Convexification of Generalized Network Flow Problem with Application to Power Systems

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***Abstract*—This paper is concerned with the minimum-cost flow problem over an arbitrary flow network. In this problem, each node is associated with some possibly unknown injection, each line has two unknown flows at its ends related to each other via a nonlinear function, and all injections and flows need to satisfy certain box constraints. This problem, named generalized network flow (GNF), is highly non-convex due to its nonlinear equality constraints. Under the practical assumption of monotonicity and convexity of the flow and cost functions, a convex relaxation is proposed, which always finds the optimal injections. This relaxation may fail to find optimal flows because the mapping from injections to flows might lead to an exponential number of solutions. However, once optimal injections are found in polynomial time, other techniques can be used to find a feasible set of flows corresponding to the injections. A primary application of this work is in optimization over power networks. Recent work on the optimal power flow (OPF) problem has shown that this non-convex problem can be solved efficiently using semidefinite programming (SDP) after two approximations: relaxing angle constraints (by adding virtual phase shifters) and relaxing power balance equations to inequality constraints. The results of this work prove two facts for the OPF problem: (i) the second relaxation (on balance equations) is not needed in practice under a very mild angle assumption, (ii) if the SDP relaxation fails to find a rank-one solution, the optimal injections (and not flows) may still be recovered from an undesirable high-rank solution.**

I. Introduction

The area of "network flows" plays a central role in operations research, computer science and engineering [1], [2]. This area is motivated by many real-word applications in assignment, transportation, communication networks, electrical power distribution, production scheduling, financial budgeting, and aircraft routing, to name only a few. Started by the classical book [3] in 1962, network flow problems have been studied extensively [4], [5], [6], [7].

The minimum-cost flow problem aims to optimize the flows over a flow network that is used to carry some commodity from suppliers to consumers. In a flow network, there is an injection of some commodity at every node, which leads to two flows over each line (arc) at its endpoints. The injection—depending on being positive or negative, corresponds to supply or demand at the node. The minimum-cost flow problem has been studied thoroughly for a lossless network, where the amount of flow entering a line equals the amount of flow leaving the line. However, since many real-world flow networks are lossy, the minimum-cost flow problem has also attracted much attention for generalized networks, also known as networks with gain [2], [8]. In this type of network, each line is associated with a constant gain relating the two flows of the line through a linear function. From the optimization perspective, network flow problems are convex and can be solved efficiently unless there are discrete variables involved [9].

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There are several real-world network flows that are lossy, where the loss is a nonlinear function of the flows. An important example is power distribution networks for which the loss over a transmission line (with fixed voltage magnitudes at both ends) is given by a parabolic function due to Kirchhoff's circuit laws [10]. The loss function could be much more complicated depending on the power electronic devices installed on the transmission line. To the best of our knowledge, there is no theoretical result in the literature on the polynomial-time solvability of network flow problems with nonlinear flow functions, except in very special cases. This paper is concerned with this general problem, named Generalized Network Flow (GNF). Note that the term "GNF" has already been used in the literature for networks with linear losses, but it corresponds to arbitrary lossy networks in this work.

GNF aims to optimize the nodal injections subject to flow constraints for each line and box constraints for both injections and flows. A flow constraint is a nonlinear equality relating the flows at both ends of a line. To solve GNF, this paper makes the practical assumption that the cost and flow functions are all monotonic and convex. The GNF problem is still highly non-convex due to its equality constraints. Relaxing the nonlinear equalities to convex inequalities gives rise to a convex relaxation of GNF. It can be easily observed that solving the relaxed problem may likely lead to a solution for which the new inequality flow constraints are not binding. One may speculate that this observation implies that the convex relaxation is not tight. However, the objective of this work is to show that as long as GNF is feasible, the convex relaxation is tight. More precisely, the convex relaxation always finds the optimal injections (and hence the optimal objective value), but probably produces wrong flows leading to non-binding inequalities. However, once the optimal injections are obtained at the nodes, a feasibility problem can be solved to find a set of feasible flows corresponding to the injections. Note that the reason why the convex relaxation does not necessarily find the correct flows is that the mapping from injections to flows is not invertible. For example, it is known in the context of power systems that the power flow equations may not have a unique solution. The main contribution of this work is to show that although GNF is NP-hard (since the flow equations can have an exponential number of solutions), the optimal injections can be found in polynomial time.

A. Application of GNF in Power Systems

The operation of a power network depends heavily on various large-scale optimization problems such as state estimation, optimal power flow (OPF), contingency constrained OPF, unit commitment, sizing of capacitor banks and network reconfiguration. These problems are highly non-convex due to the nonlinearities imposed by laws of physics [11], [12]. For example, each of the above problems has the power flow equations embedded in it, which are nonlinear equality constraints. The nonlinearity of OPF, as the most fundamental optimization problem for power systems, has been studied since 1962 leading to developing various heuristic and local-search algorithms [13], [14], [15]. These algorithms suffer from sensitivity and convergence issues, and more importantly they may convergence to a local optimum that is noticeably far from a global solution.

Recently, it has been shown in [16] that the semidefinite programming (SDP) relaxation is able to find the global solution of the OPF problem under a sufficient condition, which is satisfied for IEEE benchmark systems with 14, 30, 57, 118 and 300 buses and many randomly generated power networks. The papers [16] and [12] show that this condition holds widely in practice due to the passivity of transmission lines and transformers. In particular, [12] shows that in the case when this condition is not satisfied (see [17] for counterexamples), OPF can always be solved globally in polynomial time after two approximations: (i) relaxing angle constraints by adding a sufficient number of actual/virtual phase shifters to the network, (ii) relaxing power balance equalities at the buses to inequality constraints. OPF under Approximation (ii) was also studied in [18] and [19] for distribution networks. The paper [20] studies the optimization of active power flows over distribution networks under fixed voltage magnitudes and shows that SDP relaxation works without Approximation (i) as long as a very practical angle condition is satisfied. The idea of convex relaxation developed in [21] and [16] can be applied to many other power problems, such as voltage regulation [22], state estimation [23], calculation of voltage stability margin [24], charging of electric vehicles [25], SCOPF with variable tap-changers and capacitor banks [26], dynamic energy management [10] and electricity market [27].

Energy-related optimizations with embedded power flow equations can be regarded as nonlinear network flow problems, which are analogous to GNF. The results derived in this work for a general GNF problem lead to the following conclusions for OPF (and some of the abovementioned OPF-based optimizations):

- They generalize the result of [20] to networks with virtual phase shifters. This proves that in order to use SDP relaxation for OPF over an arbitrary power network, it may not be needed to relax power balance equalities to inequality constraints under a very mild angle assumption.
- If the SDP relaxation does not work exactly for an OPF problem, it means that the obtained bus voltages are wrong. However, it can be inferred from the results of this work on GNF that the obtained bus injection powers may still be correct, in which case the optimal voltages can be recovered by solving a separate power flow problem.

B. Notations

The following notations will be used throughout this paper:

- $\mathcal{R}$ and $\mathcal{R}_+$ denote the sets of real numbers and nonnegative numbers, respectively.
- Given two matrices M and N, the inequality $M \leq N$ means that M is less than or equal to N element-wise.
- Given a set $\mathcal{T}$, its cardinality is shown as $|\mathcal{T}|$.
- Lowercase, bold lowercase and uppercase letters are used for scalars, vectors and matrices (say x, $\mathbf{x}$ and X).

