Last-iterate Convergence in No-regret Learning: Games with Reference Effects Under Logit Demand*

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This work examines the behaviors of the online projected gradient ascent (OPGA) algorithm and its variant in a repeated oligopoly price competition under reference effects. In particular, we consider that multiple firms engage in a multi-period price competition, where consecutive periods are linked by the reference price update and each firm has access only to its own first-order feedback. Consumers assess their willingness to pay by comparing the current price against the memory-based reference price, and their choices follow the multinomial logit (MNL) model. We use the notion of stationary Nash equilibrium (SNE), defined as the fixed point of the equilibrium pricing policy, to simultaneously capture the long-run equilibrium and stability. We first study the loss-neutral reference effects and show that if the firms employ the OPGA algorithm—adjusting the price using the first-order derivatives of their log-revenues—the price and reference price paths attain last-iterate convergence to the unique SNE, thereby guaranteeing the no-regret learning and market stability. Moreover, with appropriate step-sizes, we prove that this algorithm exhibits a convergence rate of $\widetilde{\mathcal{O}}(1/t^2)$ in terms of the squared distance and achieves a constant dynamic regret. Despite the simplicity of the algorithm, its convergence analysis is challenging due to the model lacking typical properties such as strong monotonicity and variational stability that are ordinarily used for the convergence analysis of online games. The inherent asymmetry nature of reference effects motivates the exploration beyond loss-neutrality. When loss-averse reference effects are introduced, we propose a variant of the original algorithm named the conservative-OPGA (C-OPGA) to handle the non-smooth revenue functions and show that the price and reference price achieve last-iterate convergence to the set of SNEs with the rate of $\mathcal{O}(1/\sqrt{t})$. Finally, we demonstrate the practicality and robustness of OPGA and C-OPGA by theoretically showing that these algorithms can also adapt to firm-differentiated step-sizes and inexact gradients.

Key words: last-iterate convergence, price competition, reference effect, multinomial logit model

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* We note that the algorithm and its convergence for the case of loss-neutral reference effects in duopoly competition (i.e., Theorems 1 and 2) were proposed in our conference proceeding paper (Guo et al. 2024). In this work, however, we not only improve the convergence rate and regret in the loss-neural scenario but also extend all results to the oligopoly competition with potentially asymmetric reference effects.

1. Introduction

The memory-based reference effect is a well-established strategic consumer behavior, which refers to the phenomenon that consumers shape their price expectations, known as reference prices, based on their past encounters and then use them to judge the current price. Driven by the ubiquitous evidence, there has been a growing trend of studies dedicated to exploring effective pricing strategies in the presence of reference effects. However, the majority of research in this field focuses on scenarios of a monopolistic seller, and little is known about the role of reference effects within a competitive framework despite its practical importance.

In a competitive environment, a major challenge is non-transparency, as firms often lack detailed transaction data about their rivals, which prevents them from seeing a full picture of the market. Under such settings, one common approach in optimization literature is to assume access to a first-order oracle (Nesterov 2003), which typically does not require information from competitors. Despite the simplicity of gradient-based algorithms, they have proven effective in similar pricing problems. For instance, Goyal et al. (2023) consider an oligopoly price competition under the multinomial logit (MNL) demand model, aiming to derive the convergence to a Nash equilibrium, where each firm uses an online gradient descent to estimate the aggregated parameter of all other firms. Within the context of reference effects, Golrezaei et al. (2020) demonstrate the long-term market stability of a duopoly price competition under a general online mirror descent algorithm, where consumer demands are modeled using linear functions.

Enlightened by these real-world practices, we study *a repeated oligopoly price competition with reference effects, where each firm can only access its first-order oracle.* In this general multi-player game, we unify the settings of Golrezaei et al. (2020) and Goyal et al. (2023) by employing the MNL demand model to capture consumer choices under reference effects, as the logit function endogenously accounts for interactions among substitutes. Moreover, we also accommodate the potential asymmetry in the reference effects, i.e., allowing consumers to have different responsiveness to a price higher than the reference price versus a price lower with an equal amount. This asymmetry nature has its theoretical foundation in the renowned prospect theory (Kahneman and Tversky 1979) and is further justified by abundant empirical evidence (please refer to Section 2.1 for more details). Finally, due to the intertemporal characteristic of the memory-based reference effect, we consider the game in a dynamic/multi-period framework, i.e., the firms engage in repeated competitions with consecutive periods linked by reference price updates.

In this article, our goal is to investigate whether firms' prices and reference prices will be stabilized in the long run under gradient-based algorithms. The stability here means that price and reference price paths would converge to some point such that firms have no incentive to deviate from it unilaterally, which is commonly referred to as the Nash equilibrium in a standard static game. Yet, for our problem of interest, the conventional equilibrium notion is inadequate in determining the stable state, given the evolution of reference prices. For instance, even if an equilibrium is reached in one period, reference price updates will probably cause firms to deviate from this equilibrium in subsequent periods. As a result, to jointly capture the market equilibrium and stability, we consider the concept of stationary Nash equilibrium (SNE), defined as the fixed point of the equilibrium pricing policy. More precisely, at an SNE, each firm's equilibrium price with respect to its single-period revenue is equal to the given reference price, which is formally stated in Definition 2. Once firms' prices and reference prices converge to an SNE, they will remain fixed for subsequent periods.

We emphasize that the goal of this paper is different from developing a dynamic pricing policy that maximizes firms' cumulative revenue. It is noteworthy that even in the monopolistic market with a single product and full information, maximizing the cumulative revenue is highly complicated, where the optimal pricing path can be either static or cyclic in the long run (Jiang et al. 2022), and there is even no clear characterization on when each pattern would occur. Hence, in a competitive market where multiple firms interact with each other, optimizing the entire price trajectories, i.e., $\{\mathbf{p}^t\}_{t=0}^T$, becomes more intractable. Under such circumstances, a more practical objective is to examine the convergence to the SNE, which effectively ensures both equilibrium and market stability. A similar notion of equilibrium has also been studied in Golrezaei et al. (2020).

One common objective in the study of online competitions is to achieve the so-called *no-regret learning*, i.e., the regret of each player associated with the sequence of online actions produced by the algorithm, when compared to the best fixed action in hindsight, grows sub-linearly with the total number of periods. However, it is important to mention that being no-regret does not guarantee the convergence to the equilibrium by any means (Mertikopoulos et al. 2018). In our work, we seek to derive the convergence of both prices and reference prices to the SNE in the *last-iterate* sense, where the convergence is assessed based on the output in the last iteration of the algorithm. This is in contrast with the *average-iterate* convergence, which is established on the average of all iterates generated by the algorithm. Achieving the last-iterate convergence is typically more challenging than merely demonstrating the sublinear regret or the convergence in the average sense, as the former automatically proves the latter two goals.

To achieve the aforementioned goal, we employ a vanilla gradient descent algorithm called *Online Projected Gradient Ascent* (`OPGA`) for loss-neutral (symmetric) reference effects and further extend it to *Conservative Online Projected Gradient Ascent* (`C-OPGA`) to accommodate loss-averse (asymmetric) reference effects. Both algorithms only require the first-order feedback received by firms, thereby making `OPGA` and `C-OPGA` practical for implementation (see Appendix A for more details). Our conclusions show that if firms run the `OPGA` or `C-OPGA` algorithm in repeated price competitions with reference effects, they will reach SNE and achieve market stability in the long term.

1.1. Contributions

We evaluate the convergence of gradient-based learning algorithms in a repeated oligopoly price competition with reference effects, where the multi-period setting stems from the intertemporal nature of the reference price. To the best of our knowledge, this is the first work that examines the stable equilibrium in a general n-player game with the influence of reference effects.

1. *Loss-neutral Reference Effects.* We study the `OPGA` algorithm in the loss-neutral scenario, where each firm adjusts its price using the first-order derivative of its log-revenue. When the firms execute the `OPGA` with (possibly firm-differentiated) diminishing step-sizes, we show that their price and reference price paths achieve last-iterate convergence to the unique SNE, thereby assuring no-regret learning and leading to the long-run equilibrium and stability. Furthermore, when the step-sizes decrease appropriately in the order of $\Theta(\log t/t)$, we demonstrate that the prices and reference prices converge at a rate of $\widetilde{\mathcal{O}}(1/t^2)$ in terms of the squared distance and achieve a constant dynamic regret.

2. *Loss-averse and Gain-seeking Reference Effects.* The loss-averse scenario introduces significant challenges due to the non-smooth revenue function and the non-convex set of SNEs. To navigate this, we adapt the original `OPGA` into the `C-OPGA` algorithm by introducing a conservative pausing mechanism in the learning process. Specifically, the firm would temporarily pause its price update if it finds the current price close to stationarity. Under (possibly firm-differentiated) diminishing step-sizes, we show that the price and reference price paths generated by the `C-OPGA` achieve last-iterate convergence to the set of SNEs in terms of the proposed stationarity metric. In addition, when the step-sizes are chosen in the order of $\Theta(1/\sqrt{t})$, we attain a convergence rate of $\mathcal{O}(1/\sqrt{t})$. By contrast, when consumers exhibit gain-seeking reference effects towards any product, we prove that an SNE cannot exist, thus making it infeasible to simultaneously achieve the equilibrium and market stability.

3. *Convergence Analysis.* From the optimization perspective, the analyses of the `OPGA` and `C-OPGA` are related to the studies of online games with gradient feedback and discrete nonlinear systems. However, the incorporation of the MNL model renders our problem lacking favorable properties typically required in the literature, such as monotonicity, variational stability, and concavity of the objective function, making existing theories inapplicable. Additionally, unlike standard online games or nonlinear systems, the dynamic state from the reference price updates and the non-smoothness introduced by the loss-averse reference effects further perplex the convergence analysis. To address these challenges, we introduce problem-specific metrics for both `OPGA` and `C-OPGA` to assess the convergence and develop original techniques that exploit the characteristic properties of the MNL-based revenue functions. Finally, we show that our algorithms are robust in the sense that the analyses can also be adapted to the inexact gradient oracle, under which the price and reference price would converge to the neighborhood of the SNE.

4. *Managerial Insights.* We believe that achieving convergence to SNE offers a reasonable objective in practice, given the intractability of optimizing an entire price trajectory in a competitive market with limited information. By the nature of the SNE, this convergence ensures both long-run equilibrium and market stability, making the adoption of `OPGA` and `C-OPGA` a prudent and effective strategy in such a market environment. This achievement of stable equilibrium has broader implications for the ongoing debate on algorithmic collusion in price competitions, referring to the phenomenon that firms jointly charge higher prices at the expense of consumer welfare. Our findings demonstrate that `OPGA` and `C-OPGA` can effectively guide sellers toward equilibrium rather than engaging in supra-competitive behaviors, which makes these algorithms less susceptible to regulatory concerns on potential anti-competitive effects.

1.2. Organization

The rest of the paper is structured as follows. In Section 2, we conduct a literature review on topics pertinent to our study. Section 3 introduces the model and defines the partial information structure and the notion of SNE, whose properties are investigated in Section 4. Section 5 is dedicated to the loss-neutral scenario, where we propose the `OPGA` algorithm and establish its global convergence rate and the dynamic regret bound. We then introduce the `C-OPGA` algorithm for the loss-averse scenario in Section 6, with the exploration of its convergence. In Section 7, we extend the convergence results to more practical situations, allowing for firm-differentiated step-sizes and the inexact first-order oracle. Finally, we conclude this work with a discussion in Section 8. All formal proofs are documented in the supplemental materials.

2. Related Literature

Our research on the price competition with reference effects under a partial information setting is related to several streams of literature: modeling of reference effect, pricing with reference effects in monopolistic and competitive markets, and general convergence results for online games.

2.1. Modeling of Reference Effects

The concept of reference effects can be traced back to the adaptation-level theory, which states that consumers evaluate prices against the level they have adapted to. Meanwhile, the asymmetry nature of reference effects is theoretically grounded in the renowned prospect theory (Kahneman and Tversky 1979), which posits that individuals exhibit loss-averse behaviors, meaning that they weigh losses more heavily than equivalent gains. In addition to its theoretical foundation, the presence of reference effects is further corroborated by extensive empirical evidence (see, e.g., Krishnamurthi et al. (1992), Hardie et al. (1993), Kalyanaram and Winer (1995)).

