Convex Analysis of Generalized Flow Networks

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***Abstract*— This paper is concerned with the generalized network flow (GNF) problem, which aims to find a minimum-cost solution for a generalized flow network. The objective is to determine the optimal injections at the nodes as well as optimal flows over the lines of the network. In this problem, each line is associated with two flows in opposite directions that are related to each other via a given nonlinear function. Under some monotonicity and convexity assumptions, we have shown in our recent work that a convexified generalized network flow (CGNF) problem always finds optimal injections for GNF, but may fail to find optimal flows. In this paper, we develop three results to explore the possibility of obtaining optimal flows. First, we show that CGNF yields optimal flows for GNF if the optimal injection vector is a Pareto point. Second, we show that if CGNF fails to find an optimal flow vector, then the graph can be decomposed into two subgraphs, where the lines between the subgraphs are congested at optimality and CGNF finds correct optimal flows over the lines of one of these subgraphs. Third, we fully characterize the set of all optimal flow vectors. In particular, we show that this non-convex set is a subset of the boundary of a convex set, and may include an exponential number of disconnected components.**

I. Introduction

The "network flow" problem is of significant importance in computer science, operation research, and engineering [1]–[3]. This problem has immediate applications in communication networks, power and commodity distribution, financial budgeting, and production scheduling and assignment, among other fields. Since 1962, network flow problems have been extensively studied [3]–[18]. The minimum-cost flow problem aims to find optimal flows in a given network such that the overall cost of production and/or transportation is minimized. In this problem, the network is used to carry some commodity of interest between pre-specified sources and destinations.

To formalize a flow network, consider a graph consisting of nodes and lines. There is an injection of some commodity at every node, and there are two flows over each line. One flow enters the line from an endpoint and the second flow leaves from the other endpoint. Depending on the sign of its injection, each node can be considered as a supplier or consumer. This problem was developed and solved in [3] for lossless networks. Although the algorithm proposed in [3] is efficient, it does not apply to certain real-world networks because the line losses are ignored. More precisely, the flow entering a line may not be equal to the outgoing flow in practice. Driven by this practical consideration, the lossy network flow problem has drawn much attention. The paper [2] proposes a generalized network (also known as network with gain) in which each outgoing flow is proportionally related to the entering flow via a constant gain. This type of network flow problem has been studied extensively [19], [20]. Assuming that the cost functions are convex, this type of lossy network can be solved in polynomial time (up to a given accuracy) because of the convex nature of its objective and constraints [21].

Recently, [22] has introduced a more general network flow problem, referred to as Generalized Network Flow (GNF). In GNF, the output flow over each line is a nonlinear function of the input flow. This is motivated by the fact that the line losses are nonlinear in certain real-world networks, such as electrical power networks [23]. Assume that the cost and flow functions are all monotonic and convex, which is a fairly reasonable assumption in practice. The GNF problem is highly non-convex due to its nonlinear equality constraints. However, relaxing the equality constraints into convex inequality leads to a convex relaxation of the problem, named convexified generalized network flow (CGNF). The work [22] has proved that this relaxation is exact for the optimal injections but may not yield feasible (optimal) flows for GNF.

Since the optimal injections for GNF can systematically be found using CGNF, the main objective of this paper is to study the possibility of finding optimal flows. First, we prove that if the optimal injection vector is a Pareto point in its feasible region, CGNF finds optimal flows for GNF. Second, we substantiate that the flow network can be divided into two sub-networks such that: (i) CGNF obtains optimal flows over one sub-network, (ii) the lines between the two sub-networks are all congested at optimality and CGNF correctly identifies these lines. In other words, we relate the possible failure of CGNF in finding optimal flows for the whole network to certain congested lines. Moreover, we fully characterize the set of all optimal flow vectors. In particular, we show that this set may be infinite, non-convex, and disconnected, but belongs to the boundary of a convex set.

A. Notations

The following notations will be used throughout this paper:

- $\mathbb{R}$ denotes the set of real numbers.
- Given two vectors $\mathbf{x}$ and $\mathbf{y}$, the inequality $\mathbf{x} \leq \mathbf{y}$ means that $\mathbf{x}$ is less than or equal to $\mathbf{y}$ element-wise.

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- Given a set $\mathcal{T}$, its cardinality is shown as $|\mathcal{T}|$.
- Lowercase and bold lowercase are used for scalars and vectors, respectively.
- A nonconvex optimization has two types of solutions: local and global. We simply use the term "**solution**" for "**global solution**" henceforth (because local solutions are not of interest in this work).

II. Problem Statement

Consider an undirected, connected graph (network) $\mathcal{G}$ with the set of vertices/nodes $\mathcal{N} := \{1, 2, ..., m\}$ and the set of edges/lines $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$. Assume that every edge $(i, j) \in \mathcal{E}$ is associated with two unknown flows p_{ij} and p_{ji} belonging to $\mathbb{R}$. The parameters p_{ij} and p_{ji} can be regarded as the flows entering the edge (i, j) from the endpoints i and j, respectively. The parameter p_i defined as

$$p_i = \sum_{j \in \mathcal{N}(i)} p_{ij}, \qquad \forall i \in \mathcal{N} \tag{1}$$

is called "injection at node i", where $\mathcal{N}(i) \subseteq \mathcal{N}$ denotes the set of neighbors of node i in the graph $\mathcal{G}$. The injection p_i is equal to the sum of flows leaving node i. Given an edge $(i, j) \in \mathcal{E}$, assume that p_{ji} and p_{ij} are related to one another. To specify this relationship, we give an arbitrary orientation to every edge of the graph $\mathcal{G}$ and denote the resulting graph as $\vec{\mathcal{G}}$. Denote also the directed edge set of $\vec{\mathcal{G}}$ as $\vec{\mathcal{E}}$. If an edge $(i, j) \in \mathcal{E}$ belongs to $\vec{\mathcal{E}}$, we then express p_{ji} in terms of p_{ij} by a function $f_{ij}(\cdot)$.

Definition 1: Define the vectors $\mathbf{p_n}$, $\mathbf{p_e}$ and $\mathbf{p_d}$ as

$$\mathbf{p_n} = \{p_i \mid \forall i \in \mathcal{N}\} \tag{2a}$$
$$\mathbf{p_e} = \{p_{ij} \mid \forall (i, j) \in \mathcal{E}\} \tag{2b}$$
$$\mathbf{p_d} = \{p_{ij} \mid \forall (i, j) \in \vec{\mathcal{E}}\} \tag{2c}$$

The symbols $\mathbf{p_n}$, $\mathbf{p_e}$, and $\mathbf{p_d}$ are referred to as *injection vector, flow vector*, and *semi-flow vector*, respectively.