II. Problem Statement and Contributions

Consider an undirected graph (network) $\mathcal{G}$ with the vertex set $\mathcal{N} := \{1, 2, ..., m\}$ and the edge set $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$. For every $i \in \mathcal{N}$, let $\mathcal{N}(i)$ denote the set of the neighboring vertices of node i. Assume that every edge $(i, j) \in \mathcal{E}$ is associated with two unknown flows p_{ij} and p_{ji} belonging to $\mathcal{R}$. The parameters p_{ij} and p_{ji} can be regarded as the flows entering the edge (i, j) from the endpoints i and j, respectively. Define

$$p_i = \sum_{j \in \mathcal{N}(i)} p_{ij}, \qquad \forall i \in \mathcal{N} \tag{1}$$

The parameter p_i is called "nodal injection at vertex i" or simply "injection", which is equal to the sum of the flows leaving vertex i through the edges connected to this vertex. Given an edge $(i, j) \in \mathcal{E}$, we assume that the flows p_{ij} and p_{ji} are related to each other via a function $f_{ij}(\cdot)$ to be introduced later. To specify which of the flows p_{ij} and p_{ji} is a function of the other, we give an arbitrary orientation to every edge of the graph $\mathcal{G}$ and denote the resulting graph as $\vec{\mathcal{G}}$. Denote also its directed edge set as $\vec{\mathcal{E}}$. If an edge $(i, j) \in \mathcal{E}$ belongs to $\vec{\mathcal{E}}$, we then express p_{ji} as a function of p_{ij}.

Definition 1: Define the vectors $\mathbf{p}_{\rm n}$, $\mathbf{p}_{\rm e}$ and $\mathbf{p}_{\rm d}$ as follows:

$$\mathbf{p}_{\rm n} = \{p_i \mid \forall i \in \mathcal{N}\} \tag{2a}$$

$$\mathbf{p}_{\rm e} = \{p_{ij} \mid \forall (i, j) \in \mathcal{E}\} \tag{2b}$$

$$\mathbf{p}_{\rm d} = \{p_{ij} \mid \forall (i, j) \in \vec{\mathcal{E}}\} \tag{2c}$$

(the subscripts "n", "e" and "d" stand for nodes, edges and directed edges). The terms $\mathbf{p}_{\rm n}$, $\mathbf{p}_{\rm e}$ and $\mathbf{p}_{\rm d}$ are referred to as injection vector, flow vector and semi-flow vector, respectively (note that $\mathbf{p}_{\rm e}$ contains two flows per each line, while $\mathbf{p}_{\rm d}$ has only one flow per line).

Definition 2: Given two arbitrary points $\mathbf{x}, \mathbf{y} \in \mathcal{R}^n$, the box $\mathcal{B}(\mathbf{x}, \mathbf{y})$ is defined as follows:

$$B(\mathbf{x}, \mathbf{y}) = \{\mathbf{z} \in \mathcal{R}^n \;| \mathbf{x} \leq \mathbf{z} \leq \mathbf{y}\} \tag{3}$$

(note that $B(\mathbf{x}, \mathbf{y})$ is non-empty only if $\mathbf{x} \leq \mathbf{y}$).

Assume that each nodal injection p_i must be within the given interval $[p_i^{\min}, p_i^{\max}]$ for every $i \in \mathcal{N}$. We use the shorthand notation $\mathcal{B}$ for the box $\mathcal{B}(\mathbf{p}_{\rm n}^{\min}, \mathbf{p}_{\rm n}^{\max})$, where $\mathbf{p}_{\rm n}^{\min}$ and $\mathbf{p}_{\rm n}^{\max}$ are the vectors of the lower bounds $p_i^{\min}$'s and the upper bounds $p_i^{\max}$'s, respectively. This paper is concerned with the following problem.

Generalized network flow (GNF):

$$\min_{\mathbf{P}_{\mathrm{n}}\in\mathcal{B},\mathbf{P}_{\mathrm{e}}\in\mathcal{R}^{|\mathcal{E}|}} \quad \sum_{i\in\mathcal{N}} f_i(p_i) \tag{4a}$$

$$\text{subject to} \quad p_i = \sum_{j\in\mathcal{N}(i)} p_{ij}, \qquad \forall i\in\mathcal{N} \tag{4b}$$

$$p_{ji} = f_{ij}(p_{ij}), \qquad \forall (i,j)\in\vec{\mathcal{E}} \tag{4c}$$

$$p_{ij} \in [p_{ij}^{\min}, p_{ij}^{\max}], \qquad \forall (i,j)\in\vec{\mathcal{E}} \tag{4d}$$

where

1) $f_i(\cdot)$ is convex and monotonically increasing for every $i\in\mathcal{N}$.
2) $f_{ij}(\cdot)$ is convex and monotonically decreasing for every $(i,j)\in\vec{\mathcal{E}}$.
3) The limits $p_{ij}^{\min}$ and $p_{ij}^{\max}$ are given for every $(i,j)\in\vec{\mathcal{E}}$.

In the case when $f_{ij}(p_{ji})$ is equal to $-p_{ij}$ for all $(i,j)\in\vec{\mathcal{E}}$, the GNF problem reduces to the network flow problem for which every line is lossless. A few remarks can be made here:

- Given an edge $(i,j)\in\vec{\mathcal{E}}$, there is no explicit limit on p_{ji} in the formulation of the GNF problem because restricting p_{ji} is equivalent to limiting p_{ij}.
- Given a node $i\in\mathcal{N}$, the assumption of $f_i(p_i)$ being monotonically increasing is motivated by the fact that increasing the injection p_i normally elevates the cost in practice.
- Given an edge $(i,j)\in\vec{\mathcal{E}}$, p_{ij} and $-p_{ji}$ can be regarded as the input and output flows of the line (i,j), which travel in the same direction. The assumption of $f_{ij}(p_{ij})$ being monotonically decreasing is motivated by the fact that increasing the input flow normally makes the output flow higher in practice (note that $-p_{ji} = -f_{ij}(p_{ij})$).

Definition 3: Define $\mathcal{P}$ as the set of all vectors $\mathbf{p}_{\mathrm{n}}$ for which there exists a vector $\mathbf{p}_{\mathrm{e}}$ such that $(\mathbf{p}_{\mathrm{n}},\mathbf{p}_{\mathrm{e}})$ satisfies equations (4b), (4c) and (4d). The set $\mathcal{P}$ and $\mathcal{P}\cap\mathcal{B}$ are referred to as *injection region* and *box-constrained injection region*, respectively.

Regarding Definition 3, the box-constrained injection region is indeed the projection of the feasible set of GNF onto the space of the injection vector $\mathbf{p}_{\mathrm{n}}$. Now, one can express GNF geometrically as follows:

$$\text{Geometric GNF}: \quad \min_{\mathbf{p}_{\mathrm{n}}\in\mathcal{P}\cap\mathcal{B}} \sum_{i\in\mathcal{N}} f_i(p_i) \tag{5}$$

Note that $\mathbf{p}_{\mathrm{e}}$ has been eliminated in Geometric GNF. It is hard to solve this problem directly because the injection region $\mathcal{P}$ is non-convex in general. This non-convexity can be observed in Figure 2(a), which shows $\mathcal{P}$ for the two-node graph drawn in Figure 1. To address this non-convexity issue, the GNF problem will be convexified naturally next.

Convexified generalized network flow (CGNF):

$$\min_{\mathbf{P}_{\mathrm{n}}\in\mathcal{B},\mathbf{P}_{\mathrm{e}}\in\mathcal{R}^{|\mathcal{E}|}} \quad \sum_{i\in\mathcal{N}} f_i(p_i) \tag{6a}$$

$$\text{subject to} \quad p_i = \sum_{j\in\mathcal{N}(i)} p_{ij}, \qquad \forall i\in\mathcal{N} \tag{6b}$$

$$p_{ji} \geq f_{ij}(p_{ij}), \qquad \forall (i,j)\in\vec{\mathcal{E}} \tag{6c}$$

$$p_{ij} \in [p_{ij}^{\min}, p_{ij}^{\max}], \qquad \forall (i,j)\in\mathcal{E} \tag{6d}$$

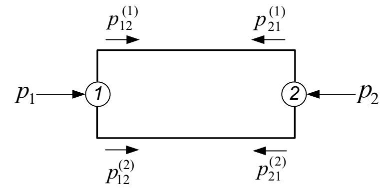


Fig. 1: The graph $\mathcal{G}$ studied in Section III-A.

where $(p_{ij}^{\min}, p_{ij}^{\max}) = (f_{ji}(p_{ji}^{\max}), f_{ji}(p_{ji}^{\min}))$ for every $(i,j)\in\mathcal{E}$ such that $(j,i)\in\vec{\mathcal{E}}$. Note that CGNF has been obtained from GNF by relaxing equality (4c) to inequality (6c) and adding limits to p_{ij} for every $(j,i)\in\vec{\mathcal{E}}$. One can write:

$$\text{Geometric CGNF}: \quad \min_{\mathbf{p}_{\mathrm{n}}\in\mathcal{P}_{\mathrm{c}}\cap\mathcal{B}} \sum_{i\in\mathcal{N}} f_i(p_i) \tag{7}$$

where $\mathcal{P}_{\mathrm{c}}$ denotes the set of all vectors $\mathbf{p}_{\mathrm{n}}$ for which there exists a vector $\mathbf{p}_{\mathrm{e}}$ such that $(\mathbf{p}_{\mathrm{n}},\mathbf{p}_{\mathrm{e}})$ satisfies equations (6b), (6c) and (6d).

Two main results to be proved in this paper are:

- **Geometry of injection region:** Given any two points $\mathbf{p}_{\mathrm{n}}$ and $\tilde{\mathbf{p}}_{\mathrm{n}}$ in the injection region, the box $\mathcal{B}(\mathbf{p}_{\mathrm{n}},\tilde{\mathbf{p}}_{\mathrm{n}})$ is entirely contained in the injection region. Similar result holds true for the box-constrained injection region.
- **Relationship between GNF and CGNF:** If $(\mathbf{p}_{\mathrm{n}}^*,\mathbf{p}_{\mathrm{e}}^*)$ and $(\bar{\mathbf{p}}_{\mathrm{n}}^*,\bar{\mathbf{p}}_{\mathrm{e}}^*)$ denote two arbitrary solutions of GNF and CGNF, then $\mathbf{p}_{\mathrm{n}}^* = \bar{\mathbf{p}}_{\mathrm{n}}^*$. Hence, although CGNF may not be able to find a feasible flow vector for GNF, it always finds the correct optimal injection vector for GNF.

The application of these results in power systems will also be discussed. Note that this work implicitly assumes that every two nodes of $\mathcal{G}$ are connected via at most one edge. However, the results to be derived later are all valid in the presence of multiple edges between two nodes. To avoid complicated notations, the proof will not be provided for this case. However, Section III-A studies a simple example with parallel lines.

III. Main Results

In this section, a detailed illustrative example will first be provided to clarify the issues and highlight the contribution of this work. In Subsections III-B and III-C, the main results for GNF will be derived, whose application in power systems will be later discussed in Subsection III-D.

A. Illustrative Example

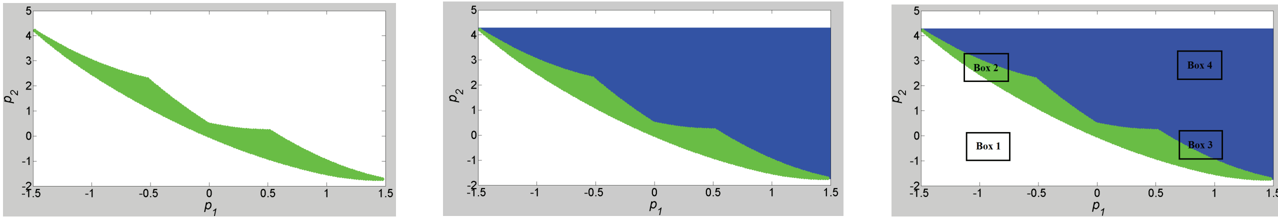
In this subsection, we study the particular graph $\mathcal{G}$ depicted in Figure 1. This graph has two vertices and two parallel edges. Let $(p_{12}^{(1)}, p_{21}^{(1)})$ and $(p_{12}^{(2)}, p_{21}^{(2)})$ denote the flows associated with the first and second edges of the graph, respectively. Consider the following GNF problem:

$$\min \quad f_1(p_1) + f_2(p_2) \tag{8a}$$

$$\text{subject to} \quad p_{21}^{(i)} = \left(p_{12}^{(i)} - 1\right)^2 - 1, \quad i = 1,2 \tag{8b}$$

$$-0.5 \leq p_{12}^{(1)} \leq 0.5, \quad -1 \leq p_{12}^{(2)} \leq 1, \tag{8c}$$

$$p_1 = p_{12}^{(1)} + p_{12}^{(2)}, \quad p_2 = p_{21}^{(1)} + p_{21}^{(2)} \tag{8d}$$



(a) (b) (c)

Fig. 2: (a) Injection region $\mathcal{P}$ for the GNF problem given in (8); (b) The set $\mathcal{P}_\text{c}$ corresponding to the GNF problem given in (8); (c) This figure shows the set $\mathcal{P}_\text{c}$ corresponding to the GNF problem given in (8) together with a box constraint $(p_1, p_2) \in \mathcal{B}$ for four different positions of $\mathcal{B}$.

with the variables $p_1, p_2, p_{12}^{(1)}, p_{21}^{(1)}, p_{12}^{(2)}, p_{21}^{(2)}$, where $f_1(\cdot)$ and $f_2(\cdot)$ are both convex and monotonically increasing. The CGNF problem corresponding to this problem can be obtained by replacing (8b) with $p_{21}^{(i)} \geq (p_{12}^{(i)} - 1)^2 - 1$ and adding the limits $p_{21}^{(1)} \leq 1.5^2 - 1$ and $p_{21}^{(2)} \leq 2^2 - 1$. One can write:

$$\text{Geometric GNF:} \quad \min_{(p_1,p_2)\in\mathcal{P}} f_1(p_1) + f_2(p_2) \tag{9a}$$

$$\text{Geometric CGNF:} \quad \min_{(p_1,p_2)\in\mathcal{P}_\text{c}} f_1(p_1) + f_2(p_2) \tag{9b}$$

where $\mathcal{P}$ and $\mathcal{P}_\text{c}$ are indeed the projections of the feasible sets of GNF and CGNF over the injection space (p_1, p_2). The green area in Figure 2(a) shows the injection region $\mathcal{P}$. As expected, this set is non-convex. In contrast, the set $\mathcal{P}_\text{c}$ is a convex set containing $\mathcal{P}$. This set is shown in Figure 2(b), which includes two parts: (i) the green area is the same as $\mathcal{P}$, (ii) the blue area is the part of $\mathcal{P}_\text{c}$ that does not exist in $\mathcal{P}$. Thus, the transition from GNF to CGNF extends the injection region $\mathcal{P}$ to a convex set by adding the blue area. Notice that $\mathcal{P}_\text{c}$ has three boundaries: (1) straight line on the top, (2) straight line on the right side, and (3) a lower curvy boundary. Since $f_1(\cdot)$ and $f_2(\cdot)$ are both monotonically increasing, the unique solution of Geometric CGNF must lie on the lower curvy boundary of $\mathcal{P}_\text{c}$. Since this lower boundary is in the green area, it is contained in $\mathcal{P}$. As a result, the unique solution of Geometric CGNF is a feasible point of $\mathcal{P}$ and therefore it is a solution of Geometric GNF. This means that CGNF finds the optimal injection vector for GNF.