Extensive research in marketing literature has been dedicated to the formulation of reference effects, from which two predominant models have emerged: memory-based and stimulus-based reference prices (see Briesch et al. (1997) and the references therein). The memory-based reference model leverages historical prices to form the benchmark, whereas the stimulus-based reference model posits that price judgments are established at the moment of purchase utilizing current external information such as the prices of substitutable products. According to the comparative analysis by Briesch et al. (1997), among different reference models, the memory-based formulation that relies on a product's own historical prices, referred to as `PASTBRSP`, offers the best fit and strongest predictive power in a multi-product setting. Consequently, our paper adopts this type of reference price formulation.

2.2. Dynamic Pricing in Monopolistic Market with Reference Effects

We observe a growth trend in the study of price optimization with reference effects within the field of operations management. The majority of classical works target the monopolistic setting, where they usually formulate the problem as a dynamic program since the memory-based reference effect naturally gives rise to the intertemporal feature. Their objectives typically involve determining the long-run market stability under the optimal strategies. For example, Popescu and Wu (2007) show that the optimal pricing policy converges and leads to market stabilization under both loss-neutral and loss-averse reference effects. More recent papers on reference effects primarily concentrate on the piecewise linear demand in a single-product case and delve into more comprehensive characterizations of myopic and optimal pricing policies (Chen et al. 2017, Chen and Nasiry 2020). Beyond the linear demand, Jiang et al. (2022) and Guo et al. (2022) employ the logit model similar to ours, though within the context of a monopoly. Jiang et al. (2022) emphasize on consumer heterogeneity and devise a numerical method for computing the optimal policy, whereas Guo et al. (2022) expand into the multi-product setting and investigate the long-run convergence behavior under the myopic and optimal pricing.

While the aforementioned studies commonly assume that the firm possesses full knowledge of its demand function, another line of research tackles the problem in the context of uncertain demand, where they couple monopolistic price optimization with reference effects and online demand learning (den Boer and Keskin 2022, Agrawal and Tang 2024). Although these works incorporate learning components, our paper distinguishes itself in two critical aspects. Firstly, the objectives for the algorithms are different. While their goals center around designing algorithms to boost the total revenue from a monopolist's perspective, our algorithm aims for the simultaneous realization of Nash equilibrium and market stability under a competitive framework. Secondly, the uncertainties to be learned reside in distinct areas. Those aforementioned works assume that a monopolist recognizes the demand structure but needs to estimate sensitivity parameters. By contrast, we consider a competitive environment where a firm knows its own demand but lacks insights into its rivals' pricing and demand functions.

2.3. Price Competition with Reference Effects

With the incorporation of reference effects, the existing literature on price competitions primarily focuses on the perfect information setting. For example, Federgruen and Lu (2016) and Guo and Shen (2024) examine the equilibrium properties in the single-period price competitions under different reference price formulations. Colombo and Labrecciosa (2021) analyze the game with reference effects in a multi-stage, continuous-time framework.

More closely pertinent to our work, Golrezaei et al. (2020) examine the long-run market stability of a duopoly price competition in a partial information setting. However, our study diverges significantly from theirs in four aspects. The first prominent distinction is our inclusion of loss-averse and gain-seeking reference effects, a factor not considered by Golrezaei et al. (2020). In particularly, we derive the convergence result in the loss-averse case and establish the non-existence of SNE under gain-seekingness. The second major improvement is our extension to the oligopoly competition, i.e., a general n-player game. Under this setting, as opposed to the linear demand used in Golrezaei et al. (2020), we adopt the logit demand for its suitability to model discrete choices among multiple products and for its better empirical performance in the presence of reference effects (Wang 2018, Jiang et al. 2022). Yet, this logit demand imposes great challenges for convergence, as a crucial step of the analysis in Golrezaei et al. (2020) hinges on demand linearity (see Golrezaei et al. (2020, Lemma 9.1)), a condition not met by the MNL model. Lastly, our approach to reference price formulation further sets us apart from Golrezaei et al. (2020). Contrary to their assumption of a uniform reference price for both products, our model incorporates the product-specific reference price, which is empirically validated by Briesch et al. (1997) to yield superior performance. These adaptations, though enriching the model's expressiveness and flexibility, render analyses in Golrezaei et al. (2020) not generalizable to our setting. For an elaboration on this reasoning, we refer readers to Appendix B.1.2.

2.4. General Convergence Results for Games

Our problem is also closely related to a stream of theoretical research in online games with first-order feedback. In this area, a typical question of interest is whether learning algorithms can achieve some equilibrium for multiple agents who aim to minimize (resp. maximize) their individual loss (resp. reward) functions. For games with continuous actions, Bravo et al. (2018) and Ba et al. (2021) show the convergence of online mirror descent to the Nash equilibrium in strongly monotone games. Lin et al. (2020) then relax the strong monotonicity assumption and examine the last-iterate convergence for games with unconstrained action sets satisfying the so-called "cocoercive" condition. Mertikopoulos and Zhou (2019) establish the convergence of the dual averaging method under a more general condition called global variational stability, which encompasses the cocoercive condition as a sub-case. We point out that in the works cited above, the concavity and smoothness of the individual reward function are invariably required for the convergence analysis. By contrast, the firm's revenue function in our problem is neither concave in its price nor satisfies any aforementioned properties, and it even becomes non-smooth under loss-averse/gain-seeking reference effects. Besides, due to the dynamic nature of reference price, the standard notion of equilibrium fails to characterize the convergence. To address the issues, we adopt the concept of SNE to jointly capture the equilibrium and market stability. For both loss-neutral and loss-averse reference effects, we demonstrate that the proposed algorithms provably converge to the SNE.

3. Problem Formulation

3.1. MNL Demand Model with Reference Effects

This study examines an oligopoly price competition that involves potentially loss-averse/gain-seeking reference price effects. We consider a market with n firms, denoted by set $N := \{1, 2, \dots, n\}$, where each firm offers a substitutable product with the choice set also labeled by N. In our discrete-time setting, firms simultaneously set prices at the beginning of each period over an infinite time horizon. Hence, in the current period, the firm does not observe its competitors' prices before setting its own price. We suppose that consumers' purchase behavior follows the multinomial logit (MNL) model, an effective framework for capturing the cross-product interactions among various alternatives. Let $\mathbf{p} = (p_i)_{i\in N}$ and $\mathbf{r} = (r_i)_{i\in N}$ be the price and reference price vectors, respectively, with $\mathbf{p}^t = (p_i^t)_{i\in N}$ and $\mathbf{r}^t = (r_i^t)_{i\in N}$ representing those at a specific time period t. The utility for purchasing product i depends on its posted price p_i and the reference price r_i, i.e.,

$$U_i(p_i, r_i) = u_i(p_i, r_i) + \varepsilon_i = a_i - b_i p_i + c_i^+ \cdot (r_i - p_i)_+ + c_i^- \cdot (r_i - p_i)_- + \varepsilon_i, \quad \forall i \in N, \tag{1}$$

where $u_i(p_i, r_i)$ is the deterministic component and ε_i is the random fluctuation that follows the i.i.d. standard Gumbel distribution. For the parameters in Eq. (1), a_i is the intrinsic value for product i, and b_i denotes its price sensitivity. In the presence of reference effects, c_i^+ and c_i^- correspond to the reference price sensitivities to gains and losses, respectively. When $r_i > p_i$, consumers would perceive gains or discounts, whereas $r_i < p_i$ is regarded as losses or surcharges. The notations $(\cdot)_+ := \max\{\cdot, 0\}$ and $(\cdot)_- := \min\{\cdot, 0\}$ are adopted to account for consumers' potentially asymmetric reactions to gains and losses, respectively.

According to the random utility maximization theory (McFadden 1974), when ε_i follows the Gumbel distribution, the market share for product i is given by

$$d_i(\mathbf{p}, \mathbf{r}) = \frac{\exp\big(u_i(p_i, r_i)\big)}{1 + \sum_{k\in N} \exp\big(u_k(p_k, r_k)\big)}, \quad \forall i \in N. \tag{2}$$

Consequently, the expected revenue for firm i can be expressed as

$$\Pi_i(\mathbf{p},\mathbf{r}) = p_i \cdot d_i(\mathbf{p},\mathbf{r}), \quad \forall i \in N. \tag{3}$$

Given the above definitions, at period t, the market share and revenue for firm $i \in N$ are denoted by $d_i\big(\mathbf{p}^t,\mathbf{r}^t\big)$ and $\Pi_i(\mathbf{p}^t,\mathbf{r}^t)$, respectively.

Acknowledging the potential asymmetry in consumer sensitivities to gains and losses, we classify reference effects into three types—*loss-averse*, *loss-neutral*, and *gain-seeking*. Here, the loss-neutral is also known as symmetric reference effects, while the loss-averse and gain-seeking are collectively referred to as the asymmetric reference effects.

Definition 1 (Types of Reference Effects). For each product $i \in N$, consumers are loss-averse when $c_i^+ < c_i^-$, loss-neutral when $c_i := c_i^+ = c_i^-$, or gain-seeking when $c_i^+ > c_i^-$.

We assume the sensitivities to be positive, i.e., b_i, c_i^+, $c_i^- > 0$ for $i \in N$, which reflects consumer behaviors toward substitutes. Moreover, we stipulate the feasible range for both price and reference price to be $\mathcal{P} = [\underline{p}, \overline{p}]$ with $\underline{p}, \overline{p} > 0$. This price constraint is well-aligned with the real-world instances of price floors and ceilings and commonly adopted in literature on price optimization with reference effects (see, e.g., Chen et al. (2017)).

We formulate the reference price using the brand-specific past prices (`PASTBRSP`) model proposed by Briesch et al. (1997), which posits that the reference price is product-specific and memory-based. For each product i, the reference price is constructed by applying exponential smoothing to its own historical prices, which is described as

$$r_i^{t+1} = \alpha r_i^t + (1-\alpha) p_i^t, \quad \forall i \in N,\ \forall t \geq 0, \tag{4}$$

where $\alpha \in [0,1]$ is a memory parameter that controls the rate at which the reference price evolves. We remark that the theories established in our work can readily be generalized to scenarios with time-varying α, although for clarity, we present the static α here.

3.2. Equilibrium and Market Stability

The goal of our paper is to analyze a gradient-based pricing update mechanism and determine whether it leads to a stable equilibrium state over the long term. Before introducing the notion of stable equilibrium considered in this work, we first define the *equilibrium pricing policy*, denoted by $\mathbf{p}^E(\mathbf{r}) = \big(p_i^E(\mathbf{r})\big)_{i\in N}$, which is a function that maps reference price to price and achieves the pure strategy Nash equilibrium in the single-period game, i.e.,

$$p_i^E(\mathbf{r}) = \arg\max_{p_i \in \mathcal{P}} \left\{\Pi_i\big((p_i, \mathbf{p}_{-i}^E(\mathbf{r})), \mathbf{r}\big)\right\} = \arg\max_{p_i \in \mathcal{P}} \left\{p_i \cdot d_i\big((p_i, \mathbf{p}_{-i}^E(\mathbf{r})), \mathbf{r}\big)\right\}, \quad \forall i \in N, \tag{5}$$

where the subscript $-i := N \backslash \{i\}$ denotes the set of all products excluding product i, and the vector $\mathbf{p}_{-i}^E(\mathbf{r}) = \big(p_j^E(\mathbf{r})\big)_{j \in N\backslash\{i\}}$ represents the equilibrium price of all products except for product i.

Definition 2 (Stationary Nash equilibrium). A stationary Nash equilibrium (SNE) is defined as the fixed point of the equilibrium pricing policy. Specifically, a price vector $\mathbf{p}^\star$ is a stationary Nash equilibrium if $\mathbf{p}^E(\mathbf{p}^\star) = \mathbf{p}^\star$, i.e., the equilibrium price is equal to its reference price.