Note that $\mathbf{p_e}$ contains two flows per line, whereas $\mathbf{p_d}$ includes one flow per line.

Definition 2: Given two arbitrary points $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, the box $\mathcal{B}(\mathbf{x}, \mathbf{y})$ is defined as

$$B(\mathbf{x}, \mathbf{y}) = \{\mathbf{z} \in \mathbb{R}^n \mid \mathbf{x} \leq \mathbf{z} \leq \mathbf{y}\} \tag{3}$$

Assume that each injection p_i and each flow p_{ij} must be within the pre-specified intervals $[p_i^{\min}, p_i^{\max}]$ and $[p_{ij}^{\min}, p_{ij}^{\max}]$, respectively, for every $i \in \mathcal{N}$ and $(i, j) \in \vec{\mathcal{E}}$. We use the shorthand notation $\mathcal{B}$ for the box $\mathcal{B}(\mathbf{p_n}^{\min}, \mathbf{p_n}^{\max})$, where $\mathbf{p_n}^{\min}$ and $\mathbf{p_n}^{\max}$ are the vectors of the lower bounds $p_i^{\min}$'s and the upper bounds $p_i^{\max}$'s, respectively. This paper is concerned with the problem to be introduced below.

Generalized network flow (GNF): This optimization is defined as

$$\underset{\substack{\mathbf{p_n} \in \mathbb{R}^{|\mathcal{N}|} \\ \mathbf{p_e} \in \mathbb{R}^{|\mathcal{E}|}}}{\text{minimize}} \quad \sum_{i \in \mathcal{N}} f_i(p_i) \tag{4a}$$
$$\text{subject to} \quad p_i = \sum_{j \in \mathcal{N}(i)} p_{ij} \qquad \forall i \in \mathcal{N} \tag{4b}$$
$$p_{ji} = f_{ij}(p_{ij}) \qquad \forall (i, j) \in \vec{\mathcal{E}} \tag{4c}$$
$$p_{ij}^{\min} \leq p_{ij} \leq p_{ij}^{\max} \qquad \forall (i, j) \in \vec{\mathcal{E}} \tag{4d}$$
$$p_i^{\min} \leq p_i \leq p_i^{\max} \qquad \forall i \in \mathcal{N} \tag{4e}$$

where

1) $f_i(\cdot)$ is convex and monotonically increasing for every $i \in \mathcal{N}$.
2) $f_{ij}(\cdot)$ is convex and monotonically decreasing for every $(i, j) \in \vec{\mathcal{E}}$.
3) The limits $p_{ij}^{\min}$ and $p_{ij}^{\max}$ are given for every $(i, j) \in \vec{\mathcal{E}}$.
4) The limits $p_i^{\min}$ and $p_i^{\max}$ are given for every $i \in \mathcal{N}$.

In the case where $f_{ij}(p_{ji})$ is equal to $-p_{ij}$ for every $(i, j) \in \vec{\mathcal{E}}$, the GNF problem reduces to the lossless network flow problem. A few remarks can be made here:

- Given an edge $(i, j) \in \vec{\mathcal{E}}$, there is no explicit limit on p_{ji} in the formulation of the GNF problem because restricting p_{ji} is equivalent to limiting p_{ij}.
- Given a node $i \in \mathcal{N}$, the assumption of $f_i(p_i)$ being monotonically increasing is motivated by the fact that increasing the injection p_i normally elevates the cost in practice.
- Given an edge $(i, j) \in \vec{\mathcal{E}}$, p_{ij} and $-p_{ji}$ can be regarded as the input and output flows of the line (i, j) traveling in the same direction. The assumption of $f_{ij}(p_{ij})$ being monotonically decreasing is motivated by the fact that increasing the input flow normally makes the output flow higher in practice (note that $-p_{ji} = -f_{ij}(p_{ij})$).

The GNF problem is non-convex and hard to solve in general. However, it has been shown in [22] that the globally optimal injection vector is unique and can be found efficiently using a convex relaxation, named convexified generalized network flow. This problem is stated below.

Convexified generalized network flow (CGNF): This optimization is defined as

$$\underset{\substack{\mathbf{p_n} \in \mathbb{R}^{|\mathcal{N}|} \\ \mathbf{p_e} \in \mathbb{R}^{|\mathcal{E}|}}}{\text{minimize}} \quad \sum_{i \in \mathcal{N}} f_i(p_i) \tag{5a}$$
$$\text{subject to} \quad p_i = \sum_{j \in \mathcal{N}(i)} p_{ij} \qquad \forall i \in \mathcal{N} \tag{5b}$$
$$p_{ji} \geq f_{ij}(p_{ij}) \qquad \forall (i, j) \in \vec{\mathcal{E}} \tag{5c}$$
$$p_{ij}^{\min} \leq p_{ij} \leq p_{ij}^{\max} \qquad \forall (i, j) \in \mathcal{E} \tag{5d}$$
$$p_i^{\min} \leq p_i \leq p_i^{\max} \qquad \forall i \in \mathcal{N} \tag{5e}$$

where $(p_{ji}^{\min}, p_{ji}^{\max})$ is equal to $(f_{ij}(p_{ij}^{\max}), f_{ij}(p_{ij}^{\min}))$ for every $(i, j) \in \vec{\mathcal{E}}$.

Note that CGNF is a convex optimization problem, which is obtained from GNF through two operations: (i) the equality constraint (4c) is relaxed to a convex inequality, (ii) upper and lower bounds are added to p_{ji} for every $(i,j) \in \vec{\mathcal{E}}$. Throughout this paper, we make the mild assumptions that GNF is feasible and that strong duality holds for CGNF. This paper is build upon the following result from [22].

Lemma 1: Let $(\mathbf{p}_\mathbf{n}^*, \mathbf{p}_\mathbf{d}^*)$ and $(\hat{\mathbf{p}}_\mathbf{n}^*, \hat{\mathbf{p}}_\mathbf{d}^*)$ denote arbitrary globally optimal solutions of GNF and CGNF problems, respectively. The relation $\mathbf{p}_\mathbf{n}^* = \hat{\mathbf{p}}_\mathbf{n}^*$ holds. ■

Lemma 1 from [22] proves that the relaxation from GNF to CGNF is exact in terms of the optimal cost and also returns a correct optimal injection vector. However, it is straightforward to contrive examples for which the relaxation produces infeasible flow vectors. More precisely, it can be verified that although the optimal injection vector $\mathbf{p}_\mathbf{n}^*$ is unique (due to the strong convexity of the objective function), there may exist an infinite, non-convex set of solutions for the optimal flow vector. This contributes to the possible failure of CGNF in finding an optimal flow vector for GNF.