To make the problem more interesting, we add the box constraint $(p_1, p_2) \in \mathcal{B}$ to GNF (and correspondingly to CGNF), where $\mathcal{B}$ is an arbitrary rectangular convex set in $\mathcal{R}^2$. The effect of this box constraint will be investigated in four different scenarios:

- Assume that $\mathcal{B}$ corresponds to Box 1 (including its interior) in Figure 2(c). In this case, $\mathcal{P} \cap \mathcal{B} = \mathcal{P}_\text{c} \cap \mathcal{B} = \phi$, meaning that Geometric GNF and Geometric CGNF are both infeasible.
- Assume that $\mathcal{B}$ corresponds to Box 2 (including its interior) in Figure 2(c). In this case, the solution of Geometric CGNF lies on the lower boundary of $\mathcal{P}_\text{c}$ and therefore it is also a solution of Geometric GNF.
- Assume that $\mathcal{B}$ corresponds to Box 3 (including its interior) in Figure 2(c). In this case, the solutions of Geometric GNF and Geometric CGNF are identical and both correspond to the lower left corner of the box $\mathcal{B}$.
- Assume that $\mathcal{B}$ corresponds to Box 4 (including its interior) in Figure 2(c). In this case, $\mathcal{P} \cap \mathcal{B} = \phi$ but $\mathcal{P}_\text{c} \cap \mathcal{B} \neq \phi$. Hence, Geometric GNF is infeasible while Geometric CGNF has an optimal solution

In summary, it can be argued that independent of the position of the box $\mathcal{B}$ in $\mathcal{R}^2$, CGNF finds the optimal injection vector for GNF as long as GNF is feasible.

B. Geometry of Injection Region

In order to study the relationship between GNF and CGNF, it is beneficial to explore the geometry of the feasible set of GNF. Hence, we investigate the geometry of the injection region $\mathcal{P}$ and the box-constrained injection region $\mathcal{P} \cap \mathcal{B}$ in this part.

Theorem 1: Consider two arbitrary points $\hat{\mathbf{p}}_\text{n}$ and $\tilde{\mathbf{p}}_\text{n}$ in the injection region $\mathcal{P}$. The box $\mathcal{B}(\hat{\mathbf{p}}_\text{n}, \tilde{\mathbf{p}}_\text{n})$ is contained in $\mathcal{P}$. ■

The proof of this theorem is based on four lemmas, and will be provided later in this subsection. To understand this theorem, consider the injection region $\mathcal{P}$ depicted in Figure 2(a) corresponding to the illustrative example given in Section III-A. If any arbitrary box is drawn in $\mathcal{R}^2$ in such a way that its upper right corner and lower left corner both lie in the green area, then the entire box must lie in the green area completely. This can be easily proved in this special case and is true in general due to Theorem 1. The result of Theorem 1 can be generalized to the box-constrained injection region, as stated below.

Corollary 1: Consider two arbitrary points $\hat{\mathbf{p}}_\text{n}$ and $\tilde{\mathbf{p}}_\text{n}$ belonging to the box-constrained injection region $\mathcal{P} \cap \mathcal{B}$. The box $\mathcal{B}(\hat{\mathbf{p}}_\text{n}, \tilde{\mathbf{p}}_\text{n})$ is contained in $\mathcal{P} \cap \mathcal{B}$.

Proof: The proof follows immediately from Theorem 1. ■

The rest of this subsection is dedicated to the proof of Theorem 1, which is based on a series of definitions and lemmas.

Definition 4: Define $\mathcal{B}_\text{d}$ as the box containing all vectors $\mathbf{p}_\text{d}$ introduced in (2c) satisfying the condition $p_{ij} \in [p_{ij}^{\min}, p_{ij}^{\max}]$ for every $(i, j) \in \vec{\mathcal{E}}$.

Definition 5: Given two arbitrary points $\bar{\mathbf{p}}_\text{d}, \tilde{\mathbf{p}}_\text{d} \in \mathcal{B}_\text{d}$, define $M(\bar{\mathbf{p}}_\text{d}, \tilde{\mathbf{p}}_\text{d})$ as follows:

- Let $M(\bar{\mathbf{p}}_\text{d}, \tilde{\mathbf{p}}_\text{d})$ be a matrix with $|\mathcal{N}|$ rows indexed by the vertices of $\mathcal{G}$ and with $|\vec{\mathcal{E}}|$ columns indexed by the edges in $\vec{\mathcal{E}}$.
- For every vertex $k \in \mathcal{N}$ and edge $(i, j) \in \vec{\mathcal{E}}$, set the $(k, (i, j))^\text{th}$ entry of $M(\bar{\mathbf{p}}_\text{d}, \tilde{\mathbf{p}}_\text{d})$ (the one in the intersection

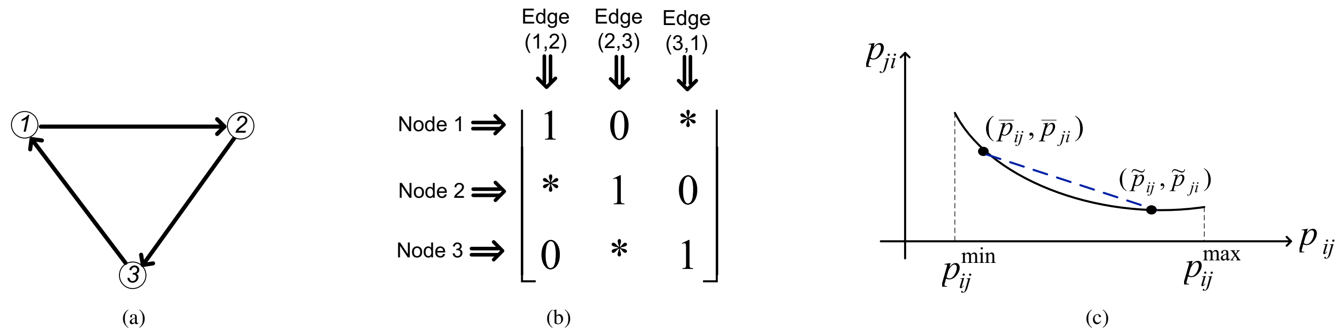


Fig. 3: (a) A particular graph $\vec{\mathcal{G}}$; (b) The matrix $M(\bar{\mathbf{p}}_{\rm d}, \tilde{\mathbf{p}}_{\rm d})$ corresponding to the graph $\vec{\mathcal{G}}$ in Figure (a); (c): The $(j,(i,j))^{\text{th}}$ entry of $M(\bar{\mathbf{p}}_{\rm d}, \tilde{\mathbf{p}}_{\rm d})$ (shown as "*") is equal to the slope of the line connecting the point $(\bar{p}_{ij}, \bar{p}_{ji})$ to $(\tilde{p}_{ij}, \tilde{p}_{ji})$.

of row k and column (i,j)) as

$$\begin{cases} 1 & \text{if} \quad k=i \\ \frac{f_{ij}(\bar{p}_{ij})-f_{ij}(\tilde{p}_{ij})}{\bar{p}_{ij}-\tilde{p}_{ij}} & \text{if} \quad k=j \text{ and } \bar{p}_{ij}\neq\tilde{p}_{ij} \\ f'_{ij}(\bar{p}_{ij}) & \text{if} \quad k=j \text{ and } \bar{p}_{ij}=\tilde{p}_{ij} \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

where $f'_{ij}(\bar{p}_{ij})$ denotes the right derivative of $f_{ij}(\bar{p}_{ij})$ if $\bar{p}_{ij} < p_{ij}^{\max}$ and the left derivative of $f_{ij}(\bar{p}_{ij})$ if $\bar{p}_{ij} = p_{ij}^{\max}$.

To illustrate Definition 5, consider the three-node graph $\vec{\mathcal{G}}$ depicted in Figure 3(a). The matrix $M(\bar{\mathbf{p}}_{\rm d}, \tilde{\mathbf{p}}_{\rm d})$ associated with this graph has the structure shown in Figure 3(b), where the "*" entries depend on the specific values of $\bar{\mathbf{p}}_{\rm d}$ and $\tilde{\mathbf{p}}_{\rm d}$. Consider an edge $(i,j) \in \vec{\mathcal{E}}$. The $(j,(i,j))^{\text{th}}$ entry of $M(\bar{\mathbf{p}}_{\rm d}, \tilde{\mathbf{p}}_{\rm d})$ is equal to

$$\frac{f_{ij}(\bar{p}_{ij})-f_{ij}(\tilde{p}_{ij})}{\bar{p}_{ij}-\tilde{p}_{ij}}, \tag{11}$$

provided $\bar{p}_{ij} \neq \tilde{p}_{ij}$. As can be seen in Figure 3(c), this is equal to the slope of the line connecting the point $(\bar{p}_{ij}, \bar{p}_{ji})$ to the point $(\tilde{p}_{ij}, \tilde{p}_{ji})$ on the parameterized curve (p_{ij}, p_{ji}), where $p_{ji} = f_{ij}(p_{ij})$. Moreover, $f'_{ij}(\bar{p}_{ij})$ is the limit of this slope as the point $(\tilde{p}_{ij}, \tilde{p}_{ji})$ approaches $(\bar{p}_{ij}, \bar{p}_{ji})$. It is also interesting to note that $M(\bar{\mathbf{p}}_{\rm d}, \tilde{\mathbf{p}}_{\rm d})$ has one positive entry, one negative entry and $m-2$ zero entries in each column (note that the slope of the line connecting $(\bar{p}_{ij}, \bar{p}_{ji})$ to $(\tilde{p}_{ij}, \tilde{p}_{ji})$ is always negative). The next lemma explains how the matrix $M(\bar{\mathbf{p}}_{\rm d}, \tilde{\mathbf{p}}_{\rm d})$ can be used to relate the semi-flow vector to the injection vector.

Lemma 1: Consider two arbitrary injection vectors $\bar{\mathbf{p}}_{\rm n}$ and $\tilde{\mathbf{p}}_{\rm n}$ in $\mathcal{P}$, associated with the semi-flow vectors $\bar{\mathbf{p}}_{\rm d}$ and $\tilde{\mathbf{p}}_{\rm d}$ (defined in (2)). The relation

$$\bar{\mathbf{p}}_{\rm n} - \tilde{\mathbf{p}}_{\rm n} = M(\bar{\mathbf{p}}_{\rm d}, \tilde{\mathbf{p}}_{\rm d}) \times (\bar{\mathbf{P}}_{\rm d} - \tilde{\mathbf{P}}_{\rm d}) \tag{12}$$

holds.

Proof: One can write:

$$\bar{p}_i - \tilde{p}_i = \sum_{j\in\mathcal{N}(i)} (\bar{p}_{ij} - \tilde{p}_{ij}), \qquad \forall i \in \mathcal{N} \tag{13}$$

By using the relations

$$\bar{p}_{ji} = f_{ij}(\bar{p}_{ij}), \quad \tilde{p}_{ji} = f_{ij}(\tilde{p}_{ij}), \quad \forall (i,j) \in \vec{\mathcal{E}}, \tag{14}$$

it is straightforward to verify that (12) and (13) are equivalent. ■

The next lemma investigates an important property of the matrix $M(\bar{\mathbf{p}}_{\rm d}, \tilde{\mathbf{p}}_{\rm d})$.

Lemma 2: Given two arbitrary points $\bar{\mathbf{p}}_{\rm d}, \tilde{\mathbf{p}}_{\rm d} \in \mathcal{B}_{\rm d}$, assume that there exists a nonzero vector $\mathbf{x} \in \mathcal{R}^m$ such that $\mathbf{x}^T M(\bar{\mathbf{p}}_{\rm d}, \tilde{\mathbf{p}}_{\rm d}) \geq 0$. If $\mathbf{x}$ has at least one strictly positive entry, then there exists a nonzero vector $\mathbf{y} \in \mathcal{R}^m_+$ such that $\mathbf{y}^T M(\bar{\mathbf{p}}_{\rm d}, \tilde{\mathbf{p}}_{\rm d}) \geq 0$.

Proof: Consider an index $i_0 \in \mathcal{N}$ such that $x_{i_0} > 0$. Define $\mathcal{V}(i_0)$ as the set of all vertices $i \in \mathcal{N}$ from which there exists a directed path to vertex i_0 in the graph $\vec{\mathcal{G}}$. Note that $\mathcal{V}(i_0)$ includes vertex i_0 itself. The first goal is to show that

$$x_i \geq 0, \qquad \forall i \in \mathcal{V}(i_0) \tag{15}$$

To this end, consider an arbitrary set of vertices $i_1, ..., i_k$ in $\mathcal{V}(i_0)$ such that $\{i_0, i_1 ..., i_k\}$ forms a direct path in $\vec{\mathcal{G}}$ as

$$i_k \to i_{k-1} \to \cdots i_1 \to i_0 \tag{16}$$

To prove (15), it suffices to show that $x_{i_1}, ..., x_{i_k} \geq 0$. For this purpose, one can expand the product $\mathbf{x}^T M(\bar{\mathbf{p}}_{\rm d}, \tilde{\mathbf{p}}_{\rm d})$ and use the fact that each column of $M(\bar{\mathbf{p}}_{\rm d}, \tilde{\mathbf{p}}_{\rm d})$ has $m-2$ zero entries to conclude that

$$x_{i_1} + \frac{f_{i_1 i_0}(\bar{p}_{i_1 i_0}) - f_{i_1 i_0}(\tilde{p}_{i_1 i_0})}{\bar{p}_{i_1 i_0} - \tilde{p}_{i_1 i_0}} x_{i_0} \geq 0 \tag{17}$$

Since x_{i_0} is positive and $f_{i_1 i_0}(\cdot)$ is a decreasing function, x_{i_1} turns out to be positive. Now, repeating the above argument for i_1 instead of i_0 yields that $x_{i_2} \geq 0$. Continuing this reasoning leads to $x_{i_1}, ..., x_{i_k} \geq 0$. Hence, inequality (15) holds. Now, define $\mathbf{y}$ as

$$y_i = \begin{cases} x_i & \text{if} \quad i \in \mathcal{V}(i_0) \\ 0 & \text{otherwise} \end{cases}, \qquad \forall i \in \mathcal{N} \tag{18}$$

In light of (15), $\mathbf{y}$ is a nonzero vector in $\mathcal{R}^m_+$. To complete the proof, it suffices to show that $\mathbf{y}^T M(\bar{\mathbf{p}}_{\rm d}, \tilde{\mathbf{p}}_{\rm d}) \geq 0$. Similar to the indexing procedure used for the columns of the matrix $M(\bar{\mathbf{p}}_{\rm d}, \tilde{\mathbf{p}}_{\rm d})$, we index the entries of the $|\vec{\mathcal{E}}|$ dimensional vector $\mathbf{y}^T M(\bar{\mathbf{p}}_{\rm d}, \tilde{\mathbf{p}}_{\rm d})$ according to the edges of $\vec{\mathcal{G}}$. Now, given an arbitrary edge $(\alpha, \beta) \in \vec{\mathcal{E}}$, the following statements hold true:

- If $\alpha, \beta \in \mathcal{V}(i_0)$, then the $(\alpha,\beta)^{\text{th}}$ entries of $\mathbf{y}^T M(\bar{\mathbf{p}}_{\rm d}, \tilde{\mathbf{p}}_{\rm d})$ and $\mathbf{x}^T M(\bar{\mathbf{p}}_{\rm d}, \tilde{\mathbf{p}}_{\rm d})$ (i.e., the entries corresponding to the edge (α,β)) are identical.

- If $\alpha \in \mathcal{V}(i_0)$ and $\beta \notin \mathcal{V}(i_0)$, then the $(\alpha,\beta)^{\text{th}}$ entry of $\mathbf{y}^T M(\bar{\mathbf{p}}_{\text{d}}, \tilde{\mathbf{p}}_{\text{d}})$ is equal to y_α.
- If $\alpha \notin \mathcal{V}(i_0)$ and $\beta \notin \mathcal{V}(i_0)$, then the $(\alpha,\beta)^{\text{th}}$ entry of $\mathbf{y}^T M(\bar{\mathbf{p}}_{\text{d}}, \tilde{\mathbf{p}}_{\text{d}})$ is equal to zero.

Note that the case $\alpha \notin \mathcal{V}(i_0)$ and $\beta \in \mathcal{V}(i_0)$ cannot happen, because if $\beta \in \mathcal{V}(i_0)$ and $(\alpha,\beta) \in \vec{\mathcal{E}}$, then $\alpha \in \mathcal{V}(i_0)$ by the definition of $\mathcal{V}(i_0)$. It follows from the above results and the inequality $\mathbf{x}^T M(\bar{\mathbf{p}}_{\text{d}}, \tilde{\mathbf{p}}_{\text{d}}) \geq 0$ that $\mathbf{y}^T M(\bar{\mathbf{p}}_{\text{d}}, \tilde{\mathbf{p}}_{\text{d}}) \geq 0$. ■

The next lemma studies the injection region $\mathcal{P}$ in the case when $f_{ij}(\cdot)$'s are all piecewise linear.

Lemma 3: Assume that the function $f_{ij}(\cdot)$ is piecewise linear for every $(i,j) \in \vec{\mathcal{E}}$. Consider two arbitrary points $\hat{\mathbf{p}}_{\text{n}}, \bar{\mathbf{p}}_{\text{n}} \in \mathcal{P}$ and a vector $\Delta\bar{\mathbf{p}}_{\text{n}} \in \mathcal{R}^m$ satisfying the relations

$$\hat{\mathbf{p}}_{\text{n}} \leq \bar{\mathbf{p}}_{\text{n}} - \Delta\bar{\mathbf{p}}_{\text{n}} \leq \bar{\mathbf{p}}_{\text{n}} \tag{19}$$

There exists a strictly positive number $\epsilon^{\max}$ with the property

$$\bar{\mathbf{p}}_{\text{n}} - \varepsilon\Delta\bar{\mathbf{p}}_{\text{n}} \in \mathcal{P}, \qquad \forall \varepsilon \in [0, \epsilon^{\max}] \tag{20}$$

Proof: The proof is based on Lemmas 1 and 2. Since the proof is lengthy and complicated, it has been moved to [28]. ■

The next lemma proves Theorem 1 in the case when $f_{ij}(\cdot)$'s are all piecewise linear.

Lemma 4: Assume that the function $f_{ij}(\cdot)$ is piecewise linear for every $(i,j) \in \vec{\mathcal{E}}$. Given any two arbitrary points $\hat{\mathbf{p}}_{\text{n}}, \tilde{\mathbf{p}}_{\text{n}} \in \mathcal{P}$, the box $\mathcal{B}(\hat{\mathbf{p}}_{\text{n}}, \tilde{\mathbf{p}}_{\text{n}})$ is a subset of the injection region $\mathcal{P}$.

Proof: With no loss of generality, assume that $\hat{\mathbf{p}}_{\text{n}} \leq \tilde{\mathbf{p}}_{\text{n}}$ (because otherwise $\mathcal{B}(\hat{\mathbf{p}}_{\text{n}}, \tilde{\mathbf{p}}_{\text{n}})$ is empty). To prove the lemma by contradiction, suppose that there exists a point $\mathbf{p}_{\text{n}} \in \mathcal{B}(\hat{\mathbf{p}}_{\text{n}}, \tilde{\mathbf{p}}_{\text{n}})$ such that $\mathbf{p}_{\text{n}} \notin \mathcal{P}$. Consider the set

$$\left\{ \gamma \;\middle|\; \gamma \in [0,1],\ \tilde{\mathbf{p}}_{\text{n}} + \gamma(\mathbf{p}_{\text{n}} - \tilde{\mathbf{p}}_{\text{n}}) \in \mathcal{P} \right\} \tag{21}$$

and denote its maximum as $\gamma^{\max}$ (the existence of this maximum number is guaranteed by the closedness and compactness of $\mathcal{P}$). Note that $\tilde{\mathbf{p}}_{\text{n}} + \gamma(\mathbf{p}_{\text{n}} - \tilde{\mathbf{p}}_{\text{n}})$ is equal to $\mathbf{p}_{\text{n}}$ at $\gamma = 1$. Since $\mathbf{p}_{\text{n}} \notin \mathcal{P}$ by assumption, we have $\gamma^{\max} < 1$. Denote $\tilde{\mathbf{p}}_{\text{n}} + \gamma^{\max}(\mathbf{p}_{\text{n}} - \tilde{\mathbf{p}}_{\text{n}})$ as $\bar{\mathbf{p}}_{\text{n}}$. Hence, $\bar{\mathbf{p}}_{\text{n}} \in \mathcal{P}$ and $\hat{\mathbf{p}}_{\text{n}} \leq \mathbf{p}_{\text{n}} \leq \bar{\mathbf{p}}_{\text{n}}$ (recall that $\gamma^{\max} < 1$). Define $\Delta\bar{\mathbf{p}}_{\text{n}}$ as $\bar{\mathbf{p}}_{\text{n}} - \mathbf{p}_{\text{n}}$. One can write:

$$\hat{\mathbf{p}}_{\text{n}} \leq \bar{\mathbf{p}}_{\text{n}} - \Delta\bar{\mathbf{p}}_{\text{n}} \leq \bar{\mathbf{p}}_{\text{n}}, \qquad \hat{\mathbf{p}}_{\text{n}}, \bar{\mathbf{p}}_{\text{n}} \in \mathcal{P} \tag{22}$$

By Lemma 3, there exists a strictly positive number $\epsilon^{\max}$ with the property

$$\bar{\mathbf{p}}_{\text{n}} - \varepsilon\Delta\bar{\mathbf{p}}_{\text{n}} \in \mathcal{P}, \qquad \forall \varepsilon \in [0, \epsilon^{\max}] \tag{23}$$

or equivalently

$$\tilde{\mathbf{p}}_{\text{n}} + (\gamma^{\max} + \varepsilon(1-\gamma^{\max}))(\mathbf{p}_{\text{n}} - \tilde{\mathbf{p}}_{\text{n}}) \in \mathcal{P}, \ \ \forall \varepsilon \in [0, \epsilon^{\max}] \tag{24}$$

Notice that

$$\gamma^{\max} + \varepsilon(1-\gamma^{\max}) > \gamma^{\max}, \quad \forall \varepsilon > 0 \tag{25}$$

Due to (24), this violates the assumption that $\gamma^{\max}$ is the maximum of the set given in (21). ■

Lemma 4 will be deployed next to prove Theorem 1 in the general case.

Proof of Theorem 1: Consider an arbitrary approximation of $f_{ij}(\cdot)$ by a piecewise linear function for every $(i,j) \in \vec{\mathcal{E}}$. As a counterpart of $\mathcal{P}$, let $\mathcal{P}_s$ denote the injection region in the piecewise-linear case. By Lemma 4, we have

$$\mathcal{B}(\hat{\mathbf{p}}_{\text{n}}, \tilde{\mathbf{p}}_{\text{n}}) \subseteq \mathcal{P}_s \tag{26}$$

Since the piecewise linear approximation can be made in such a way that the sets $\mathcal{P}$ and $\mathcal{P}_s$ become arbitrarily close to each other, the above relation implies that the interior of $\mathcal{B}(\hat{\mathbf{p}}_{\text{n}}, \tilde{\mathbf{p}}_{\text{n}})$ is a subset of $\mathcal{P}$. On the other hand, $\mathcal{P}$ is a closed set. Hence, the box $\mathcal{B}(\hat{\mathbf{p}}_{\text{n}}, \tilde{\mathbf{p}}_{\text{n}})$ must entirely belong to $\mathcal{P}$. ■

C. Relationship Between GNF and CGNF

In this subsection, the relationship between GNF and CGNF will be explored.

Theorem 2: Assume that the GNF problem is feasible. Let $(\mathbf{p}_{\text{n}}^*, \mathbf{p}_{\text{e}}^*)$ and $(\bar{\mathbf{p}}_{\text{n}}^*, \bar{\mathbf{p}}_{\text{e}}^*)$ denote arbitrary solutions of GNF and CGNF, respectively. The relation $\mathbf{p}_{\text{n}}^* = \bar{\mathbf{p}}_{\text{n}}^*$ holds. ■

Observe that since $(\bar{\mathbf{p}}_{\text{n}}^*, \bar{\mathbf{p}}_{\text{e}}^*)$ is a feasible point of CGNF, one can write

$$\bar{p}_i^* \geq p_i^{\min}, \qquad \forall i \in \mathcal{N} \tag{27}$$

The proof of Theorem 2 is lengthy and involved in the general case, but it simplifies greatly in the special case

$$\bar{p}_i^* = p_i^{\min}, \qquad \forall i \in \mathcal{N} \tag{28}$$

Hence, the general proof has been moved to [28], but its special case is provided below.

Proof of Theorem 2 under Condition (28): $(\mathbf{p}_{\text{n}}^*, \mathbf{p}_{\text{e}}^*)$ being a feasible point of GNF implies that

$$p_i^* \geq p_i^{\min}, \qquad \forall i \in \mathcal{N} \tag{29}$$

Equations (28) and (29) lead to

$$\bar{\mathbf{p}}_{\text{n}}^* \leq \mathbf{p}_{\text{n}}^* \tag{30}$$

Define the vector $\tilde{\mathbf{p}}_{\text{n}}$ as

$$\tilde{p}_i = \sum_{(i,j)\in\vec{\mathcal{E}}} \bar{p}_{ij}^* + \sum_{(j,i)\in\vec{\mathcal{E}}} f_{ij}(\bar{p}_{ij}^*), \qquad \forall i \in \mathcal{N} \tag{31}$$

Notice that $\tilde{\mathbf{p}}_{\text{n}}$ belongs to $\mathcal{P}$. It can be inferred from the definition of CGNF that

$$\tilde{\mathbf{p}}_{\text{n}} \leq \bar{\mathbf{p}}_{\text{n}}^* \tag{32}$$

Since $\tilde{\mathbf{p}}_{\text{n}}, \mathbf{p}_{\text{n}}^* \in \mathcal{P}$, it follows from Theorem 1, (30) and (32) that $\bar{\mathbf{p}}_{\text{n}}^* \in \mathcal{P}$. On the other hand, $\bar{\mathbf{p}}_{\text{n}}^* \in \mathcal{B}$. Therefore, $\bar{\mathbf{p}}_{\text{n}}^* \in \mathcal{P} \cap \mathcal{B}$, meaning that $\bar{\mathbf{p}}_{\text{n}}^*$ is a feasible point of Geometric GNF. Since the feasible set of Geometric CGNF includes that of Geometric GNF, $\bar{\mathbf{p}}_{\text{n}}^*$ must be a solution of Geometric GNF as well. The proof follows from equation (30) and the fact that $\mathbf{p}_{\text{n}}^*$ is another solution of Geometric GNF. ■

An optimal solution of CGNF comprises two parts: injection vector and flow vector. Theorem 2 states that CGNF always finds the correct optimal injection vector solving GNF. An example is provided in [28] to understand the reason why CGNF may not be able to find the correct optimal flow vector solving GNF.

D. Optimal Power Flow in Electrical Power Networks

In this subsection, the results derived earlier for GNF will be applied to power networks. Consider a group of generators (sources of energy), which are connected to a group of electrical loads (consumers) via an electrical power network (grid). This network comprises a set of transmission lines connecting various nodes to each other (e.g., a generator to a load). Figure 4(a) exemplifies a four-node power network with two generators and two loads. Each load requests certain amount of energy, and the question of interest is to find the most economical power dispatch by the generators so that the demand and network constraints are met. To formulate the problem, let $\mathcal{G}$ denote the flow network corresponding to the electrical power network, where

- Each injection p_i, $i \in \mathcal{G}$, represents either the active power produced by a generator and injected to the network or the active power absorbed from the network by an electrical load.
- Each p_{ij}, $(i, j) \in \mathcal{E}$, represents the active power entering the transmission line (i, j) from its i endpoint.

The problem of optimizing the flows in a power network is called "optimal power flow (OPF)". In this part, the goal is to optimize only active power, but most of the results to be developed later can be generalized to reactive power as well.

Let v_i denote the complex (phasor) voltage at node $i \in \mathcal{N}$ of the power network. Denote the phase of v_i as θ_i. Given an edge $(j, k) \in \mathcal{G}$, we denote the admittance of the transmission line between nodes j and k as $g_{jk} - \mathrm{i}b_{jk}$, where the symbol i denotes the imaginary unit. g_{jk} and b_{jk} are nonnegative numbers due to the passivity of the line. There are two flows entering the transmission line from its both ends. These flows are given by:

$$p_{jk} = |v_j|^2 g_{jk} + |v_j||v_k| b_{jk} \sin(\theta_{jk}) - |v_j||v_k| g_{jk} \cos(\theta_{jk}),$$
$$p_{kj} = |v_k|^2 g_{jk} - |v_j||v_k| b_{jk} \sin(\theta_{jk}) - |v_j||v_k| g_{jk} \cos(\theta_{jk})$$

where $\theta_{jk} = \theta_j - \theta_k$. As traditionally done in the power area, assume that $|v_j|$ and $|v_k|$ are fixed at their nominal values, while θ_{jk} is a variable to be designed. If θ_{jk} varies from $-\pi$ to π, then the feasible set of (p_{jk}, p_{kj}) becomes an ellipse, as illustrated in Figure 4(b). It can be seen from this figure that p_{kj} cannot be written as a function of p_{jk}. This observation is based on the implicit assumption that there is no limit on θ_{jk}. Suppose that θ_{jk} must belong to an interval $[-\theta_{jk}^{\max}, \theta_{jk}^{\max}]$ for some angle $\theta_{jk}^{\max}$. If the new feasible set for (p_{jk}, p_{kj}) resembles the partial ellipse drawn in Figure 4(c), then p_{kj} can be expressed as $f_{jk}(p_{jk})$ for a monotonically decreasing, convex function $f_{jk}(\cdot)$. This happens if

$$\theta_{jk}^{\max} \leq \tan^{-1}\left(\frac{b_{jk}}{g_{jk}}\right) \tag{33}$$

It is interesting to note that the right side of the above inequality is equal to $45.0°$, $63.4°$ and $78.6°$ for $\frac{b_{jk}}{g_{jk}}$ equal to 1, 2 and 5, respectively. Note that $\frac{b_{jk}}{g_{jk}}$ is normally greater than 5 (due to the specifications of transmission lines) and $\theta_{jk}^{\max}$ is normally less than $15°$ and very rarely as high as $30°$ due to stability and thermal limits (this angle constraint is forced either directly or through $p_{jk}^{\min}$ and $p_{jk}^{\max}$ in practice). Hence, Condition (33) is very practical. By assuming that this condition is satisfied, there exists a monotonically decreasing, convex function $f_{jk}(\cdot)$ such that

$$p_{kj} = f_{jk}(p_{jk}), \quad \forall p_{jk} \in [p_{jk}^{\min}, p_{jk}^{\max}], \tag{34}$$

where $p_{jk}^{\min}$ and $p_{jk}^{\max}$ correspond to $\theta_{jk}^{\max}$ and $-\theta_{jk}^{\max}$, respectively.

Given two disparate edges (j, k) and (j', k'), the phase differences θ_{jk} and $\theta_{j'k'}$ may not be varied independently if the graph $\mathcal{G}$ is cyclic (because the sum of the phase differences over a cycle must be zero). This is not an issue if the graph $\mathcal{G}$ is acyclic (corresponding to distribution networks) or if there is a sufficient number of phase-shifting transformers in the network. If none of these cases is true, then one could add virtual phase shifters to the power network at the cost of approximating the OPF problem. As soon as the flows (or phase differences) on various lines can be varied independently, equation (34) yields that the problem of optimizing active flows reduces to GNF. In this case, Theorems 1 and 2 can be used to study the corresponding approximated OPF problem. As a result, the optimal injections for the approximated OPF can be found via the corresponding CGNF problem. This implies two facts about the SDP and SOCP relaxations proposed in [16] and [12] for solving the OPF problem:

- The relaxations are exact without using the concept of load over-satisfaction (i.e., relaxing the flow constraints). This is the generalization of the result derived in [20].
- The relaxations always yields the optimal injections, but the produced flow vector can be wrong (meaning that the flow inequality constraints are not all binding). It is easy to contrive such examples.

In addition to active powers, voltage magnitudes and reactive powers are often variables in power systems. The following remarks can be made for a general OPF problem:

- Reactive flows can be written as linear functions of active flows. This implies that the above conclusions on OPF are valid even if the reactive power at each bus is upper bounded by a given number.
- In the case when the voltage magnitudes are variable, the flow constraint $p_{kj} = f_{jk}(p_{jk})$ needs to be replaced by $p_{kj} = f_{jk}(p_{jk}, \mathbf{x})$, where $\mathbf{x}$ is an exogenous input containing the voltage magnitudes at all buses. The technique proposed in this paper can be used to show that there is a region for $\mathbf{x}$ over which the above conclusions on OPF are valid. Due to space restrictions, the details are omitted here.

IV. Conclusions

Network flow plays a central role in operations research, computer science and engineering. Due to the complexity of this problem, the main focus has been on lossless flow networks and more recently on networks with a linear loss function. This paper studies the generalized network flow (GNF) problem, which aims to optimize the flows over a lossy flow network. It is assumed that the two flows over a line are related to each other via an arbitrary convex monotonic

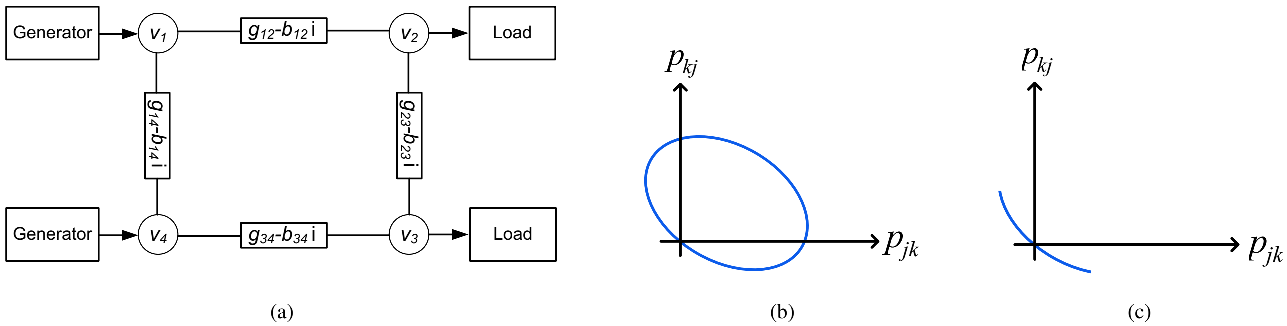


Fig. 4: (a): An example of electrical power network; (b) Feasible set for (p_{jk}, p_{kj}); (c) Feasible set for (p_{jk}, p_{kj}) after imposing lower and upper bounds on θ_{jk}.

function. The GNF problem is hard due to the presence of nonlinear equality flow constraints. If the flow constraints are relaxed to convex inequalities, these constraints may not be binding at optimality (as verified in simulations). This implies that a natural convex relaxation of GNF may lead to wrong flows. Nonetheless, this paper proves that the nodal injections obtained by solving the convex relaxation are optimal, as long as GNF is feasible. In other words, this paper proposes a polynomial-time algorithm for finding the optimal injections. Obtaining a set of flows associated with the optimal injections is a separate problem and has been considered as future work. An immediate application of this work is in power systems, where the goal is to optimize the power flows at buses and over transmission lines. Recent work on the optimal power flow problem has shown that this non-convex problem can be solved via a convex relaxation after two approximations: relaxing angle constraints (by adding virtual phase shifters) and relaxing power balance equations to inequality flow constraints. The results on GNF proves that the second relaxation (on power balance equations) is redundant under a very mild angle assumption.

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