma)$ and any power injection $P\in\mathbb{R}^{|\mathbb{V}|-1}$ such that problem* (1) *is feasible in the monotone regime, the solution to problem* (1) *in the monotone regime is strongly unique in $\mathcal{N}(\mathbb{G},\Theta,\mathcal{W})$.*
2) *For any power network $\mathbb{G}\in\mathcal{G}(\mathbb{V},\mathbb{E},\Gamma)$ and any two phase angle vectors Θ^1,Θ^2 in the monotone regime with the property $\Theta^2\in\mathcal{N}(\mathbb{G},\Theta^1,\mathcal{W})$, there exists a vector $\mathbf{y}\in\mathbb{R}^{|\mathbb{V}|}$ such that $y_1=0$ and*

$$\begin{aligned}\sin(\gamma_{k\ell}+\Theta^1_{k\ell}/2+\Theta^2_{k\ell}/2)\cdot y_k&\\ \geq \sin(\gamma_{k\ell}-\Theta^1_{k\ell}/2-\Theta^2_{k\ell}/2)\cdot y_\ell,&\\ \forall\{k,\ell\}\in\mathbb{E}\quad \text{s.t. } \Theta^1_{k\ell}-\Theta^2_{k\ell}>0,&\end{aligned}\tag{2}$$

where at least one of the inequalities above is strict.

The equivalence between statements 1 and 2 still holds true even after replacing strong uniqueness with weak uniqueness in statement 1, provided that the phase angle vector Θ^2 in statement 2 is required to satisfy $\Theta^1_{k\ell}\neq\Theta^2_{k\ell}$ for all $\{k,\ell\}\in\mathbb{E}$.

Intuitively, the above lemma studies the uniqueness of solutions through its dual form. black The dual form is preferred since the dual problem has fewer variables and its solution is easier to construct. We then derive several sufficient conditions using Lemma 1. We first show that we only need to verify statement 2 in Lemma 1 for two phase angle vectors Θ^1 and Θ^2 that induce a (weakly) feasible orientation, which we will define below. We define the orientation induced by two phase angle vectors.

Definition 6. *Suppose that Θ^1 and Θ^2 are two phase angle vectors of the graph. Then, we define the* ***induced orientation*** *of $\Delta:=\Theta^1-\Theta^2$ as $A_{k\ell}:=\mathrm{sign}(\Delta_{k\ell})$ for all $\{k,\ell\}\in\mathbb{E}$, where the sign function $\mathrm{sign}(\cdot)$ is defined as*

$$\mathrm{sign}(x):=\begin{cases}+1 & \textit{if } x>0\\ 0 & \textit{if } x=0\\ -1 & \textit{if } x<0\end{cases}.$$

In the definition of induced orientations, we assign one of the three directions $+1,-1,0$ to each edge. The first two directions are “normal” directions for directed graphs. An edge with direction $+1$ or -1 is called a **normal edge**. Edges with direction 0 are viewed as an undirected edge and reachable in both directions. In addition, edges with direction 0 are not considered when computing the in-degree and the out-degree. We only need to consider orientations induced by two different phase angle vectors Θ^1,Θ^2 such that $\hat{P}(\Theta^1)=\hat{P}(\Theta^2)$. However, a precise characterization of those orientations is difficult and we consider a larger set that contains those orientations.

Definition 7. *An orientation assigned to an undirected graph is called a* ***feasible orientation*** *if all edges are normal and each vertex except vertex* 1 *has nonzero in-degree and out-degree.*

According to the analysis in [1], the induced orientation of two solutions Θ^1 and Θ^2 in the monotone regime that are different according to Definition 4 must be a feasible orientation. Then, we give the definition of weakly feasible orientations as the counterpart for strong uniqueness.

Definition 8. *An orientation assigned to an undirected graph is called a* ***weakly feasible orientation*** *if two properties are satisfied: (i) there exists at least one normal edge, and (ii) the in-degree and the out-degree of any vertex except vertex* 1 *are both zero or both nonzero.*

Edges with direction 0 are lines with the same angular difference for the two phase angle vectors Θ^1 and Θ^2. By the same discussion as in Section II, we can view a weakly feasible orientation as a feasible orientation for the sub-graph that only has normal edges. The next lemma shows that we only need to consider weakly feasible orientations or feasible orientations when checking the conditions in statement 2 of Lemma 1.

Lemma 2. *If two different phase angle vectors $\Theta^1-\Theta^2$ in the monotone regime satisfy $\Theta^2\in\mathcal{N}(\mathbb{G},\Theta^1,\mathcal{W})$ and the induced orientation of $\Theta^1-\Theta^2$ is not weakly feasible, then there exists a vector $\mathbf{y}\in\mathbb{R}^{|\mathbb{V}|}$ such that statement 2 of Lemma 1 holds. The result holds true for the weak uniqueness property as well, provided that the induced orientation of Θ^1,Θ^2 is not a feasible orientation.*

Combining Lemmas 1 and 2, we obtain sufficient conditions for strong uniqueness and weak uniqueness.

Theorem 3. (Sufficient Conditions for Uniqueness) *Given the set of allowable perturbations $\mathcal{W}$, suppose that for any two different phase angle vectors Θ^1 and Θ^2 in the monotone regime satisfying $\Theta_2\in\mathcal{N}(\mathbb{G},\Theta^1,\mathcal{W})$, the induced orientation of $\Theta^1-\Theta^2$ is not a weakly feasible orientation. Then, the solution to problem* (1) *is strongly unique for all power networks in $\mathcal{G}$. The result holds true for the weak uniqueness as well, provided that the induced orientation of $\Theta^1-\Theta^2$ is not a feasible orientation.*

The sufficient condition given above is a generalization of Theorem 4 in [1], which ensures the weak uniqueness of solutions in the set of allowable phases. Using Theorem 3, we can derive a corollary similar to Theorem 4 in [1].

Corollary 4. *Consider an arbitrary set of allowable perturbations $\mathcal{W}$. The solution to problem (1) in the monotone regime is strongly unique for any power network $\mathbb{G}\in\mathcal{G}$ if for any weakly feasible orientation of the underlying graph $(\mathbb{V},\mathbb{E})$, there exists a directed cycle $(k_1,\ldots,k_t)$ containing at least one normal edge such that the allowable perturbations satisfy* *the inequality*

$$\sum_{\{k_i,k_{i+1}\} \text{ is normal}} \omega_{k_i k_{i+1}} < 2\pi,$$

where $k_{t+1} := k_1$. The same result holds true for the weak uniqueness if we substitute weakly feasible orientations with feasible orientations.

Now, we consider a special case where all constants $\omega_{k\ell}$ in the set of allowable perturbations are equal, i.e., there exists a constant $\omega \geq 0$ such that the set of allowable perturbation is

$$\mathcal{W}_\omega := \{\omega_{k\ell} = \omega, \forall \{k,\ell\} \in \mathbb{E}\}.$$

The problem we consider in this case is:

- What is the sufficient condition on ω such that the solution to problem (1) is unique with the allowable perturbation set $\mathcal{W}_\omega$?

We derive an upper bound on the constant ω to guarantee the uniqueness. We first define the maximal eye and the maximal girth of an undirected graph.

Definition 9. *Consider an undirected graph $(\mathbb{V},\mathbb{E})$. For any weakly feasible orientation assigned to the graph $(\mathbb{V},\mathbb{E})$, we define the minimal length of directed cycles that contain at least one normal edge as the* ***size of eye*** *of this orientation, where edges with direction 0 are considered as bi-directional edges. We define the* ***maximal eye*** *of the graph $(\mathbb{V},\mathbb{E})$ as the maximum of the size of eye over all possible weakly feasible orientations. We denote the maximal eyes of the graph $(\mathbb{V},\mathbb{E})$, a power network $\mathbb{G}$ and a group of power networks $\mathcal{G}$ as $\mathrm{e}(\mathbb{V},\mathbb{E})$, $\mathrm{e}(\mathbb{G})$ and $\mathrm{e}(\mathcal{G})$, respectively.*

Remark 1. *There always exists a directed cycle containing normal edges when the underlying graph is under a weakly feasible orientation. To understand this, we first choose an arbitrary normal edge $(k_1,k_2) \in \mathbb{E}$. Since the vertex k_2 has nonzero in-degree, it also has nonzero out-degree. Hence, there exists another vertex k_3 such that $(k_2,k_3) \in \mathbb{E}$. Continuing this procedure will result in the existence of a vertex k_t such that $v_t = k_s$ for some $s < t$. This generates a directed cycle $(k_s,k_{s+1},\dots,k_{t-1})$ containing only normal edges. Hence, the size of eye is well-defined.*

The counterpart of the maximal eye, known as the maximal girth, is defined in [1] and we restate the definition below.

Definition 10. *Consider an undirected graph $(\mathbb{V},\mathbb{E})$. For any feasible orientation assigned to the underlying graph $(\mathbb{V},\mathbb{E})$, we define the minimal size of directed cycles as the* ***girth*** *of this feasible orientation. We define the* ***maximal girth*** *of the graph $(\mathbb{V},\mathbb{E})$ as the maximum of the girth over all feasible orientations. We denote the maximal girths of the graph $(\mathbb{V},\mathbb{E})$, a power network $\mathbb{G}$ and a group of power networks $\mathcal{G}$ as $\mathrm{g}(\mathbb{V},\mathbb{E})$, $\mathrm{g}(\mathbb{G})$ and $\mathrm{g}(\mathcal{G})$, respectively.*

Remark 2. *Similar to the discussion in Remark 1, there exists at least one directed cycle when the graph is under a feasible orientation. The maximal eye can be equivalently defined as the maximum of the maximal girth over all sub-graphs that do not have degree-1 vertices.*

We provide an upper bound for ω using the maximal eye and the maximal girth, which follows from Corollary 4.

Corollary 5. *If the inequality*

$$\omega_{k\ell} < \frac{2\pi}{\mathrm{e}(\mathcal{G})}, \quad \forall \{k,\ell\} \in \mathbb{E}, \tag{3}$$

is satisfied, then the solution to problem (1) in the monotone regime is strongly unique for any power network $\mathbb{G} \in \mathcal{G}$. The same result holds true for weak uniqueness, provided that $\mathrm{e}(\mathcal{G})$ in (3) is substituted by $\mathrm{g}(\mathcal{G})$.

In Section V, we design search-based algorithms to calculate the maximal eye and the maximal girth. However, computing the maximal eye or the maximal girth is challenging for graphs with more than 100 nodes. Hence, we seek upper bounds and lower bounds for the maximal eye and the maximal girth. In this paper, we obtain a simple upper bound for both the maximal girth and the maximal eye. We define $\kappa(\mathbb{G})$ and $\kappa(\mathcal{G})$ as the sizes of the longest chordless cycles of the underlying graph of the power network $\mathbb{G}$ and any power network in the power network class $\mathcal{G}$, respectively. The upper bound on the maximal girth and eye will be provided below.

Theorem 6. *For any power network $\mathbb{G}$, it holds that*

$$\mathrm{g}(\mathbb{G}) \leq \mathrm{e}(\mathbb{G}) \leq \kappa(\mathbb{G}) \tag{4}$$

and that $\mathrm{g}(\mathcal{G}) \leq \mathrm{e}(\mathcal{G}) \leq \kappa(\mathcal{G})$.

We note that although computing the longest chordless cycle is $\mathcal{NP}$-complete [20], the computation of the longest chordless cycle is faster than the computation of the maximal eye and the maximal girth in practice.

IV. Uniqueness Theory for Three Special Cases

In this section, we consider three special cases. For each case, the power network either has a special topological structure or a special monotone regime. In the first two cases, the underlying graph of the power network is a single cycle or a 2-vertex-connected Series-Parallel (SP) graph. When the underlying graph is a single cycle, the sufficient conditions in Corollary 4 are also necessary. If the underlying graph is a 2-vertex-connected SP graph, we prove that the sufficient conditions for the weak uniqueness in Corollary 5 also ensure the strong uniqueness. In the last case, the power network is assumed to be lossless. In this case, the monotone regime of each line reaches the maximum possible size $[-\pi/2, \pi/2]$. Sinusoidal functions can then be avoided in statement 2 of Lemma 1, and therefore the verification of conditions is easier.

A. Single cycles

We first consider the case when the underlying graph $(\mathbb{V},\mathbb{E})$ is a single cycle. We first show that the weak uniqueness is equivalent to the strong uniqueness in this case.

Lemma 7. *Suppose that the underlying graph is a single cycle with the edges $(1,2),(2,3),\dots,(n,1)$. Then, given the set of allowable perturbations $\mathcal{W}$, the solution to problem (1) in the monotone regime is weakly unique if and only if it is strongly unique.*

Next, we prove that the sufficient conditions derived in Corollary 4 are also necessary for a single cycle with non-trivial monotone regime.

Theorem 8. *Suppose that the underlying graph is a single cycle with the edges* $(1,2),(2,3),\ldots,(n,1)$*, and that the set of allowable perturbations satisfies* $0<\omega_{i,i+1}\le\gamma_{i,i+1}$ *for all* $i\in[n]$*, where* $\gamma_{n,n+1}:=\gamma_{n,1}$ *and* $\omega_{n,n+1}:=\omega_{n,1}$*. The solution to problem* (1) *in the monotone regime is strongly unique for any power network* $\mathbb{G}\in\mathcal{G}(\mathbb{V},\mathbb{E},\Gamma)$ *and any power injection* $P\in\mathbb{R}^{n-1}$ *that makes problem* (1) *feasible* if and only if *the set of allowable perturbations* $\mathcal{W}$ *satisfies*

$$\sum_{i=1}^{n}\omega_{i,i+1}<2\pi,$$

where $\omega_{n,n+1}:=\omega_{n,1}$.

In contrast to requiring $\omega_{i,i+1}>0$ in the above theorem, the condition that $\omega_{i,i+1}=0$ for some i is sufficient but not necessary for the uniqueness of solutions. Under this condition, two solutions Θ^1 and Θ^2 in the monotone regime such that $\Theta^2\in\mathcal{N}(\mathbb{G},\Theta^1,\mathcal{W})$ must satisfy $\Theta^1_{i,i+1}=\Theta^2_{i,i+1}$. Hence, any solution is strongly unique with this set of allowable perturbations. However, by Theorem 8, this condition is not necessary for the uniqueness of solutions.

B. Series-Parallel graphs

In this subsection, we consider another special class of graphs, namely, the 2-vertex-connected SP graphs. The objective is to find an upper bound on the constant ω to guarantee that the solution to problem (1) is unique. Corollary 5 shows that the solution is strongly unique if $\omega<2\pi/\mathrm{e}(\mathbb{G})$ and is weakly unique if $\omega<2\pi/\mathrm{g}(\mathbb{G})$. However, for a 2-vertex-connected SP graph, we can prove a stronger theorem. We first prove that the maximal eye is equal to the maximal girth for a 2-vertex-connected SP graph. The main tool is the ear decomposition of an undirected graph [25].

Definition 11. *An* ***ear*** *of an undirected graph* $(\mathbb{V},\mathbb{E})$ *is a simple path or a single cycle. An* ***ear decomposition*** *of an undirected graph* $(\mathbb{V},\mathbb{E})$*, denoted as* $\mathcal{D}:=(L_0,\ldots,L_{r-1})$*, is a partition of* $\mathbb{E}$ *into an ordered sequence of ears such that one or two endpoints of each ear* L_k *are contained in an earlier ear, i.e., an ear* L_ℓ *with* $\ell<k$*, and the internal vertices of each ear do not belong to any earlier ear. We call* $\mathcal{D}$ *a* ***proper ear decomposition*** *if each ear* L_k *is a simple path for all* $k=1,\ldots,r-1$*. A* ***tree ear decomposition*** *is a proper ear decomposition in which the first ear is a single edge and for each subsequent ear* L_k*, there is a single ear* L_ℓ *with* $\ell<k$*, such that both endpoints of* L_k *lie on* L_ℓ*. A* ***nested ear decomposition*** *is a tree ear decomposition such that, within each ear* L_ℓ*, the set of pairs of endpoints of other ears* L_k *that lie within* L_ℓ *forms a set of nested intervals.*

The following theorem in [26] provides an equivalent characterization of 2-vertex-connected SP graphs through the ear decomposition.

Theorem 9. *A 2-vertex-connected graph is series-parallel if and only if it has a nested ear decomposition.*

With the help of the nested ear decomposition, we will prove that the maximal girth is equal to the maximal eye for 2-vertex-connected SP graphs. The intuition behind the proof is that we first choose two vertices as the "source" and the "sink" for the power flow network. black This step ensures that the first inequality in (4) holds as equality.

Lemma 10. *Suppose that* $(\mathbb{V},\mathbb{E})$ *is a* 2*-vertex-connected SP graph. Then, the following equality holds true:*

$$\mathrm{g}(\mathbb{V},\mathbb{E})=\mathrm{e}(\mathbb{V},\mathbb{E}).$$

Therefore, combining the above lemma with Corollary 5, we obtain a stronger sufficient condition for 2-vertex-connected SP graphs. This result implies that the sufficient conditions for the weak uniqueness in Corollary 5 also guarantee the strong uniqueness.

Theorem 11. *Suppose that the underlying graph* $(\mathbb{V},\mathbb{E})$ *is a* 2*-vertex-connected SP graph. The solution to problem* (1) *is strongly unique for any power network* $\mathbb{G}\in\mathcal{G}$ *in the monotone regime if*

$$\omega<\frac{2\pi}{\mathrm{g}(\mathcal{G})}.$$

C. Lossless networks

Finally, we consider the case when the power network is lossless, namely, when $\gamma_{k\ell}=\pi/2$ for all $\{k,\ell\}\in\mathbb{E}$. In this case, we prove that the strong uniqueness holds if and only if there does not exist another solution in the set of neighboring phases such that the induced orientation has strictly more strongly connected components than weakly connected components. This results makes it possible to avoid nonlinear sinusoidal functions in statement 2 of Lemma 1, and therefore the uniqueness of solutions becomes easier to verify. We first define the sub-graph induced by two phase angle vectors.

Definition 12. *Suppose that* Θ^1 *and* Θ^2 *are two different phase angle vectors, and that the orientation* A *is the induced orientation of* $\Theta^1-\Theta^2$*. Then, the* ***induced sub-graph*** *of* $\Theta^1-\Theta^2$ *is constructed as a directed sub-graph of* $(\mathbb{V},\mathbb{E},A)$ *by first deleting all edges with direction* 0 *and then deleting all degree-*1 *vertices.*

In what follows, we establish a necessary and sufficient condition for the uniqueness of the solution that does not contain sinusoidal functions.

Theorem 12. *Consider a that the set of allowable perturbations* $\mathcal{W}$*. If the monotone regime satisfies* $\gamma_{k\ell}=\pi/2$ *for all* $\{k,\ell\}\in\mathbb{E}$*, then the following two statements are equivalent:*

1) *For any power network* $\mathbb{G}\in\mathcal{G}(\mathbb{V},\mathbb{E},\Gamma)$ *and any power injection* $P\in\mathbb{R}^{|\mathbb{V}|-1}$ *such that problem* (1) *is feasible, the solution to problem* (1) *in the monotone regime is strongly unique in* $\mathcal{N}(\mathbb{G},\Theta,\mathcal{W})$*.*
2) *For any power network* $\mathbb{G}\in\mathcal{G}(\mathbb{V},\mathbb{E},\Gamma)$ *and any two phase angle vectors* Θ^1 *and* Θ^2 *in the monotone regime with the property* $\Theta^2\in\mathcal{N}(\mathbb{G},\Theta^1,\mathcal{W})$*, the induced sub-graph of* $\Theta^1-\Theta^2$ *has strictly more strongly connected components than weakly connected components.*

The equivalence between statements 1 and 2 still holds true even after replacing strong uniqueness with weak uniqueness in statement 1, provided that the phase angle vectors Θ^2 in statement 2 is required to satisfy $\Theta^1_{k\ell} \neq \Theta^2_{k\ell}$ for all $\{k, \ell\} \in \mathbb{E}$.

The result of the above theorem is stronger than the sufficient conditions in Theorem 3. This is because any (weakly) *infeasible* orientation has strictly more strongly connected components than weakly connected components. Hence, the sufficient conditions in Theorem 3 ensure that all induced orientations are (weakly) *infeasible*. Then, statement 2 of this theorem holds true and the solution becomes strongly (weakly) unique.

V. Iterative Series-Parallel Reduction

In the preceding sections, we have shown that the maximal eye and the maximal girth play important roles in the uniqueness theory. However, computing the maximal eye or maximal girth is cumbersome for large graphs. Hence, we develop an iterative reduction method to design a reduced graph, and then prove the relationship between the maximal eye or the maximal girth of the original graph and those of the reduced graph. Next, we test the performance of those algorithms on real-world problems. Search-based algorithms for computing the maximal eye and the maximal girth are given in [24].

A. Iterative Series-Parallel Reduction method

In this subsection, we propose an iterative reduction method, named as the Iterative Series-Parallel Reduction (black) method, that can reduce the size of the underlying graph for computing the maximal eye and maximal girth. The black method is different from the Series-Parallel Reduction (SPR) method introduced in [1] in two aspects. First, the purpose of the black method is to accelerate the computation of the maximal eye and the maximal girth, while the focus of SPR method is to facilitate the verification of uniqueness conditions. Second, we prove that all 2-vertex-connected SP graphs can be reduced to a single edge (K_2) without the assumption in [1] that the slack bus is the last to be reduced.

Before introducing the black method, we extend the definition of the maximal eye and the maximal girth to weighted graphs with "multiple slack buses". This generalized class of graphs appear during the reduction process. By defining the length of a cycle as the sum of the weights of the edges on the cycle, the maximal eye and the maximal girth can be generalized to weighted graphs. Next, we define (weakly) feasible orientations for graphs with "multiple slack buses", namely, the slack nodes.

Definition 13. *For a weighted undirected graph $(\mathbb{V}, \mathbb{E}, W)$, a subset of vertices $\mathbb{V}_s \subseteq \mathbb{V}$ is called the set of **slack nodes**. An orientation A assigned to the graph is called a **weakly feasible orientation** if each edge has one of the directions $\{+1, -1, 0\}$ and each vertex not in $\mathbb{V}_s$ either has nonzero in-degree and nonzero out-degree, or has zero in-degree and zero out-degree. An orientation A assigned to the graph is called a **feasible orientation** if each edge has one of the directions $\{+1, -1\}$ and each vertex not in $\mathbb{V}_s$ has nonzero in-degree and nonzero out-degree.*

Algorithm 1 Iterative Series-Parallel Reduction method

Input: Undirected unweighted graph $(\mathbb{V}, \mathbb{E})$, slack bus k
Output: Reduced undirected weighted graph $(\mathbb{V}_R, \mathbb{E}_R, W_R)$, two constants α_1, α_2 defined in Theorems 14 and 16, set of slack nodes $\mathbb{V}_s$

```
Set the initial weight for each edge to be 1.
Set the initial set of slack nodes as V_s ← {k}.
while at least one operation is implementable do
    if Type I Operations are implementable then
        Implement Type I Operation.
        Update values α1, α2 according to their definitions in Theorems 14 and 16.
        continue
    end if
    if Type II Operations are implementable then
        Implement Type II Operation.
        continue
    end if
    if Type III Operations are implementable then
        Implement Type III Operation.
        Update values α1, α2 according to their definitions in Theorems 14 and 16.
        Update the set of slack nodes V_s.
        continue
    end if
end while
Return reduced graph (V_R, E_R, W_R), set of slack nodes V_s and values α1, α2.
```

Now, we can define the maximal eye for graphs with slack nodes by taking the maximum of the size of eye over weakly feasible orientations. The maximal girth can be defined in a similar way. For power networks, the only slack node is the slack bus of the power network. Hence, the extended definitions of the maximal eye and the maximal girth are consistent with their original definitions. The black method is based on three types of operations:

- **Type I Operation.** Replacement of a set of parallel edges with a single edge that connects their common endpoints. The weight of the new single edge is the minimum over the weights of the deleted parallel edges.
- **Type II Operation.** Replacement of the two edges incident to a degree-2 vertex with a single edge, if the vertex has exactly two neighboring vertices and is not a slack node. The weight of the new edge is the sum of the weights of the two deleted edges.
- **Type III Operation.** Deletion of a vertex that has only a single neighboring vertex. If the deleted vertex is a slack node, or if the deleted vertex has degree at least 2 for the problem of computing the maximal girth, then we define its neighboring vertex as a slack node.

The update scheme of weights and slack nodes is designed to control the change of the maximal eye or the maximal girth. The black method successively reduces the size of the graph by applying Type I-III Operations; the pseudo-code of the black method is given in Algorithm 1. We note that after the reduction process, there is no parallel edge or pendant (degree-1) vertex in the reduced graph. Ignoring the weights of the edges and the set of the slack nodes, the operations in the black method can cover the operations in the classical series-parallel reduction [23], which are defined as

- **Type I' Operation.** Replacement of parallel edges with a single edge that connects their common endpoints.
- **Type II' Operation.** Replacement of the two edges incident to a degree-2 vertex with a single edge.
- **Type III' Operation.** Deletion of a pendant vertex.

Hence, the black method can be viewed as a generalization of the classical series-parallel reduction. We first consider the change of the maximal eye after each operation.

Lemma 13. *Given a weighted undirected graph $(\mathbb{V}, \mathbb{E}, W)$, let e denote its maximal eye. Assume that one of Type I-III Operations is implemented on the graph. By denoting the new graph and its maximal eye as $(\tilde{\mathbb{V}}, \tilde{\mathbb{E}}, \tilde{W})$ and $\tilde{\mathrm{e}}$, the following statements hold:*

- *If Type I Operation is implemented, then*
$$\tilde{\mathrm{e}} \leq \mathrm{e} \leq \max\{\tilde{\mathrm{e}}, W_{max} + W_{min}\},$$
where W_{max} and W_{min} are the maximal and minimal weights of the deleted parallel edges, respectively.
- *If Type II Operation is implemented, then $\mathrm{e} = \tilde{\mathrm{e}}$.*
- *If Type III Operation is implemented and the deleted vertex has degree 1, then $\mathrm{e} = \tilde{\mathrm{e}}$.*
- *If Type III Operation is implemented and the deleted vertex has degree larger than 1, then*
$$\mathrm{e} = \max\{\tilde{\mathrm{e}}, W_{max} + W_{min}\},$$
where W_{max} and W_{min} are the maximal and minimal weights of the deleted parallel edges, respectively.

Using the above lemma, we have the following theorem.

Theorem 14. *Given a power network with the underlying graph $(\mathbb{V}, \mathbb{E})$, let e denote the maximal eye of the graph. Denote the graph after reduction and its maximal eye as $(\mathbb{V}_R, \mathbb{E}_R, W_R)$ and e_R, respectively. Then, we have*
$$\max\{\mathrm{e}_R, \alpha_2\} \leq \mathrm{e} \leq \max\{\mathrm{e}_R, \alpha_1, \alpha_2\},$$
where α_1 and α_2 are the maximum of $W_{max}+W_{min}$ over Type I and Type III Operations, respectively. Here, W_{max}, W_{min} are defined in Lemma 13. If Type I or Type III Operations is never implemented, then we set α_1 or α_2 to 0.

Similarly, we can prove the relation between the maximal girth of the original graph and that of the reduced graph. We first show the change of the maximal girth after each operation.

Lemma 15. *Given a weighted undirected graph $(\mathbb{V}, \mathbb{E}, W)$, let g denote its maximal girth. Assume that one of Type I-III Operations is implemented on the graph. By denoting the new graph and its maximal girth of new graph as $(\tilde{\mathbb{V}}, \tilde{\mathbb{E}}, \tilde{W})$ and $\tilde{\mathrm{g}}$, the following statements hold:*

- *If Type I Operation is implemented, then*
$$\tilde{\mathrm{g}} \leq \mathrm{g} \leq \max\{\tilde{\mathrm{g}}, W_{max} + W_{min}\},$$
where W_{max} and W_{min} are the maximal and minimal weights of the deleted parallel edges, respectively.
- *If Type II Operation is implemented, then $\mathrm{g} = \tilde{\mathrm{g}}$.*
- *If Type III Operation is implemented and the deleted vertex has degree 1, then $\mathrm{g} = \tilde{\mathrm{g}}$.*
- *If Type III Operation is implemented, the deleted vertex is a slack node and has degree larger than 1, then*
$$\tilde{\mathrm{g}} \leq \mathrm{g} \leq \max\{\tilde{\mathrm{g}}, W_{max} + W_{min}\},$$
where W_{max} and W_{min} are the maximal and minimal weights of the deleted parallel edges, respectively.
- *If Type III Operation is implemented, the deleted vertex is not a slack node and has degree larger than 1, then*
$$\mathrm{g} = \min\{\tilde{\mathrm{g}}, W_{max} + W_{min}\},$$
where W_{max} and W_{min} are the maximal and minimal weights of the deleted parallel edges, respectively.

By the above lemma, the relationship between the maximal girth of the original graph and that of the reduced graph will be discovered below.

Theorem 16. *Given a power network with the underlying graph $(\mathbb{V}, \mathbb{E})$, let g denote its the maximal girth. By denoting the graph after reduction and its maximal girth as $(\mathbb{V}_R, \mathbb{E}_R, W_R)$ and g_R, we have*
$$\min\{\mathrm{g}_R, \alpha_2\} \leq \mathrm{g} \leq \min\{\max\{\mathrm{g}_R, \alpha_1\}, \alpha_2\},$$
where α_1 is the maximum of $W_{max} + W_{min}$ over Type I Operations and the second case of Type III Operations, and α_2 is the minimum of $W_{max} + W_{min}$ over the third case of Type III Operations. Here, W_{max}, W_{min} are defined in Lemma 13. If operations for computing α_1 or α_2 are never implemented, then we set α_1 to 0 or α_2 to $+\infty$.

Based on the numerical results in Tables I and II in [24] for large power networks, the values of α_1 and α_2 in Theorems 14 and 16 are usually smaller than e_R and g_R. Hence, we have the approximation
$$\mathrm{e} \approx \mathrm{e}_R, \quad \mathrm{g} \approx \alpha_2. \tag{5}$$
The above relations imply that for large power networks, computing the maximal eye is equivalent to computing the maximal eye of a reduced graph, while the maximal girth is already computed during the reduction process. Finally, we prove that 2-vertex-connected SP graphs can be reduced to a single edge by the black method.

Theorem 17. *If the underlying graph $(\mathbb{V}, \mathbb{E})$ of a power network is a 2-vertex-connected SP graph, then the black method reduces the underlying graph to a single edge.*

For an undirected graph without slack nodes, the classical series-parallel reduction (Type I'-III' Operations) can reduce the graph to a single edge if and only if the graph is a Generalized Series-Parallel (GSP) graph [23]. We note that 2-vertex-connected SP graphs are a special class of GSP graphs and it is unclear whether the reduction guarantee for theblack method can be extended to any GSP graphs in the presence of slack nodes.

VI. Numerical results

In this section, we verify the theoretical results of this work and test the performance of the proposed algorithms. First, we show that, using the black method, the computation of the maximal eye can be reduced to a smaller graph, while the computation of the maximal girth is finished during the process of reduction. Then, we show that Corollary 5 gives a valid sufficient condition for strong uniqueness. We use IEEE power networks in MATPOWER [27] to perform experiments. black

A. Computation of the maximal eye and the maximal girth

We first consider the computation of the maximal eye. The results are listed in Table I. Here, we use '-' to denote the case when this value does not exist, and use 'TLE' (Time Limit Exceeded) to denote the case when the algorithm does not find any leaf node in two days. The lower bounds for the maximal eye are derived by stopping the algorithm before it terminates. It can be observed that the black method can largely reduce the size of the graph, and therefore can accelerate the computing process. Moreover, the values of α_1 and α_2 are small compared to the maximal eye of the reduced graph. Hence, the approximation in equation (5) holds and the maximal eye of the original graph is equal to the maximal eye of the reduced graph. Although the algorithm achieves acceleration compared to the brute-force search method, we are only able to compute the maximal eye for graphs with up to 118 vertices. Note that since graph problems have exponential complexities, solving them for graphs having as low as 200 nodes is still beyond the current computational capabilities. However, this does not undermine the usefulness of the introduced graph parameters, since it is shown in this work that those parameters accurately decide whether the power flow problem has a unique solution.

Next, we consider the computation of the maximal girth. We use the same algorithms and the results are listed in the technical report [24]. In this case, it can be observed that α_2 is equal to 3 for large power networks. This is because the underlying graphs of large power networks considered in the table have "pendant triangles". Pendant triangles are triangles that have only one vertex connected to the rest of the graph. Furthermore, the approximation in Theorem 16 holds and the maximal girth of the original graph is equal to $\alpha_2 = 3$. Hence, the maximal girth can be computed during the reduction process. This shows that the conditions for the weak uniqueness is significantly loose and requiring $\omega_{k\ell}$ to be at most $2\pi/3$ for all edges $\{k,\ell\}$ is enough. However, for 2-vertex-connected SP graphs, we have shown that the maximal girth is equal to the maximal eye and the requirement for the weak uniqueness is the same as that for the strong uniqueness.
black
black

Power Network	Original Size	Reduced Size	α_1	α_2	e_R
Case 14	(14,20)	(2,1)	6	3	0
Case 30	(30,41)	(8,13)	4	3	8
Case 39	(39,46)	(8,12)	4	5	8
Case 57	(57,78)	(22,39)	4	-	23
Case 118	(118,179)	(44,83)	5	-	13
Case 300	(300,409)	(109,196)	8	4	$\geq$10
Case 1354	(1354,1710)	(263,500)	9	8	TLE
Case 2383	(2383,2886)	(499,949)	11	5	TLE

TABLE I: Number of vertices and edges before and after the black method for maximal eye along with values computed during the reduction process.

VII. Conclusion

In this paper, we extend the uniqueness theory of P-Θ power flow solutions developed in [1] for an AC power system. The notion of strong uniqueness is introduced to characterize the uniqueness in the common sense. We propose a general necessary and sufficient condition for the uniqueness of the solution, which depends only on the monotone regime and the network topology. These conditions can be greatly simplified in certain scenarios. When the underlying graph of the power network is a single cycle, sufficient conditions in [1] are proved to be necessary. For 2-vertex-connected SP graphs, we show that the maximal eye is equal to the maximal girth, which means that the sufficient condition for the weak uniqueness also implies the strong uniqueness. When the power network is lossless, we derive a necessary and sufficient condition that does not contain sinusoidal functions and its sufficient part is stronger than the general sufficient conditions. A reduction method, named the black method, is proposed to reduce the size of power network and accelerate the computation of the maximal eye and the maximal girth. The black method is proved to reduce a 2-vertex-connected SP graph to a single edge and the relation between the graphs before and after the reduction is analyzed. Some algorithms based on the DFS method with pruning are designed to compute the maximal eye and maximal girth.

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Appendix

A. Algorithms for computing maximal eye and maximal girth

In the appendix, we propose search-based algorithms for computing the maximal eye and the maximal girth. Our approach is based on the Depth-First Search (DFS) method and utilized the pruning technique to accelerate the computing process. We first describe a common sub-procedure that will be used in both algorithms. The sub-procedure computes the minimal directed chordless cycle containing a given edge. Given a truncation length $T \geq 1$, the sub-procedure returns the truncation length if there does not exist a directed chordless cycle that contains the given edge and has length at most T. The sub-procedure is also based on the DFS method with pruning and borrows the idea of *blocking* from [28] to accelerate the searching process. The pseudo-code of the sub-procedure is listed black

Next, we propose the algorithms for computing the maximal eye and the maximal girth. Since the algorithm of maximal girth is similar to the algorithm for maximal eye, we only present the algorithm for computing the maximal eye and offer the other one in [24]. The algorithm is also based on the DFS method with pruning, and the pseudo-code is provided black We first order all edges and gradually assign one of the directions $\{0, -1, +1\}$ to each edge following the ordering of the edges. The search space consists of the orientations for the first several edges (intermediate states) and the orientations for the entire graph (final states). One can verify that all intermediate states and final states form a trinomial[2] tree, since each orientation for the first $k < |\mathbb{E}|$ edges leads to three different orientations for the first $k+1$ edges. Then, the algorithm for computing the maximal eye searches in the same way as the classical DFS method on a directed tree. For each node, we consider the sub-graph consisting of those edges that have been assigned a direction. We compute the length of the minimal directed chordless cycle in the sub-graph, which contains the last edge in the sub-graph, using the sub-procedure. The truncation length can be decided as follows. Since a DFS method is implemented on a trinomial tree, there exists a directed path from the root node of the trinomial tree to the current node. The truncation length can be chosen as the minimal length computed on the preceding nodes of the path. When the search reaches a leaf node, we obtain an orientation for the entire graph, and the size of the eye becomes the minimal length on the path to the root node. By searching over all leaf nodes, we find the maximal eye. Similarly, one can use the pruning technique to reduce the search space. The current node is pruned if it can not be extended to a weakly feasible orientation for the entire graph, or the size of the eye of the sub-graph is smaller than the known maximal size of the eye.

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[2] A directed tree is called a trinomial tree if there is a root node and each non-leaf node has exactly three descendant nodes.

Appendix

A. Proof of Lemma 1

Proof. We only prove the strong uniqueness part since the proof for weak uniqueness is similar. For a given power network, we define the real power flow along the line $\{k,\ell\}\in\mathbb{E}$ from bus k in the direction of bus ℓ as

$$\tilde{p}_{k\ell}(\Theta) := -G_{k\ell}|v_k||v_\ell|\cos(\Theta_{k\ell}) + B_{k\ell}|v_k||v_\ell|\sin(\Theta_{k\ell}).$$

By definition, it follows that

$$\hat{P}_k(\Theta) = \sum_{\ell:\{k,\ell\}\in\mathbb{E}} \tilde{p}_{k\ell}(\Theta), \quad \forall k\in\mathbb{V}.$$

a) Proof of sufficiency: We first show by contradiction that statement 2 of the lemma is sufficient for statement 1. In particular, suppose that statement 2 holds, but the solution is not strongly unique for some graph $\mathbb{G}\in\mathcal{G}$ and some real power injection P while problem (1) is feasible. Then, there exist two different phase angle vectors Θ^1,Θ^2 such that $\Theta^2\in\mathcal{N}(\mathbb{G},\Theta^1,\mathcal{W})$ and $\hat{P}(\Theta^1)=\hat{P}(\Theta^2)$. For each line $\{k,\ell\}\in\mathbb{E}$, there exists a constant $C_{k\ell}>0$ such that

$$B_{k\ell} = C_{k\ell}\sin(\gamma_{k\ell}), \quad G_{k\ell} = C_{k\ell}\cos(\gamma_{k\ell}).$$

We calculate the change of power flow from k to ℓ as

$$\begin{aligned}
&\tilde{p}_{k\ell}(\Theta^1) - \tilde{p}_{k\ell}(\Theta^2)\\
&= -G_{k\ell}|v_k||v_\ell|[\cos(\Theta^1_{k\ell}) - \cos(\Theta^2_{k\ell})]\\
&\qquad + B_{k\ell}|v_k||v_\ell|[\sin(\Theta^1_{k\ell}) - \sin(\Theta^2_{k\ell})]\\
&= -C_{k\ell}\cos(\gamma_{k\ell})|v_k||v_\ell|[\cos(\Theta^1_{k\ell}) - \cos(\Theta^2_{k\ell})]\\
&\qquad + C_{k\ell}\sin(\gamma_{k\ell})|v_k||v_\ell|[\sin(\Theta^1_{k\ell}) - \sin(\Theta^2_{k\ell})]\\
&= (-\cos(\gamma_{k\ell})[\cos(\Theta^1_{k\ell}) - \cos(\Theta^2_{k\ell})]\\
&\qquad + \sin(\gamma_{k\ell})[\sin(\Theta^1_{k\ell}) - \sin(\Theta^2_{k\ell})])\cdot|v_k||v_\ell|C_{k\ell}\\
&= 2[\cos(\gamma_{k\ell})\sin(\Theta^1_{k\ell}/2 + \Theta^2_{k\ell}/2)\\
&\qquad + \sin(\gamma_{k\ell})\cos(\Theta^1_{k\ell}/2 + \Theta^2_{k\ell}/2)]\\
&\qquad \cdot\sin(\Delta_{k\ell}/2)|v_k||v_\ell|C_{k\ell}\\
&= 2\sin(\gamma_{k\ell} + \Theta^1_{k\ell}/2 + \Theta^2_{k\ell}/2)\cdot\mathrm{sign}(\sin(\Delta_{k\ell}/2))\\
&\qquad \cdot|\sin(\Delta_{k\ell}/2)||v_k||v_\ell|C_{k\ell}\\
&:= \delta_{k\ell}\cdot|\sin(\Delta_{k\ell}/2)v_kv_\ell|C_{k\ell},
\end{aligned}$$

where

$$\begin{aligned}
&\Delta_{k\ell} := \Theta^1_{k\ell} - \Theta^2_{k\ell}, \qquad (6)\\
&\delta_{k\ell} := 2\sin(\gamma_{k\ell} + \Theta^1_{k\ell}/2 + \Theta^2_{k\ell}/2)\mathrm{sign}(\sin(\Delta_{k\ell}/2)).
\end{aligned}$$

Note that the third equality in (6) is due to the following triangular identities:

$$\begin{aligned}
\cos(\eta) - \cos(\varphi) &= -2\sin[(\eta-\varphi)/2]\sin[(\eta+\varphi)/2],\\
\sin(\eta) - \sin(\varphi) &= 2\sin[(\eta-\varphi)/2]\cos[(\eta+\varphi)/2].
\end{aligned}$$

Since $\hat{P}_k(\Theta^1)=\hat{P}_k(\Theta^2)$ for all $k\neq 1$, we obtain

$$\begin{aligned}
&\hat{P}_k(\Theta^1) - \hat{P}_k(\Theta^2) = \sum_{\ell:\{k,\ell\}\in\mathbb{E}} \left[\tilde{p}_{k\ell}(\Theta^1) - \tilde{p}_{k\ell}(\Theta^2)\right]\\
&= \sum_{\ell:\{k,\ell\}\in\mathbb{E}} \delta_{k\ell}\cdot|\sin(\Delta_{k\ell}/2)v_kv_\ell|C_{k\ell} = 0
\end{aligned}$$

for all $k\neq 1$. Let the set $\mathbb{E}_0$ be the subset of edges such that $\Delta_{k\ell}\neq 0$ for all $\{k,\ell\}\in\mathbb{E}_0$; we assign an order to elements in $\mathbb{E}_0$. Define the matrix $M\in\mathbb{R}^{|\mathbb{V}|\times|\mathbb{E}_0|}$ and the vector $\mathbf{g}\in\mathbb{R}^{|\mathbb{E}_0|}$ as

$$M_{ki} := \delta_{k\ell}, \quad M_{\ell i} := \delta_{\ell k}, \quad \mathbf{g}_i := |\sin(\Delta_{k\ell}/2)v_kv_\ell|C_{k\ell},$$

where $\{k,\ell\}$ is the i-th edge in the set $\mathbb{E}_0$. Since $\Delta_{k\ell}\neq 0$ for all $\{k,\ell\}\in\mathbb{E}_0$ and $\Delta_{k\ell}\leq 2\gamma_{k\ell}\leq\pi$, it holds that

$$|\sin(\Delta_{k\ell}/2)| > 0, \quad \forall\{k,\ell\}\in\mathbb{E}_0.$$

Then, the vector $\mathbf{g}$ is a solution to the linear feasibility problem

$$\text{find } \mathbf{x}\in\mathbb{R}^{|\mathbb{E}_0|} \quad \text{s.t. } (M\mathbf{x})_{2:|V|} = 0, \quad \mathbf{x} > 0.$$

where $(y)_{i:j} := (y_i, y_{i+1},\dots,y_j)$ includes the i-th to the j-th entries of the vector y and inequality $x>0$ means that $x_k>0$ holds for all entries of the vector x. The notation $x\geq 1$ is defined in the same way. The above feasibility problem is equivalent to

$$\text{find } \mathbf{x}\in\mathbb{R}^{|\mathbb{E}_0|} \quad \text{s.t. } (M\mathbf{x})_{2:|V|} = 0, \quad \mathbf{x} \geq 1.$$

Then, by Farka's Lemma, the dual feasibility problem

$$\text{find } \mathbf{y}\in\mathbb{R}^{|\mathbb{V}|} \quad \text{s.t. } M^T\mathbf{y}\geq 0, \quad \mathbf{1}^TM^T\mathbf{y} > 0, \quad y_1 = 0$$

is infeasible. However, the conditions in the dual problem are the same as the conditions in statement 2 of Lemma 1. This contradicts the claim in statement 2 that there exists a vector $\mathbf{y}$ satisfying these conditions. Thus, statement 1 must hold true.

b) Proof of necessity: Next, we again show by contradiction that statement 2 of the lemma is necessary for statement 1. Assume that statement 1 holds true, and that there exist two different phase angle vectors Θ^1,Θ^2 in the monotone regime such that $\Theta^2\in\mathcal{N}(\mathbb{G},\Theta^1,\mathcal{W})$ while there does not exist $\mathbf{y}$ satisfying the conditions in statement 2. We define $\mathbb{E}_0$ as the set of edges such that $\Delta_{k\ell}\neq 0$, where $\Delta_{k\ell} := \Theta^1_{k\ell} - \Theta^2_{k\ell}$ for all $\{k,\ell\}\in\mathbb{E}_0$. We construct the matrix $M\in\mathbb{R}^{|\mathbb{V}|\times|\mathbb{E}_0|}$ as

$$M_{ki} := \delta_{k\ell}, \quad M_{\ell i} := \delta_{\ell k},$$

where $\{k,\ell\}$ is the i-th edge in the set $\mathbb{E}_0$ and

$$\delta_{k\ell} := \sin(\gamma_{k\ell} + \Theta^1_{k\ell}/2 + \Theta^2_{k\ell}/2)\mathrm{sign}(\sin(\Delta_{k\ell}/2)).$$

By the same analysis, the conditions in statement 2 turn out to be equivalent to the feasibility of the linear feasibility problem

$$\text{find } \mathbf{y}\in\mathbb{R}^{|\mathbb{V}|} \quad \text{s.t. } M^T\mathbf{y}\geq 0, \quad \mathbf{1}^TM^T\mathbf{y} > 0, \quad y_1 = 0.$$

By our assumption, the above problem is infeasible. By Farka's Lemma, there exists a solution $\mathbf{g}\in\mathbb{R}^{|\mathbb{E}_0|}$ to the feasibility problem

$$\text{find } \mathbf{x}\in\mathbb{R}^{|\mathbb{E}_0|} \quad \text{s.t. } (M\mathbf{x})_{2:|V|} = 0, \quad \mathbf{x} \geq 1$$

and also to the feasibility problem

$$\text{find } \mathbf{x}\in\mathbb{R}^{|\mathbb{E}_0|} \quad \text{s.t. } (M\mathbf{x})_{2:|V|} = 0, \quad \mathbf{x} > 0.$$

We define the matrix $C\in\mathbb{R}^{|\mathbb{V}|\times|\mathbb{V}|}$ as

$$C_{k\ell} := |\sin(\Delta_{k\ell}/2)v_kv_\ell|^{-1}\mathbf{g}_i, \quad \forall\{k,\ell\}\in\mathbb{E}_0,$$

where $\{k,\ell\}$ is the i-th edge in the set $\mathbb{E}_0$, and

$$C_{k\ell} := 1, \quad \forall\{k,\ell\}\in\mathbb{E}\backslash\mathbb{E}_0, \quad C_{k\ell} := 0, \quad \forall\{k,\ell\}\notin\mathbb{E}.$$

By the definition, it follows that $C_{k\ell} > 0$ for all $\{k,\ell\} \in \mathbb{E}$. We construct a graph $\mathbb{G}$ with the complex admittance matrix

$$Y_{k\ell} := C_{k\ell}\cos(\gamma_{k\ell}) - \mathbf{j}C_{k\ell}\sin(\gamma_{k\ell}), \quad \forall\{k,\ell\} \in \mathbb{E}.$$

Then, for all $k \neq 1$, we have

$$\hat{P}_k(\Theta^1) - \hat{P}_k(\Theta^2) = \sum_{\ell:\{k,\ell\}\in\mathbb{E}} \left[\tilde{p}_{k\ell}(\Theta^1) - \tilde{p}_{k\ell}(\Theta^2)\right]$$
$$= \sum_{\ell:\{k,\ell\}\in\mathbb{E}} \delta_{k\ell}\cdot|\sin(\Delta_{k\ell}/2)v_k v_\ell|C_{k\ell} = (M\mathbf{g})_k = 0.$$

This implies that Θ^1 and Θ^2 are both solutions to problem (1) in the monotone regime when the real power injection is

$$P := \hat{P}(\Theta^1).$$

This contradicts statement 1 that the solution is strongly unique for any real power injection. Hence, the conditions in statement 2 must be satisfied. □

B. Proof of Lemma 2

Proof. We only prove the strong uniqueness part since the proof for weak uniqueness is similar. Since the induced orientation A is not a weakly feasible orientation, there exists a vertex $i \neq 1$ such that it has nonzero out-degree and zero in-degree, or it has nonzero in-degree and zero out-degree. Without loss of generality, assume that the vertex i has nonzero out-degree and zero in-degree. We prove that the i-th unit vector $\mathbf{y} := \mathbf{e}_i$ satisfies the conditions in statement 1 of Lemma 1. It is straightforward that $y_1 = 0$. We only need to show that the inequalities in (2) hold and at least one of them is strict. We consider any edge (k,ℓ) such that $\Delta_{k\ell} > 0$. First, if $k \neq i$ and $\ell \neq i$, then both sides of the inequality (2) are zero. Next, if $k \neq i$ and $\ell = i$, then the condition $\Delta_{ki} > 0$ implies that $A_{ki} = +1$, which contradicts the assumption that i has zero in-degree. Finally, if $k = i$ and $\ell \neq i$, the goal is to prove that

$$\sin(\gamma_{i\ell} + \Theta^1_{i\ell}/2 + \Theta^2_{i\ell}/2)\cdot y_i$$
$$> \sin(\gamma_{i\ell} - \Theta^1_{i\ell}/2 - \Theta^2_{i\ell}/2)\cdot y_\ell.$$

Since $y_i = 1$ and $y_\ell = 0$, the above inequality is equivalent to

$$\sin(\gamma_{i\ell} + \Theta^1_{i\ell}/2 + \Theta^2_{i\ell}/2) > 0.$$

Recalling the assumption that $\Theta^1_{i\ell}$ and $\Theta^2_{i\ell}$ are in the monotone regime $[-\gamma_{i\ell}, \gamma_{i\ell}]$, one can write

$$\gamma_{i\ell} + \Theta^1_{i\ell}/2 + \Theta^2_{i\ell}/2 \in [0, 2\gamma_{i\ell}] \subset [0,\pi].$$

Hence, it is enough to show that

$$\gamma_{i\ell} + \Theta^1_{i\ell}/2 + \Theta^2_{i\ell}/2 \in (0, 2\gamma_{k\ell}) \subset (0,\pi).$$

If $\gamma_{i\ell} + \Theta^1_{i\ell}/2 + \Theta^2_{i\ell}/2 = 0$, then it holds that

$$\Theta^1_{i\ell} = \Theta^2_{i\ell} = -\gamma_{i\ell}.$$

This contradicts the inequality $\Delta_{i\ell} = \Theta^1_{i\ell} - \Theta^2_{i\ell} > 0$. If $\gamma_{i\ell} + \Theta^1_{i\ell}/2 + \Theta^2_{i\ell}/2 = 2\gamma_{k\ell}$, then it holds that

$$\Theta^1_{i\ell} = \Theta^2_{i\ell} = \gamma_{i\ell},$$

which also contradicts the inequality $\Delta_{i\ell} > 0$. Combining the two cases, we obtain that $\sin(\gamma_{i\ell} + \Theta^1_{i\ell}/2 + \Theta^2_{i\ell}/2) > 0$ and the inequality

$$\sin(\gamma_{i\ell} + \Theta^1_{i\ell}/2 + \Theta^2_{i\ell}/2)\cdot y_i$$
$$> \sin(\gamma_{i\ell} - \Theta^1_{i\ell}/2 - \Theta^2_{i\ell}/2)\cdot y_\ell.$$

holds strictly. It follows that $\mathbf{y} = \mathbf{e}_i$ satisfies the conditions in statement 2 of Lemma 1. □

black

C. Proof of Corollary 4

Proof. We only prove the strong uniqueness part since the proof for weak uniqueness is similar. Suppose that Θ^1 and Θ^2 are two solutions to problem (1) in the monotone regime such that $\Theta^2 \in \mathcal{N}(\mathbb{G}, \Theta^1, \mathcal{W})$. Using the results of Theorem 3, we only need to show that the induced orientation of $\Theta^1 - \Theta^2$ is not weakly feasible. Assume conversely that the induced orientation A is a weakly feasible orientation. Then, by hypothesis, there exists a directed cycle $(k_1,\dots,k_t)$ containing at least one normal edge such that

$$\sum_{k_i k_{i+1} \text{ is normal}} \omega_{k_i k_{i+1}} < 2\pi, \tag{7}$$

where $k_{t+1} := k_1$. We denote $\Delta_{k\ell} := \Theta^1_{k\ell} - \Theta^2_{k\ell}$ and it follows that

$$0 < \Delta_{k_i k_{i+1}} \leq \omega_{k_i k_{i+1}} \quad \forall i \text{ s.t. } \{k_i, k_{i+1}\} \text{ is normal}, \tag{8}$$
$$\Delta_{k_i k_{i+1}} = 0 \quad \forall i \text{ s.t. } \{k_i, k_{i+1}\} \text{ is not normal},$$

where the right part of the first inequality is because $\Theta^2 \in \mathcal{N}(\mathbb{G}, \Theta^1, \mathcal{W})$. Combining inequalities (7) and (8) yields that

$$0 < \sum_{i=1}^t \Delta_{k_i k_{i+1}} = \sum_{k_i k_{i+1} \text{ is normal}} \Delta_{k_i k_{i+1}} \tag{9}$$
$$\leq \sum_{k_i k_{i+1} \text{ is normal}} \omega_{k_i k_{i+1}} < 2\pi.$$

However, by the definition of $\Delta_{k\ell}$ and $\Theta_{k\ell}$, one can write

$$\sum_{i=1}^t \Delta_{k_i k_{i+1}} = \sum_{i=1}^t \Theta^1_{k_i k_{i+1}} - \sum_{i=1}^t \Theta^2_{k_i k_{i+1}}$$
$$= \sum_{i=1}^t \left[\Theta^1_{k_i} - \Theta^1_{k_{i+1}}\right] - \sum_{i=1}^t \left[\Theta^2_{k_i} - \Theta^2_{k_{i+1}}\right] = 0,$$

where the second last equality is the congruence relation module 2π and the last equality is because $(k_1,\dots,k_t)$ is a cycle. This contradicts equation (9). Thus, the induced orientation is not a weakly feasible orientation and the strong uniqueness holds. □

black

D. Proof of Theorem 6

Proof. To prove the first inequality, we only need to notice that any feasible orientation is also a weakly feasible orientation and the size of eye is equal to the girth when all edges are normal.

Then, we consider the second inequality. Assume conversely that the maximal eye is attained by a directed cycle with chords in the weakly feasible orientation A. Without loss of generality, assume that the directed cycle $(1,\ldots,t)$ attains the maximal eye *with fewest chords*, where $t \geq \mathrm{e}(\mathbb{G})$ and $\{1,i\} \in \mathbb{E}$ is a chord for some $i \in \{3,\ldots,t-1\}$. We consider four different cases:

1. $A_{1,i} = 0$: Consider the directed cycle
$$(1,i,i+1,\ldots,t),$$
which has at most $\mathrm{e}(\mathbb{G})$ normal edges and strictly fewer chords than $(1,\ldots,t)$. This contradicts the assumption that the cycle $(1,\ldots,t)$ is a directed cycle that attains the size of eye with fewest chords.
2. $A_{1,i} = +1$: and there exists at least one normal edge among $\{1,2\},\ldots,\{i-1,i\}$: The directed cycle
$$(1,i,i+1,\ldots,t)$$
has at most $\mathrm{e}(\mathbb{G})$ normal edges and strictly fewer chords than $(1,\ldots,t)$. This also contradicts the assumption on $(1,\ldots,t)$.
3. $A_{1,i} = +1$ and edges $\{1,2\},\ldots,\{i-1,i\}$ are not normal: Consider the directed cycle
$$(1,i,i-1,\ldots,2),$$
which has exactly one normal edge and strictly fewer chords. By the definition of the maximal eye, we know $\mathrm{e}(\mathbb{G}) \geq 1$ and the cycle $(1,i,i-1,\ldots,2)$ has at most $\mathrm{e}(\mathbb{G}) \geq 1$ normal edges. Hence, this contradicts the assumption on $(1,\ldots,t)$.
4. $A_{1,i} = -1$: Consider the orientation $\tilde{A}$ defined as
$$\tilde{A}_{k\ell} := -A_{k\ell}, \quad \forall \{k,\ell\} \in \mathbb{E}$$
and use the discussion in the first three cases.

Combining the above four cases concludes that the maximal eye of the power network $\mathbb{G}$ must be attained by a chordless cycle. Hence, the maximal eye is upper bounded by the longest chordless cycle. □

E. Proof of Lemma 7

Proof. By the definition of strong uniqueness and weak uniqueness, if a solution to problem (1) is strongly unique, than it is also weakly unique. We only need to consider the other direction. Assume conversely that there exists a solution Θ^1 in the monotone regime that is weakly unique but not strongly unique. Then, there exists another solution $\Theta^2 \in \mathcal{N}(\mathbb{G},\Theta^1,\mathcal{W})$ that is different from Θ^1 according to Definition 5. Then, the phase difference of some line is different for the two solutions. Considering the power injection balance at each bus, we know that the phase difference is different at all lines. This means that the two solutions Θ^1 and Θ^2 are different according to Definition 4, which contradicts the assumption that Θ^1 is weakly unique. □

F. Proof of Theorem 8

Proof. The sufficient part is proved in Corollary 4 and we only prove the necessary part. In this proof, bus $n+1$ is defined as bus 1. We assume that
$$\sum_{i=1}^{n} \omega_{i,i+1} \geq 2\pi$$
We construct a power network $\mathbb{G} \in \mathcal{G}$ and power injection P such that there exist two different solutions Θ^1, Θ^2 in the monotone regime and $\Theta^2 \in \mathcal{N}(\mathbb{G},\Theta^1,\mathcal{W})$. Without loss of generality, assume that
$$\sum_{i=1}^{n} \omega_{i,i+1} = 2\pi.$$
This is because the construction for
$$\tilde{W} := \left\{ \frac{2\pi}{\sum_{j=1}^{n} \omega_{j,j+1}} \cdot \omega_{i,i+1} : i \in [n] \right\}.$$
also works for the original $W = \{\omega_{i,i+1} : i \in [n]\}$ if $\sum_{j=1}^{n} \omega_{j,j+1} \geq 2\pi$. We define two phase angle vectors as
$$\Theta_1^1 := 0, \quad \Theta_i^1 := \sum_{j=2}^{i} \omega_{j,j+1}, \quad \forall i \in \{2,\ldots,n\},$$
$$\Theta_i^2 := 0, \quad \forall i \in [n].$$
Then, it follows that
$$\Theta_{i,i+1}^1 = \omega_{i,i+1}, \quad \Theta_{i,i+1}^2 = 0, \quad \forall i \in [n],$$
which means that Θ^1 and Θ^2 are both in the monotone regime. Since $\omega_{i,i+1}, \gamma_{i,i+1} \in (0,\pi/2]$, we know that $\gamma_{i,i+1}+\omega_{i,i+1} \in (0,\pi]$ and therefore, by the monotonicity of $\cos(\cdot)$ in $[0,\pi]$, we have
$$\cos(\gamma_{i,i+1}+\omega_{i,i+1}) < \cos(\gamma_{i,i+1}).$$
For each line $\{i,i+1\}$, we define the positive constant
$$C_{i,i+1} := |v_i v_{i+1}|^{-1}[-\cos(\gamma_{i,i+1}+\omega_{i,i+1}) + \cos(\gamma_{i,i+1})]^{-1}$$
and the complex admittance
$$B_{i,i+1} := \sin(\gamma_{i,i+1})C_{i,i+1}, \quad G_{i,i+1} := \cos(\gamma_{i,i+1})C_{i,i+1}.$$
We use $\tilde{p}_{i,i+1}(\Theta)$ to denote the real power flow from bus i to bus $i+1$ given the phase angle vectors Θ. Then, we can calculate that
$$\begin{aligned}
&\tilde{p}_{i,i+1}(\Theta^1) - \tilde{p}_{i,i+1}(\Theta^2) \\
&= -G_{i,i+1}|v_i v_{i+1}|[\cos(\Theta_{i,i+1}^1) - \cos(\Theta_{i,i+1}^2)] \\
&\qquad + B_{k\ell}|v_i v_{i+1}|[\sin(\Theta_{i,i+1}^1) - \sin(\Theta_{i,i+1}^2)] \\
&= -\cos(\gamma_{i,i+1})C_{i,i+1}|v_i v_{i+1}|[\cos(\omega_{i,i+1}) - 1] \\
&\qquad + \sin(\gamma_{i,i+1})C_{i,i+1}|v_i v_{i+1}|\sin(\omega_{i,i+1}) \\
&= C_{i,i+1}|v_i v_{i+1}|\cdot[-\cos(\gamma_{i,i+1}+\omega_{i,i+1}) + \cos(\gamma_{i,i+1})] \\
&= 1.
\end{aligned}$$
It follows that
$$\begin{aligned}
&\hat{P}_i(\Theta^1) - \hat{P}_i(\Theta^2) \\
&= \left[\tilde{p}_{i-1,i}(\Theta^1) - \tilde{p}_{i-1,i}(\Theta^2)\right] - \left[\tilde{p}_{i,i+1}(\Theta^1) - \tilde{p}_{i,i+1}(\Theta^2)\right] \\
&= 1 - 1 = 0.
\end{aligned}$$

If we choose $P := \hat{P}(\Theta^1)$, then Θ^1 and Θ^2 are two different solutions to problem (1) in the monotone regime such that $\Theta^2 \in \mathcal{N}(\mathbb{G}, \Theta^1, \mathcal{W})$ and that the strong uniqueness does not hold.

□

G. Proof of Lemma 10

Proof. For the notational simplicity, we denote the maximal eye and the maximal girth of the graph $(\mathbb{V}, \mathbb{E}, W)$ as e and g, respectively. Since the graph is 2-vertex-connected, there does not exist a degree-1 vertex. By Lemmas 13 and 15, Type II Operations do not change the maximal eye and the maximal girth of the graph. Moreover, the graph has a nested ear decomposition $\{L_0, L_1, \ldots, L_{r-1}\}$ by Theorem 9. Hence, we can assume that there is no degree-2 vertex except the slack bus. Assume conversely that graph $(\mathbb{V}, \mathbb{E}, W)$ is the 2-vertex-connected SP graph with minimal number of ears such that $\mathrm{e} > \mathrm{g}$. We will show that there must exist another graph with fewer ears in the ear decomposition and $\mathrm{e} > \mathrm{g}$. This will lead to a contradiction with our assumption that this graph has the minimal number of ears. If the graph has at most two ears, then the graph is a single line of a cycle and we know $\mathrm{e} = \mathrm{g}$. Hence, there exist at least three ears in the graph $(\mathbb{V}, \mathbb{E}, W)$.

a) Step 1: In this step, we prove that the graph has a pair of parallel edges that contains a leaf ear, which we will define below. Since a nested ear decomposition is also a tree decomposition, we can assign a directed tree structure to ears in the decomposition. Here, we call an ear L_k a **descendant ear** of L_ℓ if L_k is a descendant node of L_ℓ on the directed tree, or equivalently, both endpoints of ear L_k are on L_ℓ and at least one of them is different from the endpoints of L_ℓ. We also call ear L_ℓ the **precedent ear** of L_k. For any ear L_ℓ, we say that ear L_k is a **smallest descendant ear** of L_ℓ if L_k is a descendant ear of L_ℓ and there does not exist another ear L_i such that L_i is also a descendant ear of L_ℓ and the interval formed by the endpoints of L_i on L_ℓ is a strict subset of the interval formed by the endpoints of L_k. We note that each ear may have multiple smallest descendant ears. We say that an ear is a **leaf ear** if it is the smallest descendant ear of some ear and has no descendant ear. We denote the set of leaf ears as $\mathcal{L}$. Considering the directed tree structure of the ear decomposition, we know that the set $\mathcal{L}$ is not empty.

Suppose that L_k is a leaf ear with the endpoints k_1, k_2 and that L_ℓ is the precedent ear of L_k. Since we have deleted all degree-2 vertices except the slack bus, ear L_k is either a single line $\{k_1, k_2\}$ or two edges $\{k_1, k_3\}$ and $\{k_2, k_3\}$ connecting the endpoints to the slack bus k_3. Similarly, the path connecting the two endpoints of L_k on the precedent ear L_ℓ, which we denote as P_k, is either a single line or contains the slack bus. Considering the ear L_k and the path P_k, there are two cases: two parallel edges with endpoints $\{k_1, k_2\}$, or one is a single line and the other is two edges with the slack bus. If the first case occurs, we have a pair of parallel edges containing a leaf ear. Now, we consider the second case. If we exchange the two paths, i.e., let P_k be a leaf ear and L_k be a path on the precedent ear, then the structure of nested ear decomposition is not changed. Hence, without loss of generality, assume that L_k is a single line and P_k contains the slack bus. If there exists an ear L_j different from L_ℓ that also contains leaf ears, then by the uniqueness of slack bus, the first case occurs for leaf ears on ear L_j.

Hence, we simply need to consider the case when L_ℓ is the only ear that contains leaf ears. We consider the root ear L_0. By the definition of tree ear decomposition, we know that L_0 is a single line; let ℓ_1, ℓ_2 be the two endpoints of L_0. Since all vertices except the slack bus have degree at least 3 and the slack bus is not an endpoint of ears, both ℓ_1 and ℓ_2 have degree at least 3. This implies that the root ear L_0 has at least 2 descendant ears and all descendant ears have endpoints ℓ_1, ℓ_2. Let $L_{k_1}, L_{k_2}, \ldots, L_{k_m}$ be the descendant ears of L_0. For each L_{k_i}, we define a sub-graph of $(\mathbb{V}, \mathbb{E}, W)$ consisting of ear L_0 and ears that are descendant nodes of L_{k_i} in the directed tree of ears. We can verify that each sub-graph also has a nested ear decomposition and therefore contains at least one leaf ear, which implies that ear L_ℓ belongs to all sub-graphs. On the other hand, due to the tree structure, the intersection of two different sub-graphs is ear L_0 and is not a leaf ear. Hence, the leaf ears in different sub-graphs are different and $L_\ell = L_0$. It follows that all descendant ears of L_0 are leaf ears and they form at least a pair of parallel edges containing a leaf ear.

b) Step 2: In this step, we construct a nested ear decomposition of the graph $(\mathbb{V}, \mathbb{E}, W)$ such that there exists a pair of parallel edges that contains the root ear L_0 and that all edges are ears in the ear decomposition. According to Step 1, there exists a pair of parallel edges that contains a leaf ear. We denote the leaf ear in the pair of parallel edges as L_k. We consider the (undirected) cycle containing L_0 and L_k. Suppose that the cycle has a non-empty edge intersection with ears $L_{k_0}, \ldots, L_{k_t}$, where $k_0 = 0, k_t = k$ and $L_{k_{s+1}}$ is a descendant ear of L_{k_s} for $s = 0, 1, \ldots, t-1$. Notice that the endpoints of each ear L_{k_s} are on the cycle. Now, we construct a new nested ear decomposition $\tilde{L}_0, \ldots, \tilde{L}_{m-1}$ such that $L_k = \tilde{L}_0$ is the root ear. We define $\tilde{L}_0 := L_k$ and $\tilde{L}_k$ as the remaining part of the cycle. For ears L_{k_s} with $1 \le s \le t-1$, we define $\tilde{L}_{k_s}$ as the ear L_{k_s} with edges on the cycle deleted. For ears that do not intersect with the cycle, we define $\tilde{L}_i := L_i$. It is desirable to show that with the new set of ears still forms a nested ear decomposition. To this end, we analyze three cases:

- **Case I.** First, it can be verified that ears $\tilde{L}_{k_1}, \ldots, \tilde{L}_{k_{t-1}}$ are nested ears on $\tilde{L}_{k_t}$. Hence, ears $\tilde{L}_{k_0}, \ldots, \tilde{L}_{k_t}$ still form a nesting structure.
- **Case II.** Next, we consider an ear $\tilde{L}_i = L_i$ that is not changed and has both endpoints on L_{k_s} for some $s \in \{0, 1, \ldots, t-1\}$. Since $L_{k_{s+1}}$ is a descendant ear on L_{k_s}, by the definition of nested ear decomposition, we know that the endpoints of $\tilde{L}_i$ are either both on $\tilde{L}_{k_s}$ or both on $\tilde{L}_{k_t}$. For the first case, L_i is an ear on $\tilde{L}_{k_s}$ and ears on $\tilde{L}_{k_s}$ have the same nesting structure as L_{k_s}. For the second case, both endpoints of L_k locate on $\tilde{L}_{k_t}$ and are nested between the endpoints of $\tilde{L}_{k_s}$ and $\tilde{L}_{k_{s-1}}$. We note that for the case when $s = 0$, both endpoints are equal to the endpoints of L_0 and they form the smallest possible interval on $\tilde{L}_{k_t}$. Hence, ears on $\tilde{L}_{k_t}$ also have a nested structure.

- **Case III.** Finally, we consider ears that are not changed and do not have endpoints on L_{k_s} for any $s = 0, \ldots, t$. These ears still form a nested structure on the original precedent ear and the nested ear decomposition structure is not changed.

Combining the above three cases concludes that the new set of ears is also a nested ear decomposition. Moreover, the topological structure of the graph is not changed. Hence, in the new ear decomposition, the root ear $\tilde{L}_0 = L_k$ has parallel edges. Finally, we observe that the parallel edges of the root ear are also ears in the ear decomposition.

c) Step 3: Suppose that the maximal eye is achieved by the weakly feasible orientation A. In this step, we show that we can modify A such that each edge with direction 0 is incident to a degree-0 vertex and the size of eye is not changed. Here, the degree is calculated for the directed graph with orientation A and all edges with orientation 0 are not counted towards the degree. We define a partition of vertices as

$$\mathbb{V}_1 := \{k \in \mathbb{V} \mid \deg(k) > 0 \text{ or } k \text{ is the slack bus}\},$$
$$\mathbb{V}_2 := \{k \in \mathbb{V} \mid \deg(k) = 0 \text{ and } k \text{ is not the slack bus}\}$$

and a partition of edges as

$$\mathbb{E}_1 := \{\{k,\ell\} \in \mathbb{E} \mid A_{k\ell} \in \{+1,-1\}\},$$
$$\mathbb{E}_2 := \{\{k,\ell\} \in \mathbb{E} \mid A_{k\ell} = 0,\ k \in \mathbb{V}_1 \text{ and } \ell \in \mathbb{V}_1\},$$
$$\mathbb{E}_3 := \{\{k,\ell\} \in \mathbb{E} \mid A_{k\ell} = 0,\ k \in \mathbb{V}_2 \text{ or } \ell \in \mathbb{V}_2\}.$$

Then, the objective is to show that there exists a weakly feasible orientation such that the size of eye is still e and the set $\mathbb{E}_2$ is empty. For any edge $\{k,\ell\} \in \mathbb{E}_2$, we can arbitrarily assign direction $+1$ or -1 to the edge and the orientation is still weakly feasible. This is because for vertices in $\mathbb{V}_1$, the requirement on in-degree and out-degree is satisfied by other edges. More specifically, if the degree of k or ℓ is nonzero, then by the definition of weakly feasible orientation, the vertex already has nonzero in-degree and out-degree. Otherwise, if k or ℓ is the slack bus, then the in-degree and out-degree can be arbitrary. Thus, we can arbitrarily assign directions $+1$ or -1 to all edges in $\mathbb{E}_2$ and the new orientation is still weakly feasible. We define a new orientation as

$$\tilde{A}_{k\ell} := \begin{cases} +1 & \text{if } k > \ell \\ -1 & \text{otherwise} \end{cases}, \quad \forall \{k,\ell\} \in \mathbb{E}_2,$$
$$\tilde{A}_{k\ell} := A_{k\ell}, \quad \forall \{k,\ell\} \in \mathbb{E}_1 \cup \mathbb{E}_3.$$

We prove that with orientation $\tilde{A}$, the size of eye is not changed. Let $(k_1, \ldots, k_t)$ be a directed cycle in the graph with orientation $\tilde{A}$. If some edges of this cycle are in $\mathbb{E}_1 \cup \mathbb{E}_3$, then this cycle also exists in the graph with A. By assigning directions ± 1 to edges with direction 0, the lengths of the cycles are not decreased and therefore the length of $(k_1, \ldots, k_t)$ is at least e under the orientation $\tilde{A}$. If all edges of this cycle are in $\mathbb{E}_2$, then we choose the minimal index in $\{k_1, \ldots, k_t\}$, which is assumed to be k_1 without loss of generality. By the definition of $\tilde{A}$, the edge $\{k_1, k_2\}$ has orientation $\tilde{A}_{k_1 k_2} = -1$, which contradicts the fact that $(k_1, \ldots, k_t)$ is a directed cycle with $\tilde{A}$. Combining the above two cases, it can be inferred that the size of eye with orientation $\tilde{A}$ is at least e. On the other hand, e is defined to be the maximal eye. Hence, the size of eye with orientation $\tilde{A}$ is equal to e.

d) Step 4: In this step, we prove that the maximal eye is equal to the maximal girth. Suppose that the maximal eye is achieved by the weakly feasible orientation A and orientation A satisfies the conditions in Steps 2-3. We consider the set of parallel edges containing the root ear, which we denote as $\{k,\ell,1\}, \ldots, \{k,\ell,t\}$ for some $t \geq 2$. We analyze two different cases:

- **Case I.** If there exists at least one parallel edge having direction 0, then by the conditions in Step 3, we know that at least one of the endpoints k, ℓ has degree 0. This means that all parallel edges have direction 0. We construct another graph $(\tilde{\mathbb{V}}, \tilde{\mathbb{E}}, \tilde{W})$, where the parallel edges $\{k,\ell,1\}, \ldots, \{k,\ell,t\}$ are substituted by a single edge $\{k,\ell\}$ and the weight of the new edge is the minimal weight among all parallel edges, i.e.,
$$\tilde{W}_{k\ell} := \min_{s \in [t]} \ W_{k,\ell,s}.$$
Other edges are the same as those in the original graph. We construct a weakly feasible orientation $\tilde{A}$ for the new graph. For the edge $\{k,\ell\}$, we define
$$\tilde{A}_{k\ell} := 0.$$
For other edges, we define
$$\tilde{A}_{k_1\ell_1} := A_{k_1\ell_1} \quad \forall \{k_1,\ell_1\} \in \mathbb{E} \backslash \{\{k,\ell,1\}, \ldots, \{k,\ell,t\}\}.$$
Since the orientations $\tilde{A}$ and A have the same degree at each node, $\tilde{A}$ also becomes weakly feasible. Moreover, the size of eye of the graph with $\tilde{A}$ is also equal to e, which implies that the maximal eye of the new graph $\tilde{\mathrm{e}}$ is at least e. Since the new graph $(\tilde{\mathbb{V}}, \tilde{\mathbb{E}}, \tilde{W})$ has $t-1$ fewer ears, the induction assumption implies that the maximal girth of the new graph $\tilde{\mathrm{g}}$ satisfies
$$\tilde{\mathrm{g}} = \tilde{\mathrm{e}} \geq \mathrm{e}.$$
Hence, we can choose a feasible orientation $\tilde{A}^g$ such that the girth is equal to $\tilde{\mathrm{g}}$. Now, we extend the feasible orientation $\tilde{A}^g$ to be a feasible orientation of the original graph $(\mathbb{V}, \mathbb{E}, W)$. We define
$$A^g_{k_1\ell_1} := \tilde{A}^g_{k_1\ell_1} \quad \forall \{k_1,\ell_1\} \in \mathbb{E} \backslash \{\{k,\ell,1\}, \ldots, \{k,\ell,t\}\}$$
and
$$A^g_{k,\ell,s} := \tilde{A}^g_{k\ell}, \quad \forall s \in [t].$$
Since the in-degree and out-degree at points k, ℓ are still nonzero for the orientation A^g, it can be concluded that A^g is a feasible orientation for the original graph. Moreover, the girth of the original graph with orientation A^g is equal to $\tilde{\mathrm{g}}$. It follows that the maximal girth g is at least $\tilde{\mathrm{g}} \geq \mathrm{e}$. This contradicts the assumption that $\mathrm{e} > \mathrm{g}$.
- **Case II.** Next, we consider the case when all parallel edges $\{k,\ell,1\}, \ldots, \{k,\ell,t\}$ are normal edges. In this case, the goals is to construct a feasible orientation with the same size of eye by assigning directions to edges

with direction 0. We first construct a feasible orientation $\tilde{A}$. Assume that $L_0 = \{k, \ell, 1\}$ is the root ear, and define

$$\tilde{A}_{k,\ell,1} := A_{k,\ell,1},$$
$$\tilde{A}_{k,\ell,s} := -A_{k,\ell,1}, \quad \forall s \in \{2, \dots, t\}.$$

Then, we inductively define the directions of other ears using the directed tree structure of ears. For any ear L_k that has been assigned a direction, we assign its descendant ear L_ℓ with the parallel direction as the path formed by the endpoints of L_ℓ on L_k. In this way, the orientation $\tilde{A}$ is defined for all ears and the definition is unique because of the directed tree structure. Considering the structure of the nested ear decomposition, we also know that all directed cycles in orientation $\tilde{A}$ must contain the root ear. In addition, the orientation $\tilde{A}$ is feasible. This is because all internal vertices of ears have nonzero in-degree and nonzero out-degree. The only vertices that are not internal vertices of ears are the endpoints of the root ear. For the endpoints of the root ear, they also have nonzero in-degree nonzero and out-degree by the definition of directions on parallel edges. Hence, the constructed orientation $\tilde{A}$ is feasible.

We then define an orientation that combines orientations A and $\tilde{A}$ as follows:

$$A^g_{k\ell} := \begin{cases} A_{k\ell} & \text{if } A_{k\ell} \in \{+1, -1\} \\ \tilde{A}_{k,\ell} & \text{if } A_{k\ell} = 0, \end{cases}, \quad \forall \{k, \ell\} \in \mathbb{E}.$$

We prove that A^g is a feasible orientation and the girth of orientation A^g is at least e. For any vertex k that has a nonzero degree in orientation A, the vertex k has nonzero in-degree and out-degree by the definition of weakly feasible orientation. Hence, the vertex k also has nonzero in-degree and out-degree in the new orientation. If the vertex has degree 0 in the orientation A, then all edges incident to the vertex k has the same direction as in $\tilde{A}$. Since the orientation $\tilde{A}$ is feasible, the vertex k has nonzero in-degree and nonzero out-degree in the new orientation A^g. Combining the two cases, it can be concluded that the orientation A^g is feasible. Now, we estimate the girth of orientation A^g. We consider any directed cycle C in A^g. If the cycle C has normal edges in the original orientation A, then the length of cycle C is not decreased in the new orientation and therefore is at least e. If the cycle C does not have normal edges in the original orientation A, then all edges of C have the same direction as in $\tilde{A}$ and therefore is also a cycle in in $\tilde{A}$. This implies that the root ear L_0 is on the cycle C. However, the root ear is a normal edge in orientation $\tilde{A}$ and this contradicts the assumption that none of the edges of the cycle C are normal. Thus, the girth of A^g is at least e. On the other hand, the girth of a feasible orientation is bounded by the maximal girth g. This contradicts the assumption that $\mathrm{e} > \mathrm{g}$.

Combining the above two cases and using the induction method, it can be concluded that the maximal eye of a 2-vertex-connected SP graph is equal to its maximal girth.

□

H. Proof of Theorem 12

Proof. We only prove the strong uniqueness part since the proof for the weak uniqueness is similar. We only need to show that statement 2 of this theorem holds if and only if statement 2 of Lemma 1 holds.

a) Proof of sufficiency: We assume conversely that there exist two sets of phase angle vectors Θ^1 and Θ^2 satisfying statement 2 of Lemma 1 such that the induced sub-graph of $\Theta^1 - \Theta^2$ denoted as $(\mathbb{V}_0, \mathbb{E}_0, A_0)$ has the same number of strongly connected components and weakly connected components. Let $\mathbf{y}$ be a vector that satisfies conditions in statement 2 of Lemma 1. We prove that if vertices k and ℓ are in the same connected component, then $y_k = y_\ell$. By the definition of strongly connected components, there exist directed paths from k to ℓ and from ℓ to k. We first consider the directed path from k to ℓ, which we denote as $(k, k_1, \dots, k_t, \ell)$. Considering the edge $\{k, k_1\}$ and inequality (2), one can write

$$\begin{aligned} &\sin(\pi/2 + \Theta^1_{k,k_1}/2 + \Theta^2_{k,k_1}/2) \cdot y_k \\ \geq &\sin(\pi/2 - \Theta^1_{k,k_1}/2 - \Theta^2_{k,k_1}/2) \cdot y_{k_1}. \end{aligned} \tag{10}$$

By the same analysis in Lemma 2, the condition $\Delta_{k,k_1} > 0$ implies that $\Theta^1_{k,k_1}/2 + \Theta^2_{k,k_1}/2 \in (-\pi/2, \pi/2)$, which leads to

$$\begin{aligned} &\sin(\pi/2 + \Theta^1_{k,k_1}/2 + \Theta^2_{k,k_1}/2) \\ = &\sin(\pi/2 - \Theta^1_{k,k_1}/2 - \Theta^2_{k,k_1}/2) > 0. \end{aligned} \tag{11}$$

Combining the relations in (10) and (11), we obtain $y_k \geq y_{k_1}$. Considering edges $\{k_1, k_2\}, \dots, \{k_n, \ell\}$ and using the same analysis, we have

$$y_k \geq y_{k_1} \geq y_{k_2} \geq \cdots \geq y_{k_t} \geq y_\ell,$$

and therefore $y_k \geq y_\ell$. Similarly, the existence of a directed path from y_ℓ to y_k implies that $y_\ell \geq y_k$. Combining the two directions, we obtain $y_k = y_\ell$. If we further assume $\{k, \ell\} \in \mathbb{E}_0$ and $\Delta_{k\ell} > 0$, then the relation in (11) implies that inequality (2) holds with equality for $\{k, \ell\}$. By the definition of weakly connected components, there does not exist any edge in $\mathbb{E}_0$ connecting different connected components. Hence, the endpoints of all edges in $\mathbb{E}_0$ are in the same connected component and therefore inequality (2) holds with equality for all $\{k, \ell\} \in \mathbb{E}_0$ such that $\Delta_{k\ell} > 0$. Finally, by the definition of induced sub-graph, $\mathbb{E}_0$ contains all edges $\{k, \ell\} \in \mathbb{E}$ such that $\Delta_{k\ell} > 0$. It follows that inequality (2) holds with equality for all $\{k, \ell\} \in \mathbb{E}$ such that $\Delta_{k\ell} > 0$. This contradicts statement 2 of Lemma 1 that there exists at least one strict inequality in the set of inequalities (2). Hence, statement 2 of this theorem holds.

b) Proof of necessity: Assume that the conditions in statement 2 of this theorem hold. We denote the strongly connected components as $\mathcal{C}_1, \dots, \mathcal{C}_m$. Now, we define a tree structure for the set $\{\mathcal{C}_1, \dots, \mathcal{C}_m\}$. For two different strongly connected components $\mathcal{C}_s$ and $\mathcal{C}_t$, if there exists a directed path from $\mathcal{C}_s$ to $\mathcal{C}_t$, we define a directed edge from $\mathcal{C}_t$ to $\mathcal{C}_s$. Considering all strongly connected components pairs, we obtain a directed graph with the vertex set $\{\mathcal{C}_1, \dots, \mathcal{C}_m\}$. By the definition of strongly connected components, we know that there does not exist directed cycle in this directed graph and therefore this directed graph is a directed tree. Using the directed tree structure, we can choose m real numbers $c_1, \dots, c_m$ such that if there exists a directed path from $\mathcal{C}_t$ to $\mathcal{C}_s$, then it holds that $c_t > c_s$. Moreover, if vertex 1 belongs to some strongly connected component $\mathcal{C}_s$, then we can shift c_t for all $t \in [m]$ such that $c_s = 0$ and the relation between all c_t's is not changed. If vertex 1 does not belong to any strongly connected component, we do not change the value of c_t.

We construct a vector $\mathbf{y} \in \mathbb{R}^{|\mathbb{V}|}$ by

$$y_k := \begin{cases} c_s & \text{if } k \text{ is in } \mathcal{C}_s \\ 0 & \text{if } k \in \mathbb{V}\backslash\mathbb{V}_0. \end{cases}$$

Note that the set of strongly connected components gives a disjoint partition of the set $\mathbb{V}_0$. Hence, the vector $\mathbf{y}$ is well-defined. By the choice of $\{c_1, \dots, c_m\}$, the vector $\mathbf{y}$ satisfies $y_1 = 0$. Suppose that the edge $\{k, \ell\}$ belongs to $\mathbb{E}$ and $\Delta_{k\ell} > 0$. We verify that inequality (2) holds for $\{k, \ell\}$, namely,

$$\begin{aligned} &\sin(\pi/2 + \Theta^1_{k\ell}/2 + \Theta^2_{k\ell}/2) \cdot y_k \\ \geq &\sin(\pi/2 - \Theta^1_{k\ell}/2 - \Theta^2_{k\ell}/2) \cdot y_\ell. \end{aligned}$$

Recalling that the relation (11) holds for all $\{k, \ell\}$ such that $\Delta_{k\ell} > 0$, we only need to verify

$$y_k \geq y_\ell, \quad \forall \{k, \ell\} \in \mathbb{E}_0 \quad \text{s.t. } \Delta_{k\ell} > 0. \tag{12}$$

By the definition of induced sub-graph, the condition $\Delta_{k\ell} > 0$ implies that $\{k, \ell\} \in \mathbb{E}_0$. Thus, vertices k and ℓ must belong to certain strongly connected components. If k and ℓ belong to the same strongly connected component $\mathcal{C}_s$, then $y_k = y_\ell = c_s$ and inequality (12) holds. Otherwise, we assume that k and ℓ belong to two different strongly connected components $\mathcal{C}_s$ and $\mathcal{C}_t$, respectively. Since (k, ℓ) is a directed path from $\mathcal{C}_s$ to $\mathcal{C}_t$, one can write

$$y_k = c_s > c_t = y_\ell$$

and inequality (12) holds strictly. By the assumption that there are strictly more strongly connected components than weakly connected components, there exists at least one edge $\{k, \ell\} \in \mathbb{E}_0$ such that k and ℓ belong to different strongly connected components. Without loss of generality, assume that $\Delta_{k\ell} > 0$. Then, the inequality (12), or equivalently the inequality (2), holds strictly for $\{k, \ell\}$. This shows that $\mathbf{y}$ is a vector that satisfies conditions in statement 2 of Lemma 1.

□

I. Proof of Lemma 13

Proof. We prove the four claims separately.

a) Type I Operation: We first consider the inequality on the right. We denote the two endpoints as k, ℓ and the parallel edges connecting them as $\{k, \ell, 1\}, \dots, \{k, \ell, t\}$ for some $t \geq 2$. Without loss of generality, assume that the weights of parallel edges satisfy

$$W_{min} = W_{k,\ell,1} \leq \cdots \leq W_{k,\ell,t} = W_{max}.$$

Suppose that the maximal eye of graph $(\mathbb{V}, \mathbb{E}, W)$ is achieved by the weakly feasible orientation A. If there exist different directions among these parallel edges when orientation A is assigned, then we choose the first edge $\{k, \ell, 1\}$ and another edge $\{k, \ell, s\}$ such that the direction of $\{k, \ell, s\}$ is different from the direction of $\{k, \ell, 1\}$. Hence, $\{k, \ell, 1\}$ and $\{k, \ell, s\}$ form a directed cycle and two edges have different directions. Then, at least one edge is a normal edge, i.e., an edge with direction $+1$ or -1. The weight of the cycle is bounded by $W_{k,\ell,1} + W_{k,\ell,s} \leq W_{max} + W_{min}$. Thus, it holds that $\mathrm{e} \leq W_{max} + W_{min}$ in this case. Otherwise, assume that all parallel edges have the same direction when orientation A is assigned. Considering a directed cycle that contains the edge $\{k, \ell, s\}$ for some $s \in \{2, \dots, t\}$, we can substitute the edge $\{k, \ell, s\}$ with edge $\{k, \ell, 1\}$ and the length of the directed cycle is not increased. Hence, if we delete edges $\{k, \ell, 2\}, \dots, \{k, \ell, t\}$, the size of eye is not changed. On the other hand, the deletion of edges $\{k, \ell, 2\}, \dots, \{k, \ell, t\}$ is equivalent to the Type I Operation on the set of parallel edges $\{k, \ell, 1\}, \dots, \{k, \ell, t\}$. Hence, we obtain $\mathrm{e} = \tilde{\mathrm{e}}$ in this case. Combining the two cases, it follows that $\mathrm{e} \leq \max\{\tilde{\mathrm{e}}, W_{max} + W_{min}\}$.

We now prove the inequality on the left. Suppose that the maximal eye of the new graph $(\tilde{\mathbb{V}}, \tilde{\mathbb{E}}, \tilde{W})$ is achieved by the weakly feasible orientation $\tilde{A}$. By the definition of Type I Operations, the weight $\tilde{W}_{k,\ell}$ is equal to the weight $W_{k,\ell,1}$. We consider the inverse operation of Type I Operation. Namely, we add parallel edges $\{k, \ell, s\}$ with weight $W_{k,\ell,s}$ to the new graph and define the direction $\tilde{A}_{k,\ell,s} := \tilde{A}_{k,\ell,1}$ for all $s \in \{2, \dots, t\}$. Then, the orientation $\tilde{A}$ becomes a weakly feasible orientation for the original graph. By the discussion for the inequality on the right, the inverse operation will not change the size of eye. Therefore, we have a weakly feasible orientation for $(\mathbb{V}, \mathbb{E}, W)$ and the size of eye is $\tilde{\mathrm{e}}$, which implies that $\mathrm{e} \geq \tilde{\mathrm{e}}$.

b) Type II Operation: We consider the case when a Type II Operation is implemented. We denote the deleted degree-2 vertex as k. By the definition of Type II Operations, vertex k has two neighbouring vertices and we denote the two neighbouring vertices as $\ell_1 \neq \ell_2$. If A is a weakly feasible orientation for $(\mathbb{V}, \mathbb{E}, W)$, then the direction $A_{\ell_1,k}$ must be equal to the direction A_{k,ℓ_2}. Hence, treating the two edges $\{\ell_1, k\}$ and $\{k, \ell_2\}$ as a single edge with weight $W_{\ell_1,k} + W_{k,\ell_2}$ will not change the size of eye. Noticing that the claim is true for any weakly feasible orientation A, we know that $\mathrm{e} = \tilde{\mathrm{e}}$.

c) Type III Operation with a pendant vertex: Removing a pendant vertex will not affect the maximal eye, since any directed cycle does not contain pendant vertices. Thus, we conclude that $\mathrm{e} = \tilde{\mathrm{e}}$.

d) Type III Operation with a non-pendant vertex: Finally, we consider the case when the deleted vertex has degree at least 2. We denote the deleted vertex as k and denote the only neighbouring vertex as ℓ. The parallel edges connecting k and ℓ are denoted as $\{k, \ell, 1\}, \dots, \{k, \ell, t\}$ for some $t \geq 2$. Similar to the Type I Operation case, assume that the weights of parallel edges satisfy

$$W_{min} = W_{k,\ell,1} \leq \cdots \leq W_{k,\ell,t} = W_{max}.$$

We can split the deletion of vertex k into two operations. We first substitute parallel edges $\{k, \ell, 1\}, \dots, \{k, \ell, t\}$ with a single edge $\{k, \ell\}$ with weight $W_{k,\ell,1}$. Then, we delete the pendant vertex k. The two operations can be viewed as Type I and Type III Operations, respectively. Using the results in the first case and the third case, one can write

$$\tilde{\mathrm{e}} \leq \mathrm{e} \leq \max\{\tilde{\mathrm{e}}, W_{max} + W_{min}\}.$$

Hence, it remains to prove that $\mathrm{e} \geq W_{max} + W_{min}$. We can construct a weakly feasible orientation such that size of eye is $W_{max} + W_{min}$. Specifically, we define

$$A_{k,\ell,s} := +1, \quad \forall s \in \{1, \ldots, t-1\}, \quad A_{k,\ell,t} := -1$$

and all other edges are assigned the direction 0. Then, vertices k and ℓ have nonzero in-degree and out-degree, while other vertices have zero in-degree and out-degree. Hence the orientation A is weakly feasible. Now, consider directed cycles with at least one normal edge. Since parallel edges $\{k, \ell, 1\}, \ldots, \{k, \ell, t\}$ are the only normal edges, the directed cycle must contain at least one of these parallel edges. Using the facts that ℓ is the only neighbouring vertex of k and directed cycles do not have repeated vertices, vertices k and ℓ are the only two vertices of the directed cycle. Hence, the size of eye should be the the minimal length of such directed cycles, which is $W_{k,\ell,1} + W_{k,\ell,t} = W_{max} + W_{min}$. Thus, it follows that $\mathrm{e} \geq W_{max} + W_{min}$.

Combining the two parts yields that $\mathrm{e} = \max\{\tilde{\mathrm{e}}, W_{max} + W_{min}\}$. □

black

J. Proof of Lemma 15

Proof. The first three claims can be proved in the same way as Lemma 13 and we only prove the last two claims. We denote the deleted vertex as k and its only neighboring vertex as ℓ. The parallel edges connecting k and ℓ are denoted as $\{k, \ell, 1\}, \ldots, \{k, \ell, t\}$ for some $t \geq 2$. Without loss of generality, assume that the weights of parallel edges satisfy

$$W_{min} = W_{k,\ell,1} \leq \cdots \leq W_{k,\ell,t} = W_{max}.$$

a) Type III Operation for slack node: We first consider the case when the deleted vertex is a slack node. By discussing whether parallel edges $\{k, \ell, 1\}, \ldots, \{k, \ell, t\}$ have the same direction as in the first claim in Lemma 13, it holds that $\mathrm{g} \leq \max\{\tilde{\mathrm{g}}, W_{max} + W_{min}\}$.

We prove the other inequality $\tilde{\mathrm{g}} \leq \mathrm{g}$ by constructing a feasible orientation A such that the girth is $\tilde{\mathrm{g}}$. Suppose that the maximal girth of the new graph $(\tilde{\mathbb{V}}, \tilde{\mathbb{E}}, \tilde{W})$ is achieved by the feasible orientation $\tilde{A}$. We define directions for deleted parallel edge such that the orientation $\tilde{A}$ becomes a feasible orientation of the original graph $(\mathbb{V}, \mathbb{E}, W)$. We note that, by the definition of Type III Operations, the vertex ℓ is a slack node in the new graph and it may not satisfy the condition on in-degree and out-degree. If the vertex ℓ in the new graph with orientation $\tilde{A}$ has nonzero in-degree, then we define

$$\tilde{A}_{k,\ell,s} := -1, \quad \forall s \in \{1, \ldots, t\}.$$

Then, the vertex ℓ has both nonzero in-degree and nonzero out-degree. Since the vertex k is a slack node, the orientation $\tilde{A}$ becomes a feasible orientation for the original graph $(\mathbb{V}, \mathbb{E}, W)$. By the construction of $\tilde{A}$, the vertex k only has nonzero in-degree and therefore there does not exist any directed cycle containing parallel edges $\{k, \ell, 1\}, \ldots, \{k, \ell, t\}$. It follows that the girth is not changed and is equal to $\tilde{\mathrm{g}}$. If the vertex ℓ in the new graph with orientation $\tilde{A}$ has nonzero out-degree, then we can similarly define

$$\tilde{A}_{k,\ell,s} := +1, \quad \forall s \in \{1, \ldots, t\}.$$

The orientation $\tilde{A}$ also becomes a feasible orientation for the original graph and the girth is $\tilde{\mathrm{g}}$. Combining the two cases concludes that $\mathrm{e} \geq \tilde{\mathrm{g}}$.

b) Type III Operation for non-slack node: We then consider the case when the deleted vertex is not a slack node. Suppose that the maximal girth of the original graph $(\mathbb{V}, \mathbb{E}, W)$ is achieved by the feasible orientation A. Since the vertex k has nonzero in-degree and nonzero out-degree, there must exist different directions among parallel edges $\{k, \ell, 1\}, \ldots, \{k, \ell, t\}$. Hence, by the same analysis as the first claim in Lemma 13, it holds that $\mathrm{g} \leq W_{max} + W_{min}$. Now, we consider restricting the orientation A to the new graph$(\tilde{\mathbb{V}}, \tilde{\mathbb{E}}, \tilde{W})$. Since the vertex ℓ is a slack node in the new graph and the orientation A is not changed for other vertices, the orientation A becomes a feasible orientation for the new graph. Then, by the definition of the maximal girth, there exists a directed cycle in the new graph with length at most $\tilde{\mathrm{g}}$. Hence, we conclude that $\mathrm{g} \leq \tilde{\mathrm{g}}$. Combining the two inequalities, it follows that $\mathrm{g} \leq \min\{\tilde{\mathrm{g}}, W_{max} + W_{min}\}$.

Now, it remains to prove $\mathrm{g} \geq \min\{\tilde{\mathrm{g}}, W_{max} + W_{min}\}$. Suppose that the maximal girth of the new graph $(\tilde{\mathbb{V}}, \tilde{\mathbb{E}}, \tilde{W})$ is achieved by the feasible orientation $\tilde{A}$. We extend the orientation $\tilde{A}$ to be an orientation for the original graph by defining

$$A_{k,\ell,s} := +1, \quad \forall s \in \{1, \ldots, t-1\}, \quad A_{k,\ell,t} = -1.$$

Since both vertices k, ℓ have nonzero in-degree and nonzero out-degree and the orientation at other vertices is not changed, the orientation A becomes a feasible orientation for the original graph. Now, we calculate the girth of the original graph. For any directed cycle that does not contain parallel edges $\{k, \ell, 1\}, \ldots, \{k, \ell, t\}$, it is also a directed cycle in the new graph and has length at least $\tilde{\mathrm{g}}$. For any directed cycle that contains at least one of those parallel edges, vertices k and ℓ are the only two vertices of the directed cycle, since there does not exist repeated vertices on directed cycles. Hence, the length of the directed cycle is at least $W_{k,\ell,1} + W_{k,\ell,t} = W_{max} + W_{min}$. Combining the two cases yields that the girth is at least $\min\{\tilde{\mathrm{g}}, W_{max} + W_{min}\}$ and therefore $\mathrm{g} \geq \min\{\tilde{\mathrm{g}}, W_{max} + W_{min}\}$. □

black

K. Proof of Theorem 17

Proof. We prove that Type I-II Operations are enough for reducing a 2-vertex-connected SP graph to a single edge. Since Type I-II Operations do not introduce new slack nodes, there exists at most one slack node in the graph throughout the reduction process. By the assumption that the graph is a 2-vertex-connected SP graph, Theorem 9 implies that there exists a nested ear decomposition $(L_0, \ldots, L_{r-1})$ of the graph. We use the induction method on the number of ears in the ear decomposition. If there are only one ear or two ears in the ear decomposition, then the result holds trivially. We assume that any 2-vertex connected SP graphs with at most $r-1$ ears in the ear decomposition can be reduced to a single edge with Type I-II Operations.

Now, we consider the case when there are r ears in the ear decomposition. We first implement Type II Operations until there is no degree-2 vertices except the slack bus. Since Type II Operations will not change the structure of the nested ear decomposition, the new graph still has a nested ear decomposition with at most r ears in the decomposition. By the first step in the proof of Theorem 10, there exists a set of parallel edges containing the root ear or a leaf ear. We analyze two different cases:

a) Case I: Assume that there exists a set of parallel edges containing a leaf ear. We denote the leaf ear as $L_s = \{k, \ell\}$. Let L_t be the precedent ear of L_t. Then, then set of parallel ears consists of the segment $\overline{k\ell}$ on ear L_t and leaf ears on L_t. We can apply a Type I Operation to substitute the set of parallel edges with a single edge. We can view the new edge as the segment $\overline{k\ell}$ on ear L_t. Then, at least leaf ear is deleted and the new graph has a nested ear decomposition with at most $r-1$ ears. By the induction assumption, the new graph can be reduced to a single edge with Type I-II Operations. Thus, the original graph can be reduced to a single edge with Type I-II Operations.

b) Case II: Assume that there exists a set of parallel edges containing the root ear. Then by the same construction in the second step in the proof of Theorem 10, we can change the root ear to a leaf ear. Hence, we obtain a set of parallel edges containing a leaf ear and we can apply the discussion in Case I.

Combining the two cases, it follows that the result is true when there are r ears in the ear decomposition. By the induction method, the result is true for any $r \geq 1$ and the black method can reduce a 2-vertex-connected SP graph to a single edge.

□

L. Numerical results of black method for computing the maximal girth

Power Network	Original Size	Reduced Size	α_1	α_2	g_R
Case 14	(14,20)	(2,1)	6	3	0
Case 30	(30,41)	(9,14)	4	3	3
Case 39	(39,46)	(10,14)	4	3	3
Case 57	(57,78)	(22,39)	4	-	23
Case 118	(118,179)	(44,83)	5	-	4
Case 300	(300,409)	(110,197)	8	3	≥7
Case 1354	(1354,1710)	(271,509)	9	3	≥3
Case 2383	(2383,2886)	(500,950)	11	3	≥3

TABLE II: Number of vertices and edges before and after the black method for maximal girth along with values computed during the reduction process.

M. Algorithms for Computing the Maximal Girth and Eye

In the appendix, we propose search-based algorithms for computing the maximal eye and the maximal girth. Our approach is based on the Depth-First Search (DFS) method and utilized the pruning technique to accelerate the computing process. We first describe a common sub-procedure that will be used in both algorithms. The sub-procedure computes the minimal directed chordless cycle containing a given edge. Given a truncation length $T \geq 1$, the sub-procedure returns the truncation length if there does not exist a directed chordless cycle that contains the given edge and has length at most T. The sub-procedure is also based on the DFS method with pruning and borrows the idea of *blocking* from [28] to accelerate the searching process. The pseudo-code of the sub-procedure is listed in Algorithm 2.

The search space of the sub-procedure is the set of directed chordless paths with length at most T. When the current directed chordless path is a directed chordless cycle, the length of the cycle is recorded and the minimal length of known directed chordless cycles is updated. By searching over all chordless paths, we find the length of the minimal directed chordless cycle. The DFS method is initialized with the given edge, denoted as (k, ℓ), and extends the directed chordless path by adding a neighbouring vertex of the end point other than k to the path. The pruning technique becomes effective and delete the end point other than k from the path if one of the following cases occurs:

- The length of the directed chordless path is larger than T or the known minimal length of directed chordless cycles;
- All neighbouring vertices have been searched or will introduce a chord if added to the path.

Using the idea of blocking, one can efficiently check whether adding a vertex to the path will introduce a chord. This approach is based on the following observation: if the path $(k_1, \dots, k_t)$ is chordless, then any vertex k_s can only be in the neighborhood of k_{s-1}, k_{s+1}. We construct an array and, for each vertex, we record the number of vertices on the path that are in the neighborhood of the vertex. The array is updated whenever the path is updated. If there are at least two vertices on the path in the neighbourhood of a vertex not on the path, then adding the vertex to the path will introduce a chord. Hence, the cost of checking this condition for each potential vertex not on the path is a single evaluation of an array.

Next, we propose the algorithms for computing the maximal eye and the maximal girth. Since the algorithm of maximal girth is similar to the algorithm for maximal eye, we only discuss the algorithm for computing the maximal eye. The algorithm is also based on the DFS method with pruning, and the pseudo-code is provided in Algorithm 4. We first order all edges and gradually assign one of the directions $\{0, -1, +1\}$ to each edge following the ordering of the edges. The search space consists of the orientations for the first several edges (intermediate states) and the orientations for the entire graph (final states). One can verify that all intermediate states and final states form a trinomial[3] tree, since each orientation for

[3]A directed tree is called a trinomial tree if there is a root node and each non-leaf node has exactly three descendant nodes.

the first $k < |\mathbb{E}|$ edges leads to three different orientations for the first $k+1$ edges. Then, the algorithm for computing the maximal eye searches in the same way as the classical DFS method on a directed tree. For each node, we consider the sub-graph consisting of those edges that have been assigned a direction. We compute the length of the minimal directed chordless cycle in the sub-graph, which contains the last edge in the sub-graph, using the sub-procedure (Algorithm 2). The truncation length can be decided as follows. Since a DFS method is implemented on a trinomial tree, there exists a directed path from the root node of the trinomial tree to the current node. The truncation length can be chosen as the minimal length computed on the preceding nodes of the path. When the search reaches a leaf node, we obtain an orientation for the entire graph, and the size of the eye becomes the minimal length on the path to the root node. By searching over all leaf nodes, we find the maximal eye. Similarly, one can use the pruning technique to reduce the search space. The current node is pruned if it can not be extended to a weakly feasible orientation for the entire graph, or the size of the eye of the sub-graph is smaller than the known maximal size of the eye.

Algorithm 2 Truncated Minimal Chordless Cycle

Input: Directed weighted graph $(\mathbb{V}, \mathbb{E}, W)$, selected edge (k, ℓ), truncation length T
Output: Length of minimal chordless cycle c

```
Construct the neighbourhood of each vertex N : V ↦ 2^V.
Initialize blocked array block[i] ← 0 for all vertices i ∈ V.
Set the length of minimal cycle recorded c ← T.
Set current length L_cur ← W_kℓ.
Set the path P ← [k, ℓ].
Set block[k] ← 1, block[ℓ] ← 1.
if L_cur ≥ T then    ▷ Already longer than truncation length
    return c
end if
while the length of P is at least 2 do
    Get the endpoint i ← P[−1].
    Increase block for vertices in N[j] by 1.
    Get the minimal vertex j ∈ N[i] such that block[j] ≤ 1
and L_cur + W_P[−1]j < v.
    if no such vertex j exists then
                              ▷ Recursion: no next unblocked vertex
        Find the maximal index h such that P[h] ∉ {k, ℓ, i}
and P[h + 1] is not the maximal vertex in N[P[h]].
        if no such h exists then          ▷ Search finished
            break
        else
            Remove P[h + 1], . . . , P[−1] from path P.
            Decrease block of N[P[h]], . . . , N[P[−1]] by 1.
            Add the next smallest vertex in N[P[h]] to P.
            Update L_cur to be the length of path P.
            continue
        end if
    else                                  ▷ Add a new vertex
        Add vertex j to P and update L_cur.
        if k ∈ N[j] then                  ▷ find a cycle
            Calculate length c_cur ← L_cur + W_jk.
            if c_cur > 0 then
                Update c ← min{c, c_cur}.
            end if
            Recursion similarly as above.
        else
            continue
        end if
    end if
end while
return c
```

N. Algorithm for Computing The Maximal Girth

Algorithm 3 Algorithm for computing the maximal girth

Input: Undirected weighted graph $(\mathbb{V}, \mathbb{E}, W)$, slack bus k
Output: Maximal girth g

Set the maximal girth $\mathrm{g} \leftarrow 0$.
Assign an order to the set of edges $\mathbb{E}$ and denote edges as

$$\{k_1, \ell_1\}, \ \ldots, \ \{k_m, \ell_m\}.$$

Initialize the set of edges $\mathbb{E}_0 \leftarrow \{\{k_1, \ell_1\}\}$.
Initialize the set of orientations $A_{k_1,\ell_1} \leftarrow -1$.

loop
 Check the feasibility with current orientation.
 if feasibility fails **then** ▷ Recursion
  Get the maximal index j such that $A_{k_j,\ell_j} \neq 1$.
  if no such j exists **then** ▷ Terminate the algorithm
   break
  else
   Remove $\{k_{j+1}, \ell_{j+1}\}, \ldots, \{k_m, \ell_m\}$ from $\mathbb{E}_0$.
   Change orientation $A_{k_j,\ell_j} \leftarrow -A_{k_j,\ell_j}$.
   continue
  end if
 end if
 Compute the girth g_{cur} under $\mathbb{E}_0$ and A using Algorithm 2. The truncation length is set to be the girth of the precedent state.
 if $\mathrm{g}_{cur} < \mathrm{g}$ **then** ▷ Smaller than known girth
  Recursion in the same way.
 end if
 Get the next edge $\{k_i, \ell_i\}$ that is not in $\mathbb{E}_0$.
 if no such edge **then** ▷ Leaf node reached
  Update $\mathrm{g} \leftarrow \max\{\mathrm{g}, \mathrm{g}_{cur}\}$.
  Recursion in the same way.
 else
  Add the next edge $\{k_i, \ell_i\}$ that is not in $\mathbb{E}_0$.
  Assign $A_{k_j,\ell_j} \leftarrow -1$.
  continue
 end if
end loop
return g

Algorithm 4 Algorithm for Computing The Maximal Eye

Input: Undirected weighted graph $(\mathbb{V}, \mathbb{E}, W)$, slack bus k
Output: Maximal eye e

Set the maximal eye $\mathrm{e} \leftarrow 0$.
Assign an order to the set of edges $\mathbb{E}$ and denote edges as

$$\{k_1, \ell_1\}, \ \ldots, \ \{k_m, \ell_m\}.$$

Initialize the set of edges $\mathbb{E}_0 \leftarrow \{\{k_1, \ell_1\}\}$.
Initialize the set of orientations $A_{k_1,\ell_1} \leftarrow -1$.
loop
 Check the weak feasibility with current orientation.
 if weak feasibility fails **then** ▷ Recursion
  Get the maximal index j such that $A_{k_j,\ell_j} \neq 1$.
  if no such j exists **then** ▷ Terminate the loop
   break
  else
   Remove $\{k_{j+1}, \ell_{j+1}\}, \ldots, \{k_m, \ell_m\}$ from $\mathbb{E}_0$.
   Change orientation $A_{k_j,\ell_j} \leftarrow A_{k_j,\ell_j} + 1$.
   continue
  end if
 end if
 Compute the size of eye e_{cur} under $\mathbb{E}_0$ and A using Algorithm 2. The truncation length is set to be the size of eye of the precedent state.
 if $\mathrm{e}_{cur} < \mathrm{e}$ **then** ▷ Smaller than known size of eye
  Recursion in the same way.
 end if
 Get the next edge $\{k_i, \ell_i\}$ that is not in $\mathbb{E}_0$.
 if no such edge **then** ▷ Leaf node reached
  Update $\mathrm{e} \leftarrow \max\{\mathrm{e}, \mathrm{e}_{cur}\}$.
  Recursion in the same way.
 else
  Add the next edge $\{k_i, \ell_i\}$ that is not in $\mathbb{E}_0$.
  Assign $A_{k_j,\ell_j} \leftarrow -1$.
  continue
 end if
end loop
return e