Definition 3: It is said that **CGNF is equivalent to GNF** if every arbitrary global solution $(\hat{\mathbf{p}}_\mathbf{n}^*, \hat{\mathbf{p}}_\mathbf{d}^*)$ of CGNF is a solution of GNF and vice versa.

Let $(\mathbf{p}_\mathbf{n}^*, \mathbf{p}_\mathbf{d}^*)$ and $(\hat{\mathbf{p}}_\mathbf{n}^*, \hat{\mathbf{p}}_\mathbf{d}^*)$ denote arbitrary optimal solutions of GNF and CGNF problems, respectively. It can be deduced from Lemma 1 that $(\mathbf{p}_\mathbf{n}^*, \mathbf{p}_\mathbf{d}^*)$ is a solution of CGNF, but $(\hat{\mathbf{p}}_\mathbf{n}^*, \hat{\mathbf{p}}_\mathbf{d}^*)$ may not be a solution of GNF. Note that GNF and CGNF are equivalent if every flow vector returned by CGNF is feasible for GNF. The main objective of this paper is twofold: (i) studying the equivalence of GNF and CGNF, (ii) characterizing the set of all optimal flow solutions.

III. Sufficient Condition: Pareto Points

In this section, the objective is to derive a sufficient condition under which CGNF and GNF are equivalent. Define $\mathcal{P}$ as the set of all vectors $\mathbf{p}_\mathbf{n}$ for which there exists a vector $\mathbf{p}_\mathbf{e}$ satisfying (4b), (4c), and (4d). The set $\mathcal{P}$ and $\mathcal{P} \cap \mathcal{B}$ are referred to as **injection region** and **box-constrained injection region**, respectively.

Definition 4: Consider an arbitrary set $\mathcal{S} \in \mathbb{R}^n$ together with a point $\mathbf{x} \in \mathcal{S}$. The point $\mathbf{x}$ is called **Pareto** if there does not exist another point $\mathbf{y} \in \mathcal{S}$ that is less than or equal to $\mathbf{x}$ entry-wise. $\mathbf{x} \in \mathcal{S}$ is called an **interior point** if $\mathcal{S}$ constrains a ball around this point. $\mathbf{x} \in \mathcal{S}$ is called a **boundary point** if it is not an interior point.

It is easy to show that a Pareto point is a boundary point, but the converse statement is not always true. In what follows, we will prove that the equivalence of GNF and CGNF can be related to the Pareto points of the injection region.

Theorem 1: Let $\mathbf{p}_\mathbf{n}^*$ denote the unique optimal injection vector for GNF (or CGNF). If $\mathbf{p}_\mathbf{n}^*$ is a Pareto point of the injection region $\mathcal{P}$, then CGNF is equivalent to GNF.

Proof: To prove by contradiction, assume that $\hat{\mathbf{p}}_\mathbf{e}^*$ is an optimal flow vector for CGNF that is not feasible for GNF. Define a new flow vector $\hat{\mathbf{p}}_\mathbf{e}^\mathbf{c}$ as follows:

$$\hat{p}_{ij}^c = \hat{p}_{ij}^*, \qquad \forall (i,j) \in \vec{\mathcal{E}} \tag{6a}$$

$$\hat{p}_{ji}^c = f_{ij}(\hat{p}_{ij}^*), \qquad \forall (i,j) \in \vec{\mathcal{E}} \tag{6b}$$

Let $\hat{\mathbf{p}}_\mathbf{n}^\mathbf{c}$ denote the injection vector corresponding to $\hat{\mathbf{p}}_\mathbf{e}^\mathbf{c}$. Since $\hat{p}_{ji}^* = f_{ij}(\hat{p}_{ij}^*)$ for every $(i,j) \in \vec{\mathcal{E}}$, it can be concluded that $\hat{\mathbf{p}}_\mathbf{n}^\mathbf{c} \leq \mathbf{p}_\mathbf{n}^*$. On the other hand, we have $\hat{\mathbf{p}}_\mathbf{n}^\mathbf{c} \neq \mathbf{p}_\mathbf{n}^*$ because $\mathbf{p}_\mathbf{e}^*$ was assumed not to be feasible for GNF. Since $\hat{\mathbf{p}}_\mathbf{n}^\mathbf{c}$ belongs to $\mathcal{P}$ by design, the point $\mathbf{p}_n^*$ cannot be a Pareto point of $\mathcal{P}$ due to the relation $\hat{\mathbf{p}}_\mathbf{n}^\mathbf{c} \leq \mathbf{p}_\mathbf{n}^*$. This contradiction completes the proof. ■

Two examples will be provided below to elaborate on the result of Theorem 1. For simplicity in developing the technical results and with no loss of generality, it is assumed throughout the paper that there is at most one edge between every two nodes. But the examples offered here consider multiple edges between two nodes.

Example 1: Consider the 2-node graph $\mathcal{G}$ depicted in Figure 1(a). Let $(p_{12}^{(1)}, p_{21}^{(1)})$ and $(p_{12}^{(2)}, p_{21}^{(2)})$ denote the flows associated with the first and second edges of the graph, respectively. Consider the GNF problem

$$\text{minimize} \quad f_1(p_1) + f_2(p_2) \tag{7a}$$

$$\text{subject to} \quad p_{21}^{(i)} = \left(p_{12}^{(i)} - 1\right)^2 - 1 \quad i = 1,2 \tag{7b}$$

$$-0.5 \leq p_{12}^{(1)} \leq 0.5 \tag{7c}$$

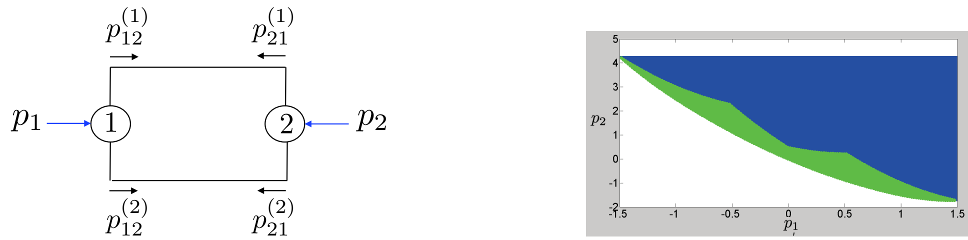
$$-1 \leq p_{12}^{(2)} \leq 1, \tag{7d}$$

$$p_1 = p_{12}^{(1)} + p_{12}^{(2)} \tag{7e}$$

$$p_2 = p_{21}^{(1)} + p_{21}^{(2)} \tag{7f}$$

with the variables $p_1, p_2, p_{12}^{(1)}, p_{21}^{(1)}, p_{12}^{(2)}, p_{21}^{(2)}$, where $f_1(\cdot)$ and $f_2(\cdot)$ are both convex and monotonically increasing. The CGNF problem corresponding to this problem can be obtained by relaxing (7b) to $p_{21}^{(i)} \geq (p_{12}^{(i)} - 1)^2 - 1$ and adding the bounds $p_{21}^{(1)} \leq 1.5^2 - 1$ and $p_{21}^{(2)} \leq 2^2 - 1$. The injection regions of GNF and CGNF problems (without the box constraint $\mathbf{p}_\mathbf{n} \in \mathcal{B}$) are drawn in Figure 1(b). The green area is the injection region $\mathcal{P}$ of the GNF problem, and the union of the green and blue areas is the injection region of the CGNF problem. It can be observed that every point on the lower curvy boundary of the feasible set is a Pareto point. Therefore, if the box $\mathcal{B}$ induced by the lower and upper constraints on p_1 and p_2 intersects with any part of the lower boundary of the green area, CGNF always finds optimal flow vectors for GNF, leading to the equivalence of GNF and CGNF (see Figures 3(a) and (b) for possible scenarios).

Example 2: Theorem 1 states that CGNF and GNF are equivalent if the optimal injection vector is Pareto. As stated before, a Pareto point lies on the boundary of the injection region. A question arises as to whether the condition "Pareto point" can be replaced by "boundary point" in Theorem 1.



(a) (b)

Fig. 1: (a) The 2-node graph $\mathcal{G}$ studied in Example 1; (b): injection region $\mathcal{P}$ for the 2-node graph $\mathcal{G}$ in Example 1.

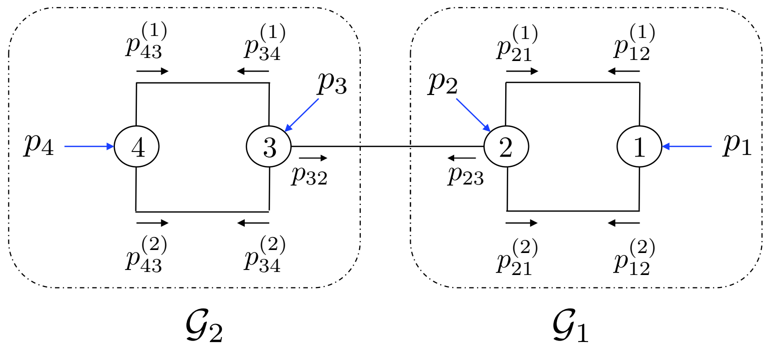


Fig. 2: The 4-node graph $\mathcal{G}$ studied in Example 2.

(a) (b)

Fig. 3: (a) Injection region of the subgraph $\mathcal{G}_1$ in Example 2; (b): injection region of the subgraph $\mathcal{G}_2$ in Example 2.

We will provide an example here to show that the optimal injection being a boundary point does not necessarily guarantee the equivalence of GNF and CGNF. To this end, consider the 4-node graph $\mathcal{G}$ given in Figure 2. This graph can be decomposed into two subgraphs $\mathcal{G}_1$ and $\mathcal{G}_2$, where each subgraph has the same topology as the 2-node graph studied in Example 1. Assume that the flow over the line $(2,3)$ is restricted to zero, by imposing the constraints $p_{23}^{\min} = p_{23}^{\max} = p_{32}^{\min} = p_{32}^{\max} = 0$. This implies that the link $(2,3)$ is redundant, whose removal splits the graph $\mathcal{G}$ into two disjoint subgraphs $\mathcal{G}_1$ and $\mathcal{G}_2$. Accordingly, the vector $\mathbf{p}_\mathbf{n}^*$ can be broken down into two parts as

$$\mathbf{p}_\mathbf{n}^* = [\mathbf{p}_\mathbf{n}^*(\mathcal{G}_\mathbf{1})^\mathbf{T} \ \mathbf{p}_\mathbf{n}^*(\mathcal{G}_\mathbf{2})^\mathbf{T}]^\mathbf{T} \tag{8}$$

where $\mathbf{p}_\mathbf{n}^*(\mathcal{G}_\mathbf{1})$ and $\mathbf{p}_\mathbf{n}^*(\mathcal{G}_\mathbf{2})$ denote the optimal values of the sub-vectors $[p_1 \ p_2]^T$ and $[p_3 \ p_4]^T$, respectively. Let $\mathcal{P}(\mathcal{G}_1) \in \mathbb{R}^2$ and $\mathcal{P}(\mathcal{G}_2) \in \mathbb{R}^2$ denote the injection regions associated with the subgraphs $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. Note that $\mathcal{P}(\mathcal{G}_1)$ and $\mathcal{P}(\mathcal{G}_2)$ could both resemble the green area in Figure 1(b). We make two assumptions here:

- *Assumption 1:* As demonstrated in Figure 3(a), the box constraints on p_1 and p_2 are such that $\mathbf{p}_\mathbf{n}^*(\mathcal{G}_\mathbf{1})$ becomes a Pareto point located on the lower boundary of $\mathcal{P}(\mathcal{G}_1)$. In this case, it is guaranteed from Theorem 1 that if CGNF is solved just over $\mathcal{G}_1$, it always finds feasible flows for this subgraph.
- *Assumption 2:* As demonstrated in Figure 3(b), the box constraints on p_3 and p_4 are such that $\mathbf{p}_\mathbf{n}^*(\mathcal{G}_\mathbf{2})$ becomes an interior point of $\mathcal{P}(\mathcal{G}_2)$, corresponding to the lower left corner of the box. In this case, assume that if CGNF is solved just over $\mathcal{G}_2$, it cannot always find feasible flows for this subgraph (see [22] for such an example).

Since the link $(2,3)$ is not allowed to carry any flow, it is easy to show that CGNF solved over $\mathcal{G}$ finds feasible flows for the lines between nodes 1 and 2, but may result in wrong flows for the lines between nodes 3 and 4. Hence, CGNF and GNF are not equivalent. On the other hand, it is straightforward to inspect that $\mathcal{P}$ is the product of two regions as

$$\mathcal{P} = \mathcal{P}(\mathcal{G}_1) \times \mathcal{P}(\mathcal{G}_2) \tag{9}$$

Now, since $\mathbf{p_n^*}(\mathcal{G}_1)$ is on the boundary of $\mathcal{P}(\mathcal{G}_1)$ but $\mathbf{p_n^*}(\mathcal{G}_2)$ is in the interior of $\mathcal{P}(\mathcal{G}_2)$, it can be deduced that

- $\mathbf{p_n^*}$ is on the boundary of the injection region $\mathcal{P}$.
- $\mathbf{p_n^*}$ is not a Pareto point of the injection region $\mathcal{P}$.

In summary, although $\mathbf{p_n^*}$ is a boundary point for $\mathcal{G}$, CGNF is not equivalent to GNF. This is due to the connection of a well-behaved subgraph $\mathcal{G}_1$ to a problematic subgraph $\mathcal{G}_2$ via a redundant link with no flow. It will be shown in the next section that whenever CGNF fails to work for an arbitrary graph $\mathcal{G}$, the network can be decomposed into two subgraphs $\mathcal{G}_1$ and $\mathcal{G}_2$ such that the flows over $\mathcal{G}_1$ are all feasible.

IV. Network Decomposition

So far, we have shown that if the optimal injection vector is a Pareto point, GNF is equivalent to CGNF. In this section, we consider the case where the optimal injection vector is not necessarily Pareto but lies on the boundary of the injection region. The objective is to prove that the network $\mathcal{G}$ can be decomposed into two subgraphs $\mathcal{G}_1$ and $\mathcal{G}_2$ such that: (i) the flows obtained from CGNF are optimal (feasible) for GNF for those lines in $\mathcal{G}_1$ and between $\mathcal{G}_1$ and $\mathcal{G}_2$, (ii) the flows over the lines between $\mathcal{G}_1$ and $\mathcal{G}_2$ all hit their limits at optimality.

Since $f_i(p_i)$ can be approximated by a differentiable function arbitrarily precisely, with no loss of generality, assume that $f_i(p_i)$ is differentiable with a nonzero derivative for every $i \in \mathcal{N}$. Let $\mathbf{p_n^*}$ denote the unique optimal injection vector for both GNF and CGNF. Moreover, let $\underline{\lambda}_i^*$ and $\bar{\lambda}_i^*$ denote any optimal Lagrange multipliers corresponding to the constraints $p_i^{\min} \leq p_i$ and $p_i \leq p_i^{\max}$ in the convex CGNF problem. Define

$$\lambda_i^* = f_i'(p_i^*) - \underline{\lambda}_i^* + \bar{\lambda}_i^*, \qquad \forall i \in \mathcal{N} \tag{10}$$

Definition 5: Define $\mathcal{N}_1$ as the set of all vertices $i \in \mathcal{N}$ such that $\lambda_i^* > 0$, and $\mathcal{N}_2$ as the complement of $\mathcal{N}_1$ in the set $\mathcal{N}$. Also, define $\mathcal{G}_1$ and $\mathcal{G}_2$ as the subgraphs of $\mathcal{G}$ induced by the vertex subsets $\mathcal{N}_1$ and $\mathcal{N}_2$, respectively. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ denote the edge sets of $\mathcal{G}_1$ and $\mathcal{G}_2$.

Theorem 2: Let $(\mathbf{p_n^*}, \mathbf{p_d^*})$ and $(\mathbf{p_n^*}, \hat{\mathbf{p}}_\mathbf{d}^*)$ denote arbitrary globally optimal solutions of GNF and CGNF problems, respectively. The following relations hold:

$$p_{ij}^* = \hat{p}_{ij}^*, \qquad \forall (i,j) \in \mathcal{E}_1 \tag{11a}$$
$$p_{ji}^* = \hat{p}_{ji}^* = p_{ji}^{\max}, \qquad \forall (i,j) \in (\mathcal{N}_1 \times \mathcal{N}_2) \cap \mathcal{E} \tag{11b}$$

Proof: Since every solution of GNF is a solution of CGNF as well, $(\mathbf{p_n^*}, \mathbf{p_d^*})$ and $(\mathbf{p_n^*}, \hat{\mathbf{p}}_\mathbf{d}^*)$ are both solutions of CGNF. Now, it follows from the duality theorem that $(\mathbf{p_n^*}, \mathbf{p_d^*})$ and $(\mathbf{p_n^*}, \hat{\mathbf{p}}_\mathbf{d}^*)$ are both minimizers of the optimization problem

$$\underset{\mathbf{p_n}, \mathbf{p_e}}{\text{minimize}} \quad \sum_{i \in \mathcal{N}} \lambda_i^* p_i \tag{12a}$$
$$\text{subject to} \quad p_i = \sum_{j \in \mathcal{N}(i)} p_{ij} \qquad \forall i \in \mathcal{N} \tag{12b}$$
$$f_{ij}(p_{ij}) \leq p_{ji} \qquad \forall (i,j) \in \vec{\mathcal{E}} \tag{12c}$$
$$p_{ij}^{\min} \leq p_{ij} \leq p_{ij}^{\max} \qquad \forall (i,j) \in \mathcal{E} \tag{12d}$$

Substituting (12b) into (12a) yields that (p_{ij}^*, p_{ji}^*) and $(\hat{p}_{ij}^*, \hat{p}_{ji}^*)$ are both optimal solutions of the 2-variable optimization problem

$$\underset{(p_{ij}, p_{ji}) \in \mathbb{R}^2}{\text{minimize}} \quad \lambda_i^* p_{ij} + \lambda_j^* p_{ji} \tag{13a}$$
$$\text{subject to} \quad f_{ij}(p_{ij}) \leq p_{ji} \tag{13b}$$
$$p_{ij}^{\min} \leq p_{ij} \leq p_{ij}^{\max} \tag{13c}$$
$$p_{ji}^{\min} \leq p_{ji} \leq p_{ji}^{\max} \tag{13d}$$

for every $(i,j) \in \vec{\mathcal{E}}$. Since the objective function of the above optimization problem is linear, two observations can be made here:

- The inequality (13b) must be binding at optimality as long as $\lambda_i^* > 0$ or $\lambda_j^* > 0$.
- (p_{ij}, p_{ji}) becomes equal to $(p_{ij}^{\min}, p_{ji}^{\max})$ at optimality if $\lambda_i^* > 0$ and $\lambda_j^* \leq 0$.
- (p_{ij}, p_{ji}) becomes equal to $(p_{ij}^{\max}, p_{ji}^{\min})$ at optimality if $\lambda_j^* > 0$ and $\lambda_i^* \leq 0$.

The proof follows immediately from the above properties. ■

Theorem 2 states that CGNF finds correct values for the flows of those lines inside $\mathcal{G}_1$ and between $\mathcal{G}_1$ and $\mathcal{G}_2$. In addition, the flows over the lines between $\mathcal{G}_1$ and $\mathcal{G}_2$ all hit their limits at optimality.

Corollary 1: Let $(\mathbf{p_n^*}, \mathbf{p_d^*})$ and $(\mathbf{p_n^*}, \hat{\mathbf{p}}_\mathbf{d}^*)$ denote arbitrary globally optimal solutions of GNF and CGNF problems, respectively. If there exists a vertex $i \in \mathcal{N}$ such that $\hat{p}_i^* > p_i^{\min}$, then $\mathbf{p_d^*}$ and $\hat{\mathbf{p}}_\mathbf{d}^*$ must be identical in at least one entry.

Proof: Consider a vertex $i \in \mathcal{N}$ such that $\hat{p}_i^* > p_i^{\min}$. It follows from (10) that λ_i^* is positive. Now, Definition 5 yields that the subgraph $\mathcal{G}_1$ is nonempty. The proof is an immediate consequence of Theorem 2. ■

Definition 6: A line $(i,j) \in \mathcal{E}$ of the network $\mathcal{G}$ is called **congested** if the GNF problem has a solution $(\mathbf{p_n^*}, \mathbf{p_d^*})$ such that p_{ij}^* is equal to $p_{ij}^{\max}$ or p_{ji}^* is equal to $p_{ji}^{\max}$.

Corollary 2: Assume that there exists a vertex $i \in \mathcal{N}$ such that $\hat{p}_i^* > p_i^{min}$. If the network $\mathcal{G}$ has no congested line, then GNF and CGNF are equivalent.

Proof: Due to the proof of Corollary 1, the set $\mathcal{N}_1$ is nonempty. On the other hand, since the network $\mathcal{G}$ has no congested line by assumption, it can be concluded from Theorem 2 that $(\mathcal{N}_1 \times \mathcal{N}_2) \cap \mathcal{E}$ is an empty set. Therefore, $\mathcal{N}_1$ must be equal to $\mathcal{N}$, which implies the equivalence of GNF and CGNF due to Theorem 2. ■

Corollary 2 states that whenever CGNF fails to find feasible flows for all lines of the network, some lines must be congested at optimality.

V. Characterization of Flow Vectors

In this section, we aim to find the set of all optimal flow vectors for GNF. We fully characterize this set and show that it may be nonconvex and disconnected.

Definition 7: Define $f_{ji}(p_{ji}) = f_{ij}^{-1}(p_{ji})$ for every $(i,j) \in \vec{\mathcal{E}}$. This makes the flow constraint (4c) equivalent to

$$p_{ji} = f_{ij}(p_{ij}), \quad \forall (i,j) \in \mathcal{E} \tag{14}$$

where $f_{ij}(\cdot)$ is convex and monotonically decreasing.

Before presenting our next result in its full generality, we illustrate the key ideas in two examples below.

Example 3: Consider the graph $\mathcal{G}$ given in Figure 4(a), which consists of one cycle and four nodes. Assume that CGNF and GNF may not be equivalent. Let $(\mathbf{p_n^*}, \mathbf{p_e^*})$ denote an arbitrary solution of GNF, where $\mathbf{p_n^*}$ is obtained from CGNF and $\mathbf{p_e^*}$ is to be found. The objective of this example is to demonstrate that all optimal flows in the network can be uniquely characterized in terms of a single flow. Consider the unknown flow p_{12}^*. One can write:

$$p_{23}^* = p_2^* - f_{12}(p_{12}^*) \tag{15a}$$
$$p_{34}^* = p_3^* - f_{23}\left(p_2^* - f_{12}(p_{12}^*)\right) \tag{15b}$$
$$p_{41}^* = p_4^* - f_{34}\left(p_3^* - f_{23}\left(p_2^* - f_{12}(p_{12}^*)\right)\right) \tag{15c}$$

It follows from the above equations that all flows in the network can be cast as functions of p_{12}^*, and in addition $p_{12} = p_{12}^*$ is a solution to the level-set problem $F(p_{12}, p_2^*, p_3^*, p_4^*) = p_1^*$, where $F(p_{12}, p_2, p_3, p_4)$ is defined as

$$p_{12} + f_{41}\left(p_4 - f_{34}\left(p_3 - f_{23}\left(p_2 - f_{12}(p_{12})\right)\right)\right) \tag{16}$$

It can be verified that

- Due to (15), each of the flows $p_{23}^*, p_{34}^*, p_{41}^*$ is a concave, increasing function of p_{12}^*. Hence, the flow constraints $p_{ij}^{\min} \le p_{ij}^* \le p_{ij}^{\max}$, $(i,j) \in \vec{\mathcal{E}}$, can all be equivalently translated into a single constraint $\tilde{p}_{12}^{\min} \le p_{12}^* \le \tilde{p}_{12}^{\max}$, for some constants $\tilde{p}_{12}^{\min}$ and $\tilde{p}_{12}^{\max}$.
- The function $F(p_{12}, p_2, p_3, p_4)$ is convex (but not necessarily monotonic) with respect to its argument p_{12}.

As illustrated in Figure 4(b), the level-set problem $F(p_{12}, p_2^*, p_3^*, p_4^*) = p_1^*$ has up to two disjoint solutions, and each or both of them could be optimal flows for GNF, depending on which one of the level-set solutions satisfies the constraint $\tilde{p}_{12}^{\min} \le p_{12}^* \le \tilde{p}_{12}^{\max}$.

Example 4: Consider the graph $\mathcal{G}$ given in Figure 5(a), which consists of two cycles and four nodes. Let $(\mathbf{p_n^*}, \mathbf{p_e^*})$ denote an arbitrary solution of GNF, where $\mathbf{p_n^*}$ is obtained from CGNF and $\mathbf{p_e^*}$ is to be found. The objective of this example is to demonstrate that all optimal flows in the network can be uniquely characterized in terms of two flows. Consider the unknown flows p_{12}^* and p_{13}^*. One can write

$$p_{24}^* = p_2^* - f_{12}(p_{12}^*) \tag{17a}$$
$$p_{34}^* = p_3^* - f_{13}(p_{13}^*) \tag{17b}$$
$$p_{14}^* = p_1^* - p_{12}^* - p_{13}^* \tag{17c}$$

It follows from the above equations that all flows in the network can be cast as functions of (p_{12}^*, p_{13}^*), and in addition $(p_{12}, p_{13}) = (p_{12}^*, p_{13}^*)$ is a solution to the level-set problem $F(p_{12}, p_{13}, p_1^*, p_2^*, p_3^*) = p_4^*$, where

$$\begin{aligned} F(p_{12}, p_{13}, p_1, p_2, p_3) = {} & f_{24}\left(p_2 - f_{12}(p_{12})\right) \\ & + f_{34}\left(p_3 - f_{13}(p_{13})\right) \\ & + f_{14}\left(p_1 - p_{12} - p_{13}\right) \end{aligned} \tag{18}$$

is a convex function with respect to (p_{12}, p_{13}) but not necessarily monotonic. On the other hand, the equations in (17) can be used to translate the box constraints on all flows to certain constraints only on p_{12}^* and p_{13}^*:

$$\tilde{p}_{12}^{\min} \le p_{12}^* \le \tilde{p}_{12}^{\max} \tag{19a}$$
$$\tilde{p}_{13}^{\min} \le p_{13}^* \le \tilde{p}_{13}^{\max} \tag{19b}$$
$$p_{14}^{\min} \le p_1^* - p_{12}^* - p_{13}^* \le p_{14}^{\max} \tag{19c}$$

for some numbers $\tilde{p}_{12}^{\min}, \tilde{p}_{12}^{\max}, \tilde{p}_{13}^{\min}, \tilde{p}_{13}^{\max}$. Let C_1 and C_2 denote the sets of all points (p_{12}^*, p_{13}^*) satisfying the level-set problem $F(p_{12}^*, p_{13}^*, p_1^*, p_2^*, p_3^*) = p_4^*$ and the reformulated flow constraints (19), respectively. The set of all optimal flow solutions (p_{12}^*, p_{13}^*) can be expressed as $\mathcal{C}_1 \cap \mathcal{C}_2$, where $\mathcal{C}_1$ is the boundary of a convex set (corresponding to $F(\cdot)$) and $\mathcal{C}_2$ is a polytope. As illustrated in Figure 5(b), $\mathcal{C}_1$ is the boundary of a convex set, and therefore its intersection with a polytope (e.g., a box) could form up to 4 disconnected components. In summary, the optimal flow vectors for GNF may constitute a nonconvex infinite set, consisting of as high as 4 disconnected components.

A. Algebraic Characterization of Flows

It is straightforward to show that if the graph $\mathcal{G}$ is a tree, the optimal flow vector is unique and can be easily obtained from the optimal injection vector $\mathbf{p_n^*}$. Hence, the main challenge is to deal with mesh flow networks. To this end, consider an arbitrary spanning tree of the m-node graph $\mathcal{G}$, denoted as $\mathcal{G}_t$. Let $\mathbf{p_{dt}}$ denote a sub-vector of the semi-flow vector $\mathbf{p_d}$ associated with those edges of $\mathcal{G}$ that do not exist in $\mathcal{G}_t$.

Lemma 2: There exist convex or convex monotonic functions $F_{ij} : \mathbb{R}^{|\mathcal{E}|} \to \mathbb{R}$, for every $(i,j) \in \mathcal{E}$, for which the following statements hold:

1) Given every arbitrary feasible solution $(\mathbf{p_n}, \mathbf{p_e})$ of the GNF problem, the relations

$$p_{ij} = F_{ij}(\mathbf{p_{dt}}, p_1, p_2, \ldots, p_{m-1}), \qquad (i,j) \in \mathcal{E} \tag{20}$$

are satisfied.

2) The function $F(\mathbf{p_{dt}}, p_1, p_2, \ldots, p_{m-1})$ defined as

$$\sum_{j \in \mathcal{N}(m)} F_{mj}(\mathbf{p_{dt}}, p_1, p_2, \ldots, p_{m-1}) \tag{21}$$

is convex.

Proof: The proof is in line with the technique used in Examples 3 and 4. The details are omitted for brevity. ■

Definition 8: Define $\mathcal{C}_1$ as the set of all vectors $\mathbf{p_{dt}}$ satisfying the level-set problem $F(\mathbf{p_{dt}}, p_1^*, p_2^*, \ldots, p_{m-1}^*) =$

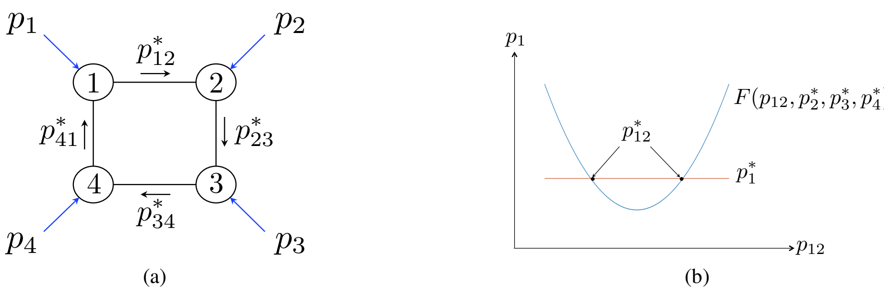


Fig. 4: (a) The 1-cycle graph studied in Example 3; (b): visualization of the level-set problem used to find optimal flows for Example 3.

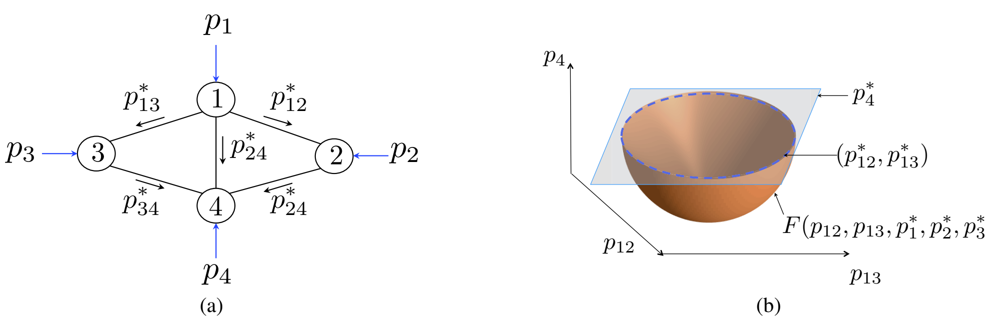


Fig. 5: (a) The 2-cycle graph studied in Example 4; (b): visualization of the level-set problem used to find optimal flows for Example 4.

p_m^*. Also, define $\mathcal{C}_2$ as the set of all vectors $\mathbf{p_{dt}}$ satisfying the inequalities

$$p_{ij}^{\min} \leq F_{ij}(\mathbf{p_{dt}}, p_1^*, p_2^*, ..., p_{m-1}^*) \leq p_{ij}^{\max}, \quad (i,j) \in \mathcal{E} \tag{22}$$

Theorem 3: A flow vector $\mathbf{p_e^*}$ is optimal for GNF if and only if

$$\mathbf{p_{dt}^*} \in \mathcal{C}_1 \cap \mathcal{C}_2 \tag{23a}$$

$$p_{ij}^* = F_{ij}(\mathbf{p_{dt}^*}, p_1^*, p_2^*, ..., p_{m-1}^*), \qquad (i,j) \in \mathcal{E} \tag{23b}$$

Proof: The proof is based on Lemma 2 and the technique used in Examples 3 and 4. The details are omitted for brevity. ■

Theorem 3 states that: (i) the set of optimal flow vectors can be characterized in terms of the unique optimal injection vector as well as the flow sub-vector $\mathbf{p_{dt}}$, (ii) the set of optimal flow sub-vectors $\mathbf{p_{dt}^*}$ is the collection of all points in the intersection of $\mathcal{C}_1$ and $\mathcal{C}_2$. Moreover, in light of Lemma 2, $\mathcal{C}_1$ is the boundary of a convex set. Although $\mathcal{C}_2$ was shown to be a polytope in Examples 3 and 4, it is non-convex in general. The problem of finding a sufficient condition on $\mathcal{G}$ to guarantee the convexity of $\mathcal{C}_2$ is left for future work. Since $\mathcal{C}_1$ is the boundary of a convex set, it occurs that the intersection of $\mathcal{C}_2$ with $\mathcal{C}_1$ may lead to as high as $2^{|\mathcal{E}|-|\mathcal{N}|+1}$ disconnected components, all lying on the boundary of a convex set (note that $|\mathcal{E}| - |\mathcal{N}| + 1$ is the size of the vector $\mathbf{p_{dt}}$).

VI. Conclusions

The network flow problem appears in many real-world applications and plays a key role in engineering, computer science, operation research, and sociology, among others. In this paper, we consider a nonlinear version of the classical network flow problem, referred to as generalized network flow (GNF), where there is an injection at each node, leading to two incoming and outgoing flows over each line. We assume that the flows over each line are related to one another via a nonlinear function. Under the assumptions of convexity and monotonicity of cost and flow functions, we have shown in our recent work that although GNF is highly nonconvex, optimal injections (not necessarily optimal flows) can be found by means of a convexified generalized network flow (CGNF) problem. The current paper investigates how optimal flows may be obtained, by developing three results. First, we show that CGNF produces optimal flows for GNF, as long as the optimal injection vector is a Pareto point. Second, we prove that if CGNF returns a wrong (infeasible) flow vector for GNF, then the network can be decomposed into two subgraphs such that: (i) the flows found by CGNF for one of the subgraphs are all correct, and (ii) the flows obtained by CGNF for the lines between the subgraphs are all correct and at their limits. Third, we characterize the set of all optimal flow vectors. In particular, we show that this set may be infinite, non-convex, and disconnected, but belongs to the boundary of a convex set.

References

[1] A. V. Goldberg, E. Tardos, and R. E. Tarjan, "Network flow algorithms," *Flows, Paths and VLSI (Springer, Berlin)*, pp. 101–164, 1990.
[2] W. S. Jewell, "Optimal flow through networks with gains," *Operations Research*, vol. 10, pp. 476–499, 1962.

[3] L. R. Ford and D. R. Fulkerson, "Flows in networks," *Princeton University Press*, 1962.

[4] M. Klein, "A primal method for minimal cost flows with applications to the assignment and transportation problems," *Management Science*, vol. 14, pp. 205–220, 1967.

[5] R. K. Ahuja, T. L. Magnanti, and J. B. Orlin, "Network flows: theory, algorithms, and applications," *Prentice-Hall*, 1993.

[6] D. Bertsimas and M. Sim, "Robust discrete optimization and network flows," *Mathematical Programming*, vol. 98, pp. 49–71, 2003.

[7] D. Bertsimas and S. Stock-Paterson, "The traffic flow management rerouting problem in air traffic control: A dynamic network flow approach," *Transportation Science*, vol. 34, pp. 239–255, 2000.

[8] M. S. Bazaraa, J. J. Jarvis, and H. D. Sherali, "Linear programming and network flows," *John Wiley & Sons*, 1990.

[9] J. Edmonds and R. M. Karp, "Theoretical improvements in algorithmic efficiency for network flow problems," *Journal of the ACM*, vol. 19, pp. 248–264, 1972.

[10] K. E. Nygard, P. R. Chandler, and M. Pachter, "Dynamic network flow optimization models for air vehicle resource allocation," *American Control Conference*, 2001.

[11] D. Goldfarb and J. Hao, "Polynomial-time primal simplex algorithms for the minimum cost network flow problem," *Algorithmica*, vol. 8, pp. 145–160, 1992.

[12] D. Bienstock, S. Chopra, O. Gunluk, and C. Y. Tsai, "Minimum cost capacity installation for multicommodity network flows," *Mathematical Programming*, vol. 81, pp. 177–199, 1998.

[13] J. Lavaei and S. H. Low, "Zero duality gap in optimal power flow problem," *IEEE Transactions on Power Systems*, vol. 27, no. 1, pp. 92–107, 2012.

[14] B. Zhang, A. Lam, A. Dominguez-Garcia, and D. Tse, "An optimal and distributed method for voltage regulation in power distribution systems," to appear in *IEEE Transactions on Power Systems*, 2014.

[15] S. Sojoudi and S. H. Low, "Optimal charging of plug-in hybrid electric vehicles in smart grids," in *IEEE Power and Energy Society General Meeting*, 2011.

[16] J. A. Momoh, *Electric Power System Applications of Optimization*. New York: Markel Dekker, 2001.

[17] J. Lavaei, D. Tse, and B. Zhang, "Geometry of power flows and optimization in distribution networks," *IEEE Transactions on Power Systems*, vol. 29, no. 2, pp. 572–583, 2014.

[18] D. Bienstock, M. Chertkov, and S. Harnett, "Chance-constrained optimal power flow: Risk-aware network control under uncertainty," *SIAM Review*, vol. 56, no. 3, pp. 461–495, 2014.

[19] H. Brannlund, J. A. Bubenko, D. Sjelvgren, and N. Andersson, "Optimal short term operation planning of a large hydrothermal power system based on a nonlinear network flow concept," *IEEE Transactions on Power Systems*, vol. 1, pp. 75–81, 1986.

[20] J. L. Goffin, J. Gondzio, R. Sarkissian, and J. P. Vial, "Solving nonlinear multicommodity flow problems by the analytic center cutting plane method," *Mathematical Programming*, vol. 76, pp. 131–154, 1996.

[21] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge, 2004.

[22] S. Sojoudi and J. Lavaei, "Convexification of generalized network flow problem with application to power systems," *IEEE Conference on Decision and Control*, 2013, Extended version available at http://www.ee.columbia.edu/~lavaei/Generalized_Net_Flow.pdf.

[23] M. Kraning, E. Chu, J. Lavaei, and S. Boyd, "Dynamic network energy management via proximal message passing," *Foundations and Trends in Optimization*, vol. 1, no. 2, pp. 1–54, 2013.