From Eq. (5) and Definition 2, we observe that an SNE possesses the following two properties:

- **Equilibrium.** The revenue function for every firm $i \in N$ satisfies $\Pi_i\big((p_i, \mathbf{p}^\star_{-i}), \mathbf{p}^\star\big) \leq \Pi_i\big(\mathbf{p}^\star, \mathbf{p}^\star\big)$ for all $p_i \in \mathcal{P}$, i.e., when the reference price $\mathbf{r}$ and the price of other firms $\mathbf{p}_{-i} = \big(p_j\big)_{j \in N \setminus \{i\}}$ are both set at the SNE, the best-response price for firm i is equal to its SNE price $p_i^\star$. Therefore, in such a case, firm i has no incentive to deviate from the SNE.
- **Stability.** If the price and the reference price attain the SNE at some period t, they would remain unchanged in the following periods, i.e., $\mathbf{p}^t = \mathbf{r}^t = \mathbf{p}^\star$ implies that $\mathbf{r}^\tau = \mathbf{p}^\tau = \mathbf{p}^\star, \forall \tau \geq t+1$.

Together, the SNE jointly characterizes the equilibrium and the market stability, implying that when the market reaches the SNE, the firms have no incentive to deviate and will maintain the same price in subsequent competitions.

As our study integrates the aspects of reference effects and competition, it is pertinent to two special cases that exclusively consider one of these aspects. When $c_i^+ = c_i^- = 0$, our problem reduces to the standard price competition without reference effects, a setting studied by Goyal et al. (2023). Specifically, they explore the sequential price competition and mainly focus on learning to price—firms gradually adjust their prices by learning competitors' aggregated parameters, with the goal of reaching a Nash equilibrium. However, since Goyal et al. (2023) do not incorporate reference effects and assume uniform price sensitivities, i.e., $b_i \equiv b$ for all $i \in N$, their analyses and findings are greatly different from ours. When $|N| = 1$, indicating there is no competition, our problem is equivalent to price optimization in the monopolistic market with reference effects. The relevant literature includes Jiang et al. (2022) and Guo et al. (2022), where they investigate long-term market behaviors—either convergent or cyclic—under the optimal dynamic pricing policy. Yet, both studies diverge significantly from ours as they are conducted within a perfect information setting typical for monopolies and thus lack the learning components central to this study.

3.3. First-order Feedback

We assume that the firms can access a first-order oracle as a feedback mechanism, which is a common assumption in the optimization literature (Nesterov 2003). For every firm $i \in N$, this oracle outputs the derivative of its log-transformed revenue function, i.e., $\partial \log\big(\Pi_i(\mathbf{p}, \mathbf{r})\big)/\partial p_i = 1/p_i + (b_i + c_i)\left[d_i(\mathbf{p}, \mathbf{r}) - 1\right]$ in the loss-neutral scenario. Under loss-averse/gain-seeking reference effects, the oracle returns two distinct values by substituting c_i in the derivative with both c_i^+ and c_i^-. To access this oracle, the firm does not require any information about its competitors, including their prices, reference prices, and market shares, as well as the scheme behind reference price updates. It suffices for each firm i to know its previously posted price, own sensitivity parameters (b_i, c_i^+, c_i^-), and market share $d_i(\mathbf{p}, \mathbf{r})$. In addition, if the firms have a good understanding of their own sensitivity parameters from operational experiences, the first-order feedback reduces to the *bandit feedback*, since knowing only the revenue $\Pi_i(\mathbf{p}, \mathbf{r})$ is sufficient for firm i to compute $\partial \log\big(\Pi_i(\mathbf{p}, \mathbf{r})\big)/\partial p_i$. Otherwise, it is also feasible for firms to estimate their sensitivity parameters while protecting their privacy. This can be achieved through temporary cooperation between firms, during which one firm applies a slight perturbation to its price while the others maintain the previous prices. We provide a detailed elaboration and a step-by-step methodology for this idea in Appendix A. Admittedly, any parameter estimation may incur some errors, the impact of which is further explored as an extension in Section 7.2.

4. Properties of SNE

To guarantee that long-run equilibrium and market stability are achievable, it is important to investigate whether an SNE always exists. In this section, we focus on the existence, structure, and uniqueness of SNE under different types of reference effects. This endeavor is far from trivial as the revenue function is non-concave and potentially non-smooth due to the MNL choice model and loss-averse/gain-seeking reference effects. By Definition 2, an SNE is equivalent to the fixed-point of the equilibrium pricing policy $\mathbf{p}^E(\cdot)$ defined in Eq. (5). Using the optimality condition for the Nash equilibrium in the single-period game, we reveal that the existence of an SNE is contingent on the presence of gain-seekingness. The formal proof of Proposition 1 is deferred to Appendix C.1.

Proposition 1 **(Existence and Structure of SNE)**. *Let $\mathcal{S}$ be the set of SNE(s). Then, the following statements hold:*

- *If there exists any gain-seeking product, an SNE never exists, i.e., $\mathcal{S}$ is empty.*
- *Otherwise, with only loss-averse and loss-neutral products, an SNE always exists, and $\mathcal{S}$ can be expressed as*

$$\mathcal{S} = \left\{ \mathbf{p}^\star \;\middle|\; p_i^\star \in \left[\frac{1}{\big(b_i + c_i^-\big)\cdot\big(1 - d_i(\mathbf{p}^\star, \mathbf{p}^\star)\big)}, \frac{1}{\big(b_i + c_i^+\big)\cdot\big(1 - d_i(\mathbf{p}^\star, \mathbf{p}^\star)\big)} \right],\ \forall i \in N \right\}. \tag{6}$$

As indicated by Proposition 1, the presence of one or more gain-seeking products implies the non-existence of SNE, suggesting that the long-run equilibrium and market stability cannot be achieved simultaneously. Therefore, *the remaining part of the paper will be devoted to the loss-neutral and loss-averse scenarios.* To avoid ambiguity, we provide the precise definition for each scenario: the *loss-neutral scenario* refers to when all products have loss-neutral reference effects; the *loss-averse scenario* refers to when at least one product displays loss-averse reference effects, with the others being either loss-neutral or loss-averse.

It is noteworthy that the characterization in Eq. (6) is not a simple box constraint since the two boundaries of the interval are functions depending on $\mathbf{p}^\star$. Under the loss-neutral scenario where $c_i^- = c_i^+$, the interval collapses into a single function of $\mathbf{p}^\star$, suggesting the potential uniqueness of the SNE. In the next proposition, we confirm this uniqueness and further provide a bound for the set $\mathcal{S}$ in terms of problem parameters, with its proof documented in Appendix C.2.

Proposition 2 **(Uniqueness of SNE).** *In loss-averse and loss-neutral scenarios where SNE(s) always exists, its uniqueness depends on the presence of any loss-averse product. Specifically,*

- *The SNE is unique, i.e., $\mathcal{S}$ is a singleton, if and only if all products are loss-neutral.*
- *Otherwise, with any loss-averse product, there always exists a continuum of SNEs, and $\mathcal{S}$ can be a non-convex set.*

Furthermore, any SNE $\mathbf{p}^\star \in \mathcal{S}$ can be bounded as

$$\frac{1}{b_i + c_i^-} < p_i^\star < \frac{1}{b_i + c_i^+} + \frac{1}{b_i} W\left(\frac{b_i}{b_i + c_i^+} \exp\left(a_i - \frac{b_i}{b_i + c_i^+}\right)\right), \quad \forall i \in N, \tag{7}$$

where $W(\cdot)$ is the Lambert W function (see definition in Eq. (C.25)).

In Figure 1, we illustrate the non-convexity of the set $\mathcal{S}$ using a two-product example, where both exhibit loss-averse reference effects (i.e., $c_i^+ < c_i^-$ for $i \in \{1,2\}$, as outlined in the figure caption). The shaded area in Figure 1a depicts the set of SNEs, with the four colored curves corresponding to the boundaries of intervals defined in Eq. (6). Upon closer inspection of the region surrounding the upper green curve, as displayed in Figure 1b, the non-convexity of $\mathcal{S}$ becomes evident, marked by the black dashed line that represents a straight path between two vertices.

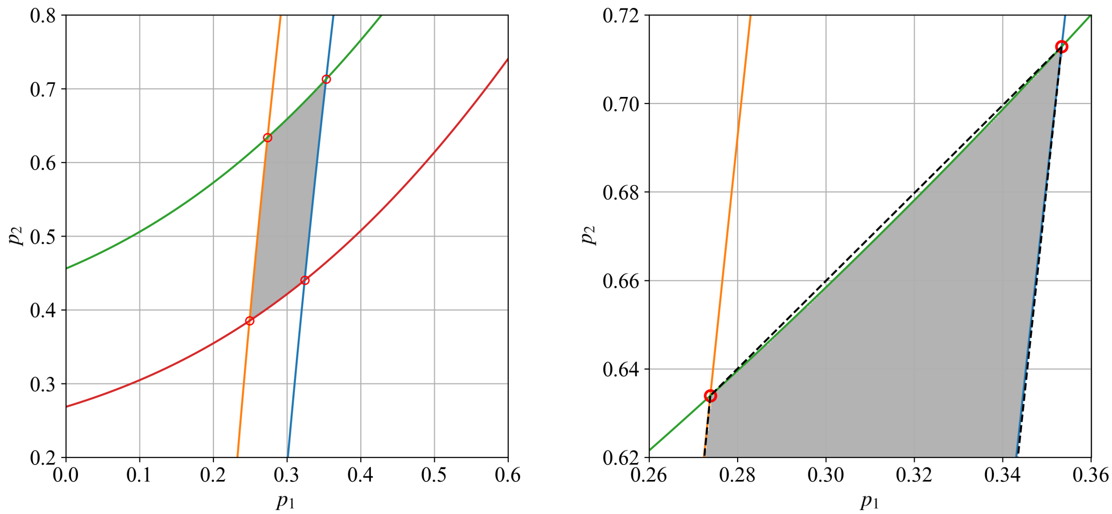
Without loss of generality, we assume that the feasible price range $\mathcal{P}^n = [\underline{p}, \overline{p}]^n$ is sufficiently large to contain the set of SNE(s), i.e., $\mathcal{S} \subseteq [\underline{p}, \overline{p}]^n$. Proposition 2 provides a quantitative characterization for this assumption: it suffices to choose the price lower bound $\underline{p}$ to be any real number between $\big(0, \min_{i\in N}\{1/(b_i + c_i^-)\}\big]$, and the price upper bound $\overline{p}$ can be set to any value satisfying that

$$\overline{p} \geq \max_{i \in N} \left\{ \frac{1}{b_i + c_i^+} + \frac{1}{b_i} W\left(\frac{b_i}{b_i + c_i^+} \exp\left(a_i - \frac{b_i}{b_i + c_i^+}\right)\right)\right\}. \tag{8}$$

This assumption is mild since the bound in Eq. (8) is independent of both the price and reference price and does not grow exponentially with respect to any parameters. Hence, there is no need for $\overline{p}$ to be excessively large. For example, when $a_1 = a_2 = 10$ and $b_1 = b_2 = c_1^+ = c_2^+ = 1$, the upper bound from Eq. (8) becomes $\overline{p} \geq 7.3785$.

Figure 1 **Illustration for the Non-convexity of $\mathcal{S}$ in the Loss-averse Scenario. The Shaded Region Represents $\mathcal{S}$ and the Colored Curves Represent the Boundaries of $\mathcal{S}$.**

(Parameters: $(a_1, b_1, c_1^+, c_1^-) = (5.88, 4.20, 1.17, 3.63)$ and $(a_2, b_2, c_2^+, c_2^-) = (5.32, 1.16, 1.77, 4.12)$.)



(a) Global view. (b) Enlarged local view.

5. Loss-neutral: Online Projected Gradient Ascent

We first consider the loss-neutral scenario and explore the convergence behavior of the price and reference price paths under the *Online Projected Gradient Ascent* (`OPGA`) method, as outlined in Algorithm 1. Under the setting of first-order feedback, the firm is incapable of computing either the equilibrium price, as defined in Eq. (5), or the best-response price, i.e., the optimal response to the competitors' previous-period prices and all firms' current reference prices. While these two commonly studied pricing strategies are impractical due to the lack of information about competitors, our `OPGA` algorithm provides a feasible and effective way for firms to enhance revenues based on gradient feedback.

In the `OPGA` algorithm, firms update their prices using the first-order derivatives of log-revenues (see Lines 3–5). We note that the derivative of the log-revenue in Eq. (10) differs from that of the standard revenue by a scaling factor equal to the revenue itself, i.e.,

$$\frac{\partial \log\big(\Pi_i(\mathbf{p},\mathbf{r})\big)}{\partial p_i} = \frac{1}{\Pi_i(\mathbf{p},\mathbf{r})} \cdot \frac{\partial\big(\Pi_i(\mathbf{p},\mathbf{r})\big)}{\partial p_i}, \quad \forall i \in N. \tag{9}$$

Therefore, the price updates can be equivalently viewed as an adaptively regularized gradient ascent using the standard revenue function, where the regularizer is $1/\Pi_i(\mathbf{p}^t, \mathbf{r}^t)$ for firm i at time t. When executing the algorithm, firms do not need to know their own reference prices, and the updates of reference prices in Line 7 are automatically handled by the market. For now, we assume all firms have the same sequence of step-sizes $\{\eta^t\}_{t\geq 0}$ in the learning process. In Section 7.1, we extend our results to a more general setting where each firm i can adopt different step-sizes $\{\eta_i^t\}_{t\geq 0}$.

Algorithm 1: Online Projected Gradient Ascent (`OPGA`)

Input: Initial reference price $\mathbf{r}^0$, initial price $\mathbf{p}^0$, and step-sizes $\{\eta^t\}_{t\geq 0}$.

for $t = 0, 1, 2, \ldots$ **do**

  for $i \in N$ **do**

    Compute the derivative from price and market share of firm i at period t:

$$D_i^t \leftarrow \frac{\partial \log\left(\Pi_i(\mathbf{p}^t, \mathbf{r}^t)\right)}{\partial p_i} = \frac{1}{p_i^t} + (b_i + c_i) \cdot d_i(\mathbf{p}^t, \mathbf{r}^t) - (b_i + c_i). \tag{10}$$

    Update posted price: $p_i^{t+1} \leftarrow \mathtt{Proj}_{\mathcal{P}}\,(p_i^t + \eta^t D_i^t)$.

  end

  Update reference price: $\mathbf{r}^{t+1} \leftarrow \alpha \mathbf{r}^t + (1-\alpha)\mathbf{p}^t$.

end

While our problem is related to two research topics—multi-player online games and discrete nonlinear systems, the existing theories and techniques in those areas are insufficient for the convergence analysis of the `OPGA` algorithm. Below, we briefly introduce how our problem can be transformed into these more standard formulations and highlight the analytical challenges involved. For a more comprehensive explanation, we direct readers to Appendix B.

- **Standard $2n$-player Online Game.** By adding n virtual firms to represent the reference price updates, we can convert the original n-player game with underlying dynamic states (i.e., reference prices are non-stationary in time) into a standard $2n$-player without such states. However, analyzing this $2n$-player game poses a great challenge due to the absence of certain favorable properties in the objective functions of the real firms (i.e., revenues). These properties, such as strong monotonicity (Lin et al. 2020) or variational stability (Mertikopoulos and Zhou 2019), are typically crucial in proving the convergence of online games. Moreover, while the real firms have the flexibility to dynamically adjust their step-sizes, the learning rate for the virtual firms is fixed to the constant $(1-\alpha)$, where α is the memory parameter for reference price updates. This disparity also hinders the direct application of the existing results from multi-agent online learning literature, as their results typically require the step-sizes of all agents to either diminish at comparable rates or remain at a sufficiently small constant (e.g., Mertikopoulos and Zhou (2019)). For details on the transformation to the standard $2n$-player game and further discussions on the two main difficulties in the analysis, please refer to Appendix B.1.

- **Nonlinear Dynamical System.** The second approach involves translating the `OPGA` algorithm into a discrete nonlinear dynamical system by treating $(\mathbf{p}^{t+1}, \mathbf{r}^{t+1})$ as a vector-valued function of $(\mathbf{p}^t, \mathbf{r}^t)$, i.e., $(\mathbf{p}^{t+1}, \mathbf{r}^{t+1}) = \mathbf{f}(\mathbf{p}^t, \mathbf{r}^t)$ for some function $\mathbf{f}(\cdot)$. In this context, analyzing the convergence of the `OPGA` is equivalent to examining the stability of the fixed point of $\mathbf{f}(\cdot)$, which is related to the spectral radius of the Jacobian matrix $\nabla \mathbf{f}(\mathbf{p}^\star, \mathbf{p}^\star)$ (see Arrowsmith et al. (1990)). Nevertheless, the SNE lacks a closed-form expression, making it difficult to calculate the eigenvalues of $\nabla \mathbf{f}(\mathbf{p}^\star, \mathbf{p}^\star)$. In addition, function $\mathbf{f}(\cdot)$ is non-smooth due to the presence of the projection operator and can become non-stationary when the firms adopt time-varying step-sizes, such as diminishing step-sizes. Finally, typical results in dynamical systems only guarantee the local convergence (Khalil 2002), i.e., the asymptotic stability of the fixed point, whereas our goal is to establish the global convergence of both the price and reference price. We provide more details on the transition to a dynamical system and its challenges in Appendix B.2.

Despite the aforementioned challenges, we manage to show the global convergence of Algorithm 1 by developing a novel analysis distinct from all existing approaches. Our result is based on the last iterate of the algorithm, i.e., $\lim_{t\to\infty} \mathbf{p}^t = \lim_{t\to\infty} \mathbf{r}^t = \mathbf{p}^\star$. This last-iterate (or point-wise) convergence indicates that the `OPGA` algorithm provably achieves no-regret learning, i.e., in the long run, the algorithm performs at least as well as the best fixed action in hindsight. However, we remark that the reverse direction is not necessarily true: being no-regret does not guarantee the convergence at all, let alone the convergence to an equilibrium (Mertikopoulos et al. 2018, 2019). In fact, the players may exhibit entirely unpredictable and chaotic behaviors under a no-regret policy (Palaiopanos et al. 2017), with the only exception being the finite games where players compete for finitely many rounds, a category that our problem does not fall into.

5.1. Convergence Results for `OPGA`

In this section, we investigate the convergence properties of the `OPGA` algorithm. We first establish the global last-iterate convergence to the unique SNE in Theorem 1. Subsequently, we show in Theorem 2 that this convergence exhibits a rate of $\widetilde{\mathcal{O}}(1/t^2)$ in terms of squared distance, given that the step-sizes are selected appropriately. Lastly, in Theorem 3, we demonstrate that Algorithm 1 achieves a constant regret.

THEOREM 1 **(Global Convergence of `OPGA`).** *In the loss-neutral scenario, suppose all firms adopt Algorithm 1 with non-increasing step-sizes* $\{\eta^t\}_{t\geq 0}$ *such that* $\lim_{t\to\infty} \eta^t = 0$ *and* $\sum_{t=0}^{\infty} \eta^t = \infty$. *Then, their price and reference price paths converge to the unique stationary Nash equilibrium.*

As discussed above, the existing techniques for online games and dynamical systems are not applicable to our problem. In response, we develop an original method that leverages the equilibrium structure of the MNL model. Below, we provide a high-level proof sketch of Theorem 1, with the full proof deferred to Appendix D. To measure the convergence, we introduce the following metrics:

$$\kappa(\mathbf{p}) := \sum_{i\in N} \frac{|p_i^\star - p_i|}{b_i + c_i}, \quad \kappa_\epsilon(\mathbf{p}) := \sum_{i\in N} \max\left\{\frac{|p_i^\star - p_i|}{b_i + c_i} - \epsilon, 0\right\}, \tag{11}$$

where ϵ can take any positive value. By definition, $\kappa(\mathbf{p})$ is a weighted ℓ^1-distance function that directly measures the proximity of $\mathbf{p}$ to $\mathbf{p}^\star$, and $\kappa_\epsilon(\mathbf{p})$ is the soft version of $\kappa(\mathbf{p})$ that allows some tolerance $\epsilon > 0$ for each product $i \in N$, which satisfies that $\kappa(\mathbf{p}) \le \kappa_\epsilon(\mathbf{p}) + n\epsilon$ for all $\mathbf{p} \in \mathcal{P}^n$. Due to the choice of diminishing step-sizes, we demonstrate that the reference price path always approaches the price path in the long run. Then, using the characteristic property of the MNL demand, we show that the gradient ascent step of the price moves toward the SNE in terms of $\kappa_\epsilon(\cdot)$ for sufficiently large t, i.e., when the reference price and price are reasonably close. This helps us establish a recursive relation between $\kappa_\epsilon(\mathbf{p}^{t+1})$ and $\kappa_\epsilon(\mathbf{p}^t)$, which further guarantees that $\lim_{t\to\infty} \kappa(\mathbf{p}^t) \le \lim_{t\to\infty} \kappa_\epsilon(\mathbf{p}^t) + n\epsilon = \mathcal{O}(\epsilon)$. Since ϵ can be arbitrarily close to zero, we conclude the global convergence of the price path to the SNE. Finally, as the reference price results from the exponential smoothing of historical prices, its convergence follows from that of the price path.

We note that the only prerequisite for the convergence is the diminishing step-sizes satisfying $\lim_{t\to\infty} \eta^t = 0$ and $\sum_{t=0}^\infty \eta^t = \infty$. This condition is widely seen in the online game literature (see, e.g., Mertikopoulos and Zhou (2019), Ba et al. (2021)). Since the firms are likely to become more acquainted with their competitors through repeated interactions, it is reasonable to assume that they would be more conservative in price adjustments over time, leading to a gradual reduction in learning rates. Though a uniform sequence of step-sizes is used here, we generalize Theorem 1 to accommodate firm-differentiated step-sizes in Section 7.1.

While Theorem 1 shows the asymptotic convergence of the `OPGA` under general diminishing step-sizes, it does not tell the relation between the convergence rate and the choice of step-sizes. In the next theorem, we demonstrate that under specified step-sizes, the `OPGA` achieves a non-asymptotic convergence rate of $\widetilde{\mathcal{O}}(1/t^2)$ in terms of squared distance, where the symbol $\widetilde{\mathcal{O}}(\cdot)$ hides the logarithm terms. Note that the notation $\|\cdot\|$ specifically refers to ℓ^2-norm.

THEOREM 2 **(Convergence Rate of `OPGA`)**. *In the loss-neutral scenario, suppose all firms adopt Algorithm 1 with step-sizes $\eta^t = \frac{C_\eta \log(t+1)}{t+1}$ for $t \ge 2$. Then, there exist constants T_1 and $\widehat{C}_\kappa$ such that when $C_\eta > 2\overline{p}^2/\log 2$, it holds for all $t > \max\{2T_1, 10\}$ that*

$$\kappa(\mathbf{p}^t) = \sum_{i\in N} \frac{|p_i^\star - p_i^t|}{b_i + c_i} \le \widehat{C}_\kappa \frac{\log t}{t} = \widetilde{\mathcal{O}}\left(\frac{1}{t}\right). \tag{12}$$

Furthermore, in terms of the squared distance, there exist constants C_p and C_r such that for all $t > \max\{2T_1, 10\}$, it holds that

$$\left\|\mathbf{p}^\star - \mathbf{p}^t\right\|^2 \le C_p \left(\frac{\log t}{t}\right)^2 = \widetilde{\mathcal{O}}\left(\frac{1}{t^2}\right), \quad \left\|\mathbf{p}^\star - \mathbf{r}^t\right\|^2 \le C_r \left(\frac{\log t}{t}\right)^2 = \widetilde{\mathcal{O}}\left(\frac{1}{t^2}\right). \tag{13}$$

The constants T_1, $\widehat{C}_\kappa$, C_p, and C_r are explicitly defined in Table EC.1.

The proof of Theorem 2 refines that of Theorem 1 by optimizing the choice of step-sizes. In particular, we separate the learning process into two stages, where the first stage accounts for the convergence of reference price to price, and the second stage refers to the latter periods when the recursion on $\{\kappa_\epsilon(\mathbf{p}^t)\}_{t\geq 0}$ is applicable. Utilizing an inductive argument, we first show in Lemma EC.2 that the difference $\|\mathbf{p}^t - \mathbf{r}^t\|$ decreases at a similar rate as the step-sizes. Then, during the second stage where the reference price and price are close, we unroll the recursion and establish a convergence rate of $\widetilde{\mathcal{O}}(1/t)$ for $\{\kappa(\mathbf{p}^t)\}_{t\geq 0}$, which further implies $\|\mathbf{p}^\star - \mathbf{p}^t\|^2 = \widetilde{\mathcal{O}}(1/t^2)$ since $\kappa(\cdot)$ is a weighted ℓ^1-norm. Finally, the convergence rate of the reference price path can be determined through a triangular inequality, i.e., $\|\mathbf{p}^\star - \mathbf{r}^t\|^2 \leq 2\|\mathbf{p}^\star - \mathbf{p}^t\|^2 + 2\|\mathbf{p}^t - \mathbf{r}^t\|^2$. The full proof of Theorem 2 is relegated to Appendix E.

For the convergence rate in Eq. (13), we remark that the requirement $t > \max\{2T_1, 10\}$ can be further relaxed by taking larger constants C_p and C_r, since both $\|\mathbf{p}^\star - \mathbf{p}^t\|^2$ and $\|\mathbf{p}^\star - \mathbf{r}^t\|^2$ are upper-bounded by $n(\overline{p} - \underline{p})^2$. Additionally, the omission of step-sizes condition $C_\eta > 2\overline{p}^2/\log 2$ would not change the order of the convergence rate but only impact the constants C_p and C_r.

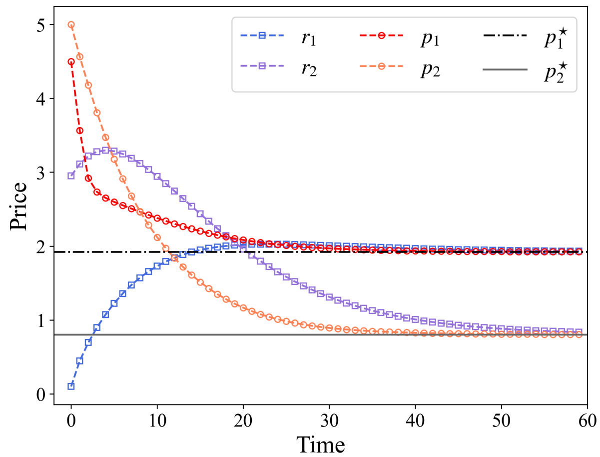
Theorem 2 demonstrates a faster convergence rate compared to the rate of $\mathcal{O}(1/t)$ in Golrezaei et al. (2020) for their duopoly competition with reference effects under linear demand. Besides, despite the absence of many desirable properties such as concavity and variational stability, our $\widetilde{\mathcal{O}}(1/t^2)$ rate is better than the results for standard games with such properties, e.g., Mertikopoulos and Zhou (2019), Lin et al. (2020).

Remark 1 (Constant step-sizes). The convergence analysis presented in this work can be readily extended to accommodate constant step-sizes, i.e., $\eta^t \equiv \eta$ for some $\eta > 0$. Specifically, we can show that the price and reference price converge to a $\mathcal{O}(\eta)$-neighborhood of the SNE, which is slightly weaker than the precise convergence under diminishing step-sizes. The reason is that, due to the reference price update, constant step-sizes would allow oscillations of the price path around the SNE, a phenomenon that has already been illustrated in Figure 2b. The magnitude of such oscillations is bounded by the size of η.

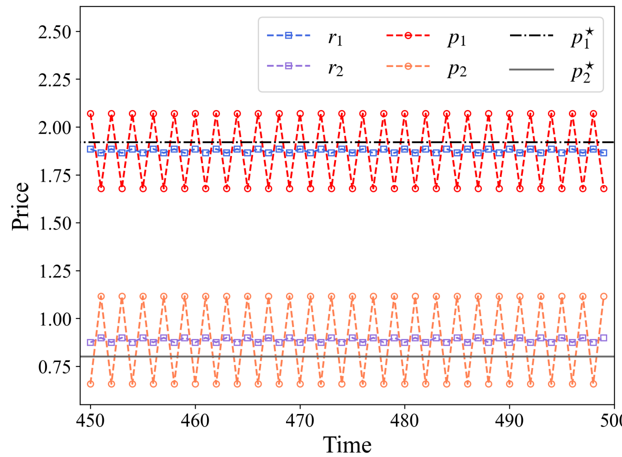
We conduct numerical experiments to illustrate the convergence behavior of the `OPGA` algorithm. Without loss of generality, we consider the two-product case with $N = \{1, 2\}$. Figures 2a and 2b present a pair of examples that only differ in the step-sizes. Their long-run behaviors highlight the crucial role of $\{\eta^t\}_{t\geq 0}$ in attaining convergence. In particular, Figures 2a verifies Theorem 1 by demonstrating that the price and reference price paths converge to the unique SNE when the chosen diminishing step-sizes fulfill the criteria specified by Theorem 1. By contrast, the over-large constant step-sizes in Figure 2b impede convergence and lead to cyclic patterns in the long run.

Figure 2 **Price and Reference Price Paths Under** `OPGA`

(Parameters: $(a_1, b_1, c_1) = (8.70, 2.00, 0.82)$, $(a_2, b_2, c_2) = (4.30, 1.20, 0.32)$, $(r_1^0, r_2^0) = (0.10, 2.95)$, $(p_1^0, p_2^0) = (4.50, 5.00)$, and $\alpha = 0.90$.)



(a) Convergence with $\eta^t = \frac{\log(t+1)}{t+1}$.



(b) Cyclic pattern with $\eta^t = 1$.

In the remaining part of this section, we derive the regret of Algorithm 1. The standard (static) regret compares an online algorithm with the best fixed decision in hindsight. The rationale behind the static metric is that this best fixed decision performs adequately well across all iterations. Nevertheless, given the evolving underlying state (i.e., reference price) in our context, such an assumption becomes overly optimistic. To address this limitation, we choose the *dynamic regret* as the performance measure, defined as the difference between cumulative revenue attained by an online algorithm and a sequence of best decisions in hindsight. The dynamic regret is strictly stronger than the static regret in a non-stationary environment, as the latter only benchmarks against the single best fixed action over all rounds. Let $\text{D-Regret}_i(T)$ denote the dynamic regret for firm i over T periods, which is defined as

$$\text{D-Regret}_i(T) := \sum_{t=1}^{T} \left[\max_{p_i \in \mathcal{P}} \left\{ \Pi_i\big((p_i, \mathbf{p}_{-i}^t), \mathbf{r}^t\big) \right\} - \Pi_i(\mathbf{p}^t, \mathbf{r}^t) \right]. \tag{14}$$

Theorem 3 **(Dynamic Regret Bound)**. *In the loss-neutral scenario, if all firms adopt Algorithm 1 with step-sizes $\{\eta^t\}_{t\geq 0}$ satisfying $\lim_{t\to\infty} \eta^t = 0$ and $\sum_{t=0}^{\infty} \eta^t = \infty$, the dynamic regret of each firm grows in a sublinear rate, i.e.,*

$$\lim_{T\to\infty} \frac{1}{T} \times \text{D-Regret}_i(T) = 0, \quad \forall i \in N. \tag{15}$$

Furthermore, if the step-sizes are specified as $\eta^t = \frac{C_\eta \log(t+1)}{t+1}$ for $t \geq 2$, there exist constants T_1 and $C_{R,i}$ such that when $C_\eta > 2\overline{p}^2/\log 2$, it holds that

$$\text{D-Regret}_i(T) \leq \overline{p} \cdot \max\{2T_1, 10\} + 2C_{R,i} = \mathcal{O}(1), \quad \forall T \geq 1, \ \forall i \in N, \tag{16}$$

where constants T_1 and $C_{R,i}$ are explicitly defined in Table EC.1.

Theorem 3 shows that the `OPGA` algorithm yields a sublinear dynamic regret for any non-increasing step-sizes satisfying $\lim_{t\to\infty}\eta^t = 0$ and $\sum_{t=0}^{\infty}\eta^t = \infty$, the same condition as described in Theorem 1. Moreover, it achieves a constant regret when the step-sizes are selected as $\eta^t = \frac{C_\eta \log(t+1)}{t+1}$ for all $t \geq 2$, which is in accordance with the specifications in Theorem 2. In proving Theorem 3, we let $p_i^{B,t} := \arg\max_{p_i \in \mathcal{P}} \left\{\Pi_i\big((p_i, \mathbf{p}_{-i}^t), \mathbf{r}^t\big)\right\}$. Then, using the smoothness of the revenue function, we can upper-bound the regret at period t through

$$\max_{p_i\in\mathcal{P}} \left\{\Pi_i\big((p_i,\mathbf{p}_{-i}^t),\mathbf{r}^t\big)\right\} - \Pi_i(\mathbf{p}^t,\mathbf{r}^t) \leq \frac{\partial \Pi_i\big((p_i^{B,t},\mathbf{p}_{-i}^t),\mathbf{r}^t\big)}{\partial p_i}\cdot\big(p_i^t - p_i^{B,t}\big) + \frac{h_i}{2}(p_i^{B,t}-p_i^t)^2 \leq \frac{h_i}{2}(p_i^{B,t}-p_i^t)^2,$$

where h_i denotes the smoothness parameter and the second inequality is due to the single-period optimality of $p_i^{B,t}$. We further upper-bound the quadratic term $(p_i^{B,t} - p_i^t)^2$ by the squared distance $\|\mathbf{p}^t - \mathbf{p}^\star\|^2$ and $\|\mathbf{r}^t - \mathbf{p}^\star\|^2$, which allows us to apply the results from Theorems 1 and 2. This also confirms that the convergence to SNE is stronger than purely being no-regret. The complete proof of Theorem 3 can be found in Appendix F.

6. Loss-averse: Conservative Online Projected Gradient Ascent

As demonstrated in Proposition 1, the presence of any gain-seeking reference effect precludes the existence of SNE, making it impossible to obtain the equilibrium and market stability at the same time (some numerical experiments are provided in Appendix K). Therefore, this section concentrates on the loss-averse scenario, where consumers exhibit loss-aversion toward at least one product, i.e., $c_i^- \geq c_i^+$ for all $i \in N$, with at least one inequality being strict.

From Proposition 2, the set of SNEs under loss-averse reference effects is not confined to a single point but forms a continuum that can be non-convex. Thus, while it is natural to assess the convergence using the distance between the current price and the unique SNE point $\mathbf{p}^\star$ in the loss-neutral case, such distance-based metrics are no longer well-defined in the loss-averse scenario due to the non-convexity of the set $\mathcal{S}$. Inspired by the criteria for stationarity found in the non-convex non-smooth optimization literature (see, e.g., Li et al. (2020)), we propose a novel metric to evaluate the convergence under loss-averse reference effects, taking both price and reference price into account. For any given pair of $(\mathbf{p},\mathbf{r})$, we define the metric $\widetilde{\kappa}(\mathbf{p},\mathbf{r})$ as follows

$$\widetilde{\kappa}(\mathbf{p},\mathbf{r}) := \|\mathbf{p}-\mathbf{r}\| + \sum_{i\in N} \mathtt{dist}\Big(0, \mathtt{Hull}\big\{D_i^-(\mathbf{p},\mathbf{r}), D_i^+(\mathbf{p},\mathbf{r})\big\}\Big), \tag{17}$$

where $\mathtt{dist}(\cdot,\cdot)$ denotes the Euclidean distance function, $\mathtt{Hull}\{\cdot,\cdot\}$ refers to the convex hull of the input points, and the functions $D_i^-(\cdot,\cdot)$ and $D_i^+(\cdot,\cdot)$ are defined as

$$D_i^-(\mathbf{p},\mathbf{r}) := \frac{1}{p_i} + (b_i + c_i^-)\cdot d_i(\mathbf{p},\mathbf{r}) - (b_i + c_i^-), \tag{18a}$$

$$D_i^+(\mathbf{p},\mathbf{r}) := \frac{1}{p_i} + (b_i + c_i^+)\cdot d_i(\mathbf{p},\mathbf{r}) - (b_i + c_i^+). \tag{18b}$$

In the loss-averse scenario, given that $c_i^- \geq c_i^+$, the term $\texttt{dist}\big(0, \texttt{Hull}\{D_i^-(\mathbf{p},\mathbf{r}), D_i^+(\mathbf{p},\mathbf{r})\}\big)$ actually measures the distance between 0 and the interval $\big[D_i^-(\mathbf{p},\mathbf{r}), D_i^+(\mathbf{p},\mathbf{r})\big]$. The metric $\widetilde{\kappa}(\mathbf{p},\mathbf{r})$ consists of two components, where the first part quantifies the distance between the price and the reference price, and the second part evaluates the stationarity of the current price. For any specified pair of $(\mathbf{p},\mathbf{r})$ with $p_i \neq r_i$, it is evident from Eq. (18) that exactly one of $D_i^+(\mathbf{p},\mathbf{r})$ and $D_i^-(\mathbf{p},\mathbf{r})$ is equal to the derivative of log-revenue for firm i. Meanwhile, the other quantity can be regarded as the *virtual derivative* assuming the opposite reference price sensitivity (with $d_i(\mathbf{p},\mathbf{r})$ still being the true market share). For instance, when $p_i < r_i$, the effective reference price sensitivity should be c_i^+, and thereby $D_i^+(\mathbf{p},\mathbf{r}) = \partial \log\big(\Pi_i(\mathbf{p},\mathbf{r})\big)/\partial p_i$ is the true derivative. The other function $D_i^-(\mathbf{p},\mathbf{r})$ is the virtual derivative and can be obtained by differentiating $\log\big(\Pi_i(\mathbf{p},\mathbf{r})\big)$ as if the effective reference price sensitivity is c_i^-. We remark that when $p_i = r_i$, the terms $D_i^+(\mathbf{p},\mathbf{r})$ and $D_i^-(\mathbf{p},\mathbf{r})$ correspond to the left-hand and right-hand derivatives of its log-revenue, respectively.

In the next proposition, we show that $\widetilde{\kappa}(\mathbf{p},\mathbf{r})$ is a well-defined metric for determining the convergence to SNEs. The proof of this proposition can be found in Appendix C.3.

Proposition 3. *The set of SNEs can be equivalently expressed as $\mathcal{S} = \{\mathbf{p} \in \mathcal{P}^n \mid \widetilde{\kappa}(\mathbf{p},\mathbf{p}) = 0\}$, where $\widetilde{\kappa}(\cdot,\cdot)$ is the metric defined in Eq.* (17).

Thus, given the compactness of $\mathcal{P}^n$ and the continuity of $\widetilde{\kappa}(\cdot,\cdot)$, Proposition 3 ensures that a price vector $\mathbf{p}$ is sufficiently close to $\mathcal{S}$ if and only if $\widetilde{\kappa}(\mathbf{p},\mathbf{p})$ is being small enough. Motivated by this implication, we introduce the concept of ϵ-approximated SNE, abbreviated as ϵ-SNE.

Definition 3 (ϵ-SNE). For any $\epsilon > 0$, a price vector $\mathbf{p}$ is an ϵ-SNE if $\widetilde{\kappa}(\mathbf{p},\mathbf{p}) < \epsilon$.

Below, we focus on designing an algorithm to find an ϵ-SNE for any given $\epsilon > 0$. According to Proposition 3, when ϵ is sufficiently small, the identified point is guaranteed to be a good approximation for SNEs. To deal with the non-smoothness induced by the loss-averse reference effects, we propose a modified algorithm called Conservative Online Projected Gradient Ascent (`C-OPGA`), as detailed in Algorithm 2. The name of this algorithm reflects its more conservative approach compared to the original `OPGA`, where the `C-OPGA` incorporates pausing criteria (see Line 5 in Algorithm 2) to refine the process of price updates.

Next, we walk through the details of the `C-OPGA` algorithm. During each period, the firm computes both true and virtual derivatives of its log-revenue (see Eq. (19)). We note that even though firm i may not discern whether $D_i^{t,+}$ or $D_i^{t,-}$ is the true derivative in the partial information environment, it can still compute both quantities through the feedback $d_i(\mathbf{p}^t,\mathbf{r}^t)$ with the knowledge of reference price sensitivities. These derivatives are then input into the pausing criteria in Line 5, where the mechanism can be understood as follows: When the term $\texttt{dist}\big(0, \texttt{Hull}\{D_i^{t,-}, D_i^{t,+}\}\big)$ falls below a

Algorithm 2: Conservative Online Projected Gradient Ascent (`C-OPGA`)

1 **Input:** Initial reference price $\mathbf{r}^0$, initial price $\mathbf{p}^0$, step-sizes $\{\eta^t\}_{t\geq 0}$, and threshold ϵ.

2 **for** $t=0,1,2,\dots$ **do**

3 **for** $i \in N$ **do**

4 Compute the true and virtual derivatives from price and market share at period t:

$$D_i^{t,+} \leftarrow D_i^{+}(\mathbf{p}^t,\mathbf{r}^t) = \frac{1}{p_i^t} + (b_i + c_i^{+}) \cdot d_i(\mathbf{p}^t,\mathbf{r}^t) - (b_i + c_i^{+}), \tag{19a}$$

$$D_i^{t,-} \leftarrow D_i^{-}(\mathbf{p}^t,\mathbf{r}^t) = \frac{1}{p_i^t} + (b_i + c_i^{-}) \cdot d_i(\mathbf{p}^t,\mathbf{r}^t) - (b_i + c_i^{-}). \tag{19b}$$

5 **if** $D_i^{t,+} > -\epsilon$ *and* $D_i^{t,-} < \epsilon$ **then** `// Pausing criteria.`

6 Maintain posted price: $p_i^{t+1} \leftarrow p_i^t$.

7 **else**

8 Construct the update direction as an arbitrary convex combination of the true and virtual derivatives at period t:

$$D_i^t \leftarrow w_i^t D_i^{t,+} + (1-w_i^t) D_i^{t,-}, \tag{20}$$

where the coefficient w_i^t takes any value within the interval $[0,1]$.

9 Update posted price: $p_i^{t+1} \leftarrow \mathtt{Proj}_{\mathcal{P}}\,(p_i^t + \eta^t D_i^t)$.

10 **end**

11 **end**

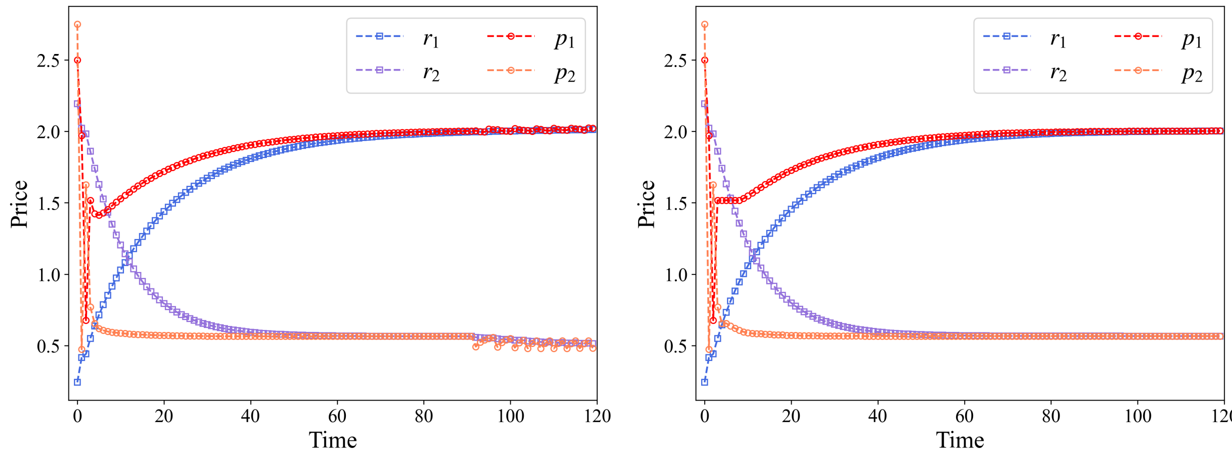
12 Update reference price: $\mathbf{r}^{t+1} \leftarrow \alpha \mathbf{r}^t + (1-\alpha)\mathbf{p}^t$.

13 **end**

predetermined threshold ϵ, it triggers a pausing mechanism that stops the price update for the current round (see Lines 5–6). Otherwise, when the pausing criteria are not met, the firm adjusts its price according to the convex combination of the true and virtual derivatives (see Lines 7–9). Finally, the algorithm concludes with the market performing the reference price updates.

There are several benefits of introducing the pausing mechanism in the `C-OPGA` algorithm. First and foremost, in the case of non-smooth objective functions, the standard vanilla sub-gradient ascent algorithm, which updates the price using the true sub-derivative without any pausing criteria, is not an ascent method even under sufficiently small step-sizes (Boyd et al. 2003). In addition, acquiring this true sub-derivative mandates firms to collect additional information about their reference prices to determine the effective sensitivity between c_i^+ and c_i^-. We further illustrate the advantage of the pausing mechanism by the following two examples in Figure 3, which compares the vanilla sub-gradient ascent to the `C-OPGA` algorithm under the same set of parameters and

Figure 3 **Comparison of Trajectories Between Vanilla Sub-gradient Ascent and C-OPGA**
(Parameters: $(a_1, b_1, c_1^+, c_1^-) = (8.19, 1.48, 0.34, 1.50)$, $(a_2, b_2, c_2^+, c_2^-) = (4.59, 1.80, 0.31, 1.14)$, $(r_1^0, r_2^0) = (0.24, 2.19)$, $(p_1^0, p_2^0) = (2.50, 2.75)$, $\alpha = 0.90$, and $\eta^t = 1/\sqrt{t+1}$.)



(a) Cyclic pattern with vanilla sub-gradient ascent. (b) Convergence with C-OPGA.

initializations. The price and reference price paths generated by the vanilla sub-gradient ascent (see Figure 3a) initially display signs of convergence, but start to oscillate in a cyclic pattern around the 90th period. This instability is typically observed when the price and the reference price are in close proximity. Under such conditions, the sign of the difference $(r_i^t - p_i^t)$ may alternate in consecutive periods, thereby inducing a jump discontinuity in the derivative due to the discrepancy between c_i^+ and c_i^-. By contrast, in Figure 3b, the convergence of price and reference price paths under the C-OPGA highlights the effectiveness of its pausing criteria. Indeed, it is easy to verify that the paths generated by the C-OPGA converge to an interior point of the SNE set.

Remark 2. In Algorithm 2, while all firms are assumed to use a uniform threshold ϵ, such an assumption is made only for the brevity of presentation. It is worth mentioning that the convergence results in Section 6.1 extend directly to cases with product-specific and time-varying thresholds.

6.1. Convergence Results for C-OPGA

Compared to the loss-neutral scenario, the loss-averse reference effects introduce additional challenges in convergence analysis due to the non-smoothness of the revenue function. Despite these complexities, we successfully develop an original method that achieves global last-iterate convergence of the C-OPGA, as presented in the following theorem.

Theorem 4 **(Global Convergence of C-OPGA)**. *In the loss-averse scenario, let the step-sizes* $\{\eta^t\}_{t\geq 0}$ *be a non-increasing sequence such that* $\lim_{t\to\infty} \eta^t = 0$ *and* $\sum_{t=0}^{\infty} \eta^t = \infty$. *Then, for any* *reasonably small $\epsilon > 0$, the price and reference price paths generated by Algorithm 2 with the step-sizes $\{\eta^t\}_{t\geq 0}$ and threshold ϵ converge to an $\tilde{C}_\kappa\epsilon$-SNE, where constant $\tilde{C}_\kappa$ is explicitly defined in Table EC.2.*

We summarize the core idea behind the proof of Theorem 4 below, with full details relegated to Appendix G. Note that the diminishing step-sizes ensure the reference price path converges to the price path in the long run. Hence, the bottleneck of the convergence lies in the second portion of $\widetilde{\kappa}(\mathbf{p},\mathbf{r})$ defined in Eq. (17), i.e., whether $\mathbf{p}$ in the current iterate is close to the stationarity. This motivates us to study the following surrogate metric based only on the price

$$\widetilde{\kappa}(\mathbf{p}) = \sum_{i\in N} \mathtt{dist}\Big(0, \mathtt{Hull}\big\{G_i^-(\mathbf{p},\mathbf{p}), G_i^+(\mathbf{p},\mathbf{p})\big\}\Big), \tag{21}$$

where $G_i^-(\mathbf{p},\mathbf{r}) := D_i^-(\mathbf{p},\mathbf{r})/(b_i + c_i^-)$ and $G_i^+(\mathbf{p},\mathbf{r}) := D_i^+(\mathbf{p},\mathbf{r})/(b_i + c_i^+)$ are the scaled true/virtual derivatives. We demonstrate that there exists some period $T_\epsilon > 0$ such that the sequence $\{\widetilde{\kappa}(\mathbf{p}^t)\}_{t\geq T_\epsilon}$ is non-increasing. Moreover, for any $t \geq T_\epsilon$, if the pausing criteria are not triggered for all products, there must exist a strict decrement from $\widetilde{\kappa}(\mathbf{p}^t)$ to $\widetilde{\kappa}(\mathbf{p}^{t+1})$. Otherwise, when all firms pause their price update, we show that $\mathbf{p}^t$ is already close to the set of SNEs, i.e., $\widetilde{\kappa}(\mathbf{p}^t,\mathbf{r}^t) = \mathcal{O}(\epsilon)$. Together, we conclude that Algorithm 2 converges globally to an approximate SNE.

Several insights can be drawn from Theorem 4. Firstly, since $\tilde{C}_\kappa$ is merely a constant, Theorem 4 implies that the `C-OPGA` algorithm can converge to any desired level of accuracy by selecting a sufficiently small threshold ϵ. Recall that firms can use time-variant thresholds ϵ for their pausing criteria (see Remark 2). Therefore, a pragmatic strategy for firms to achieve a highly accurate approximation is to gradually decrease their thresholds along the competition. Secondly, we note that the "reasonably small" condition for ϵ is explicitly specified in Eq. (G.25). Lastly, though the `C-OPGA` adopts a common sequence of step-sizes for all firms, the convergence result remains valid even when firms have different step-sizes. This generalization is formalized in Section 7.1.

In the next theorem, we optimize the step-sizes to obtain the rate of convergence for the `C-OPGA`.

Theorem 5 **(Convergence Rate of `C-OPGA`)**. *In the loss-averse scenario, suppose all firms adopt Algorithm 2 with step-sizes $\eta^t = \frac{C_\eta}{\sqrt{t+1}}$ and a reasonably small threshold ϵ, where C_η is some general constant. Then, there exists $\tilde{T} = \mathcal{O}(1/\epsilon^2)$ such that*

$$\widetilde{\kappa}(\mathbf{p}^t,\mathbf{r}^t) \leq \left(\frac{1}{2\max_{i\in N}\big\{(b_i + c_i^-)\tilde{\ell}_{r,i}\big\}} + \sum_{i\in N}\frac{2\max_{k\in N}\big\{b_k + c_k^-\big\}}{b_i + c_i^+}\right)\epsilon, \quad \forall t \geq \tilde{T}, \tag{22}$$

where $\widetilde{\kappa}(\cdot)$ is defined in Eq. (17), and constants $\tilde{T}$ and $\tilde{\ell}_{r,i}$ are explicitly defined in Table EC.2.

The main idea of the proof for Theorem 5 is outlined as follows. We first consider general sublinear step-sizes taking the form of $\eta^t = C_\eta(t+1)^{-\beta}$ with $\beta \in (0,1]$. Under these step-sizes, we can explicitly determine the period T_ϵ, after which the sequence $\{\widetilde{\kappa}(\mathbf{p}^t)\}_{t\geq T_\epsilon}$ becomes non-increasing, along with the amount of decrement from $\widetilde{\kappa}(\mathbf{p}^t)$ to $\widetilde{\kappa}(\mathbf{p}^{t+1})$ for any $t \geq T_\epsilon$. Finally, by balancing between the size of T_ϵ and the decreasing speed of $\{\widetilde{\kappa}(\mathbf{p}^t)\}_{t\geq T_\epsilon}$, we conclude that $\beta = 1/2$ yields the best convergence rate. The formal proof is deferred to Appendix H.

Theorem 5 demonstrates that an $\mathcal{O}(\epsilon)$-SNE can be found in $\mathcal{O}(1/\epsilon^2)$ iterations, equivalently suggesting a convergence rate of $\mathcal{O}(1/\sqrt{t})$ for `C-OPGA` in terms of metric $\widetilde{\kappa}(\mathbf{p},\mathbf{r})$. This is slower than the rate established in Theorem 2 for `OPGA` in the loss-neutral scenario. We speculate the reasons to be two-folded. First, since loss-averse reference effects result in non-smooth revenue functions, it is typical for non-smooth problems to exhibit a slower rate. Second, in the loss-averse case where the SNEs are no longer unique, the sub-gradients in different iterations may point toward different SNEs in $\mathcal{S}$, leading to potential oscillations in the price and reference price trajectories (an example of severe oscillations under the small step-sizes is illustrated in Figure 3a).

Remark 3. Given the convergence rate in Theorem 5, we can similarly show its dynamic regret as Theorem 3. For a fixed and reasonably large value of T, Theorem 5 implies a cumulative regret of $\mathcal{O}(1/\epsilon^2 + \epsilon T)$, where the first part corresponds to the regret before period $\tilde{T} = \mathcal{O}(1/\epsilon^2)$, and the second part comes from the regret after period $\tilde{T}$, i.e., when `C-OPGA` already converges to an $\mathcal{O}(\epsilon)$-SNE. Therefore, the optimal choice is $\epsilon = T^{-1/3}$, yielding the cumulative regret of $\mathcal{O}(T^{2/3})$ over T periods.

7. Extensions

In this section, we examine two extensions of our algorithms that can enhance their applicability in realistic environments. The rigorous proofs for this section can be found in Appendices I and J.

7.1. Firm-differentiated Step-sizes

Previously, we assumed that firms adopt a common sequence of step-sizes $\{\eta^t\}_{t\geq 0}$, which implies the need for some initial communication among the firms. Below, we show that our theoretical findings for algorithms `OPGA` and `C-OPGA` can be extended to the case where each firm i uses its own private step-sizes $\{\eta_i^t\}_{t\geq 0}$. This gives firms the flexibility to implement the algorithm without sharing the choice of step-sizes with their competitors, thereby preserving the privacy of such information.

Theorem 6 **(Convergence with Firm-differentiated Step-sizes)**. *Suppose that each firm $i \in N$ takes its own non-increasing step-sizes $\{\eta_i^t\}_{t\geq 0}$ such that $\lim_{t\to\infty} \eta_i^t = 0$ and $\sum_{t=0}^{\infty} \eta_i^t = \infty$. Then, it follows that:*

- *In the loss-neutral scenario, the price and reference price paths generated by Algorithm 1 converge to the unique SNE, where the convergence rate is determined by the slowest decay rate among the step-size sequences.*

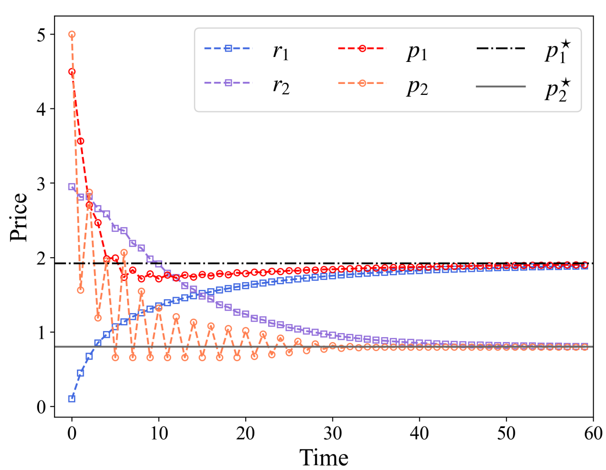
- *In the loss-averse scenario, the price and reference price paths generated by Algorithm 2 with threshold ϵ converge to an $\mathcal{O}(\epsilon)$-SNE, where the convergence rate is determined by both the slowest and fastest decay rates among the step-size sequences.*

While Theorem 6 addresses the convergence for general scenarios of firm-differentiated step-sizes, we use the example of sublinear step-sizes to illustrate how the convergence rate depends on the slowest and fastest decay rates among the step-size sequences. Consider the step-sizes $\eta_i^t = C_{\eta,i}(t+1)^{-\beta_i}$ with $0 < \beta_1 \le \beta_2 \cdots \le \beta_n < 1$, i.e., firm 1 takes the step-sizes with the slowest decay rate and firm n takes the step-sizes with the fastest decay rate. Then, in the loss-neutral scenario, the convergence rate of `OPGA` has the order $\mathcal{O}\big(t^{-\beta_1}\big)$. In the loss-averse scenario, the total number of iterations required to achieve an ϵ-SNE has the order $\mathcal{O}\big(\epsilon^{-1/\beta_1} + \epsilon^{-1/(1-\beta_n)}\big)$. Additionally, we also comment on the special case where the step-sizes of firms only differ by a constant multiplier, i.e., there exist $\{\eta^t\}_{t\ge 0}$ and $(C_{\eta,i})_{i\in N}$ such that $\eta_i^t = \eta^t C_{\eta,i}$ for every $t \ge 0$ and $i \in N$. In this case, the orders of the convergence rates remain the same as those stated in Theorems 2 and 5 for the corresponding loss-neutral and loss-averse scenarios.

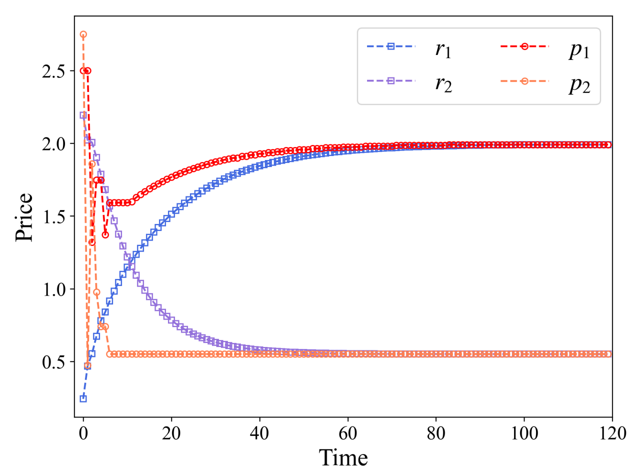
To complement the theorem, we draw the price and reference price paths under firm-differentiated step-sizes. Figure 4a showcases the paths generated by the `OPGA` algorithm in the loss-neutral scenario, which represents the firm-differentiated counterpart of Figure 2a. Instead of the uniform step-sizes seen in Figure 2a, two firms adopt $\eta_1^t = \frac{\log(t+1)}{t+1}$ and $\eta_2^t = \frac{5}{\sqrt{t+1}}$, respectively. Likewise, Figure 4b corresponds to the loss-averse scenario, which follows the same setup as the example in Figure 3b but with firm 2 having step-sizes $\eta_2^t = \frac{\log(t+1)}{t+1}$. Together, these two plots validate the convergence of both price and reference price to the SNE(s) under firm-differentiated step-sizes.

Figure 4 **Convergence Under Firm-differentiated Step-sizes**

(Sub-figure (a) shares parameters with Figure 2; sub-figure (b) shares parameters with Figure 3.)



(a) Loss-neutrality with $\eta_1^t = \frac{\log(t+1)}{t+1}$ and $\eta_2^t = \frac{5}{\sqrt{t+1}}$.



(b) Loss-aversion with $\eta_1^t = \frac{1}{\sqrt{t+1}}$ and $\eta_2^t = \frac{\log(t+1)}{t+1}$.

7.2. Inexact First-order Oracle

In the preceding sections, we assume that firms can access the exact first-order oracle. Now, we consider a more practical setting where computing this oracle may require the knowledge of the sensitivity parameters and overall market size. While it is feasible to obtain (b_i, c_i^+, c_i^-) through the perturbation approach (see Appendix A) and approximate market size using historical data, such estimations would bring additional errors to the first-order derivative, rendering the previous convergence results not directly applicable. Indeed, if the errors are disruptive enough, they could impede the convergence of our algorithms. However, if the errors are uniformly bounded by some small threshold δ, the following theorem demonstrates the convergence of both price and reference price paths to a neighborhood of the SNE(s).

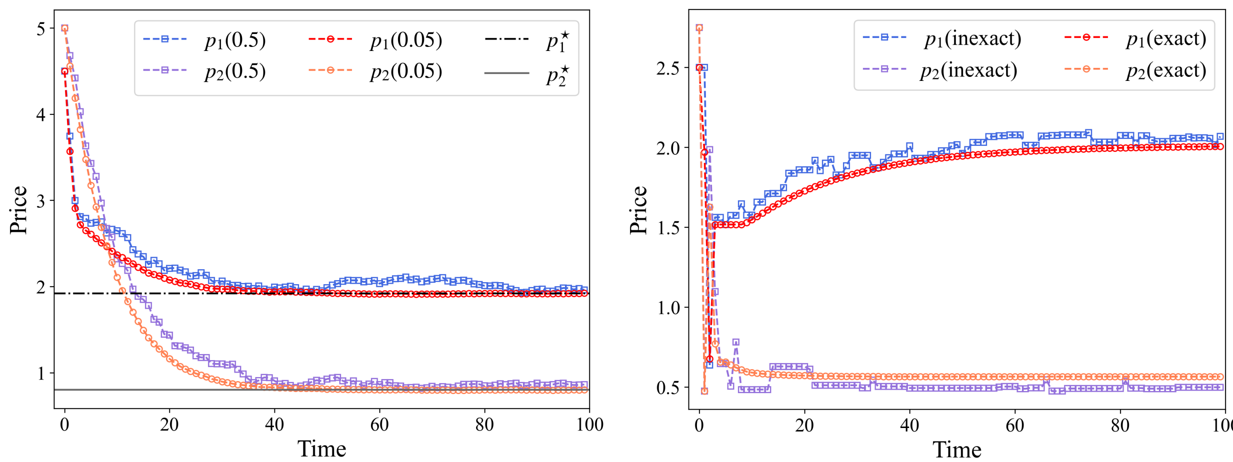
THEOREM 7 **(Convergence with Inexact First-order Oracle)**. *Suppose that the firms can only access an inexact first-order oracle such that the errors are uniformly bounded by some $\delta > 0$. Let the step-sizes $\{\eta^t\}_{t\geq 0}$ be a non-increasing sequence such that $\lim_{t\to\infty} \eta^t = 0$ and $\sum_{t=0}^{\infty} \eta^t = \infty$. Then, the price and reference price paths generated by Algorithm 1 (or Algorithm 2 with threshold ϵ) converge to an $\mathcal{O}(\delta)$-neighborhood of the unique SNE in the loss-neutral scenario (or an $\mathcal{O}(\delta+\epsilon)$-SNE in the loss-averse scenario), where the convergence rate has the same order as the setting of exact first-order oracle.*

We remark that the inexact first-order oracle studied in our work is different from the stochastic gradient, which generally assumes a zero-mean white noise with finite variance. In stochastic gradient case, it is possible to derive the convergence to a limiting point in expectation or with high probability. By contrast, in our case, the noise in the first-order derivative is a kind of approximation errors without distributional properties. For instance, assuming the loss-neutral scenario, if firm i over-estimates the value of $b_i + c_i$, then the derivative computed by firm i would be constantly smaller than the true derivative (see the expression in Eq. (10)). Thus, under the step-sizes specified in Theorem 7, we expect the price and reference price paths to approach the neighborhood of the SNE(s) but might continue to fluctuate around that area without admitting a limiting point.

Our experiments demonstrate the behaviors of Algorithms 1 and 2 under inexact first-order oracles, where we manually add noises to the derivatives used by the two firms. The noises are independently sampled from a normal distribution with a mean of 0.5 and a variance of 1, which are then truncated by certain threshold δ. Figure 5a focuses on the loss-neutral scenario, where we run `OPGA` twice on the same instance with truncation thresholds set at 0.5 and 0.05, respectively. Comparing the two cases, we find that the volatility of the paths is proportional to the magnitude of gradient noises. Moreover, we observe that the price paths under the inexact gradient oracle approach the SNE at a similar rate as those in the exact case depicted in Figure 2a. For the loss-averse scenario, Figure 5b shows the performances of `C-OPGA` with the inexact and exact oracles. The resulting patterns align with those in the loss-neutral scenario. Finally, it is important to note that since the noises have non-zero mean, the paths in both sub-figures do not converge to SNE in expectation; rather, they consistently remain either above or below the SNE.

Figure 5 **Convergence of `OPGA` and `C-OPGA` Under Inexact First-order Oracles**

(Sub-figure (a) considers the loss-neutral scenario with parameters and step-sizes same as Figure 2a; sub-figure (b) considers the loss-averse scenario with parameters and step-sizes same as Figure 3b.)



(a) `OPGA`: Two instances with gradient noises bounded by 0.5 and 0.05, respectively.

(b) `C-OPGA`: One instance under exact oracle and the other under inexact oracle with noises bounded by 0.4.

8. Conclusion

Despite the growing attention given to reference effects, this well-established consumer behavior remains relatively unexplored in competitive frameworks, particularly within partial information settings. Our paper bridges this gap by examining the oligopoly price competition with the first-order gradient feedback. The problem is structured as an online game with an underlying dynamic state—reference price. We analyze the gradient-based algorithms, i.e., `OPGA` and its variant `C-OPGA`, and provide theoretical guarantees for their global last-iterate convergence to the SNE(s), which indicates that firms can simultaneously achieve the equilibrium and market stability in the long run. With loss-neutral reference effects, we show that the `OPGA` algorithm has the convergence rate of $\widetilde{\mathcal{O}}(1/t^2)$ in terms of squared distance and attains a constant regret given proper step-sizes. For the loss-averse scenario, the price and reference price paths generated by the `C-OPGA` converge to the set of SNEs in the rate of $\mathcal{O}(1/\sqrt{t})$. We further demonstrate the robustness of our algorithms by showing that those findings can be extended to more practical scenarios where firms use different step-sizes and operate with inexact gradients.

This work sheds light on the long-run behavior of gradient-based algorithms in price competitions with reference effects, and it paves the way for several exciting future research directions. First, this paper unfolds within a deterministic strategy profile, i.e., considering only the pure strategy Nash equilibrium, which is a more common notion found in the literature and proves to be more practical and straightforward for firms to implement. However, it is also worthwhile to investigate mixed-strategy learning in online games. Another interesting direction is to consider different reference price models, such as the stimulus-based reference price. This perspective argues that the price judgment is formed at the point of purchase by using the current external information such as the price of other products. Finally, the reference effect as a strategic behavior can appear in other sequential decision-making problems such as repeated auctions (Han et al. 2020) and multi-period inventory control (Qin et al. 2022). Investigating how reference effects influence the optimal strategies in these applications presents an intriguing research topic.

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Online Supplement for "Last-iterate Convergence in No-regret Learning: Games with Reference Effects Under Logit Demand"

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The supplemental materials are structured as follows. In Appendix A, we present a detailed discussion on the partial information structure, specifically focusing on the feasibility of accessing the first-order oracle. In Appendix B, we elaborate on the reasons why the alternative methods mentioned in Section 5 do not apply to our problem, and some supporting proofs are deferred to Appendix M. In Appendix C, we provide the proofs for all propositions. Then, Appendices D, E, and F correspond to the proofs for Theorems 1, 2, and 3, respectively, all of which pertain to the `OPGA` algorithm in the loss-neutral scenario. For the loss-averse scenario, Appendices G and H are dedicated to the proofs for Theorems 4 and 5, respectively. Next, Appendices I and J show the convergence of our algorithms with firm-differentiated step-sizes and inexact gradients, corresponding to Theorems 6 and 7, respectively. In Appendix K, we provide numerical experiments to illustrate the performance of `OPGA` in gain-seeking scenarios. In Appendix L, we present all supporting lemmas used in proofs. Finally, in Appendix N, we provide a summary for all constants used in the paper.

Appendix A Discussion on Partial Information Structure

In Section 3.3, we have introduced the partial information setting considered in this paper, where each firm i can access a first-order oracle but does not necessarily know the information about its competitors. The oracle for firm i outputs the derivative of its log-transformed revenue function, i.e., $\partial \log\big(\Pi_i(\mathbf{p},\mathbf{r})\big)/\partial p_i = 1/p_i + (b_i + c_i)\,[d_i(\mathbf{p},\mathbf{r}) - 1]$ in the loss-neutral scenario. Below, we provide further discussions and discuss the feasibility of obtaining the first-order information.

Since each firm i naturally knows its realized revenue $\Pi_i(\mathbf{p}^t,\mathbf{r}^t)$ after period t, firm i can directly deduce its market share through $d_i(\mathbf{p}^t,\mathbf{r}^t) = \Pi_i(\mathbf{p}^t,\mathbf{r}^t)/p_i^t$. Therefore, if firm i precisely knows $b_i + c_i$ as prior knowledge, it can compute the derivative $\partial \log\big(\Pi_i(\mathbf{p}^t,\mathbf{r}^t)\big)/\partial p_i$ solely based on the market feedback, making our partial information setting equivalent to the *bandit feedback*. Even when the sensitivity parameters are not known prior, it is feasible for firm i to estimate $b_i + c_i$ through temporary collaboration with other firms. Below, we describe the estimation scheme.

First, it is legitimate to assume that each firm has insights into its reference price, inferred from the historical pricing data. Then, let firms engage in the following two-period cooperation:

Period 1: Let every firm set the price equal to its current reference price, i.e., $p_k = r_k,\ \forall k \in N$.

Period 2: Let firm i slightly perturb its price to $p_i = r_i + \delta$ with small δ, whereas other firms retain the previous prices, i.e., $p_j = r_j,\ \forall j \in N \backslash \{i\}$.

We remark that during the cooperation, firms do not need to disclose their prices to the competitors if they prefer to preserve confidentiality. Below, we provide a detailed explanation of why this two-step approach is sufficient for parameter estimation. First, the left-hand and right-hand derivatives of the market share with respect to price change can be computed as follows

$$\lim_{\delta \to 0} \frac{d_i\big((p_i + \delta, \mathbf{p}_{-i}), \mathbf{r}\big) - d_i(\mathbf{p}, \mathbf{r})}{\delta} = \begin{cases} -(b_i + c_i^-) \cdot d_i(\mathbf{p}, \mathbf{r})\big(1 - d_i(\mathbf{p}, \mathbf{r})\big) & \text{if } \delta \to 0_+, \\ -(b_i + c_i^+) \cdot d_i(\mathbf{p}, \mathbf{r})\big(1 - d_i(\mathbf{p}, \mathbf{r})\big) & \text{if } \delta \to 0_-. \end{cases} \tag{A.1}$$

Due to the structural property of the MNL model, we observe that the derivatives in Eq. (A.1) only involve firm i's own information. In our partial information setting, the market share $d_i(\mathbf{p}, \mathbf{r})$ is accessible to firm i, as it can be derived from the realized sales volumes. Thus, firm i can approximate its sensitivity parameters using its market shares from periods 1 and 2, i.e., $d_i\big(\mathbf{r}, \mathbf{r}\big)$ and $d_i\big((r_i + \delta, \mathbf{r}_{-i}), \mathbf{r}\big)$, such that

$$\frac{d_i(\mathbf{r}, \mathbf{r}) - d_i\big((r_i + \delta, \mathbf{r}_{-i}), \mathbf{r}\big)}{\delta \cdot d_i(\mathbf{r}, \mathbf{r})\big(1 - d_i(\mathbf{r}, \mathbf{r})\big)} \approx \begin{cases} b_i + c_i^- & \text{if } \delta > 0, \\ b_i + c_i^+ & \text{if } \delta < 0. \end{cases} \tag{A.2}$$

Since the first-order oracle used in Algorithms 1 and 2 only relies on the sensitivity parameters through $b_i + c_i^-$ and $b_i + c_i^+$, knowing these two sums through Eq. (A.2) is sufficient for the firms.

In this two-period cooperation, firms are not required to disclose their proprietary information to competitors, thereby ensuring confidentiality and competitive advantage are upheld. Moreover, it would be considered reasonable for all firms to set their prices equal to the current reference prices, as these prices mirror consumers' price expectations, indicating a practical pricing strategy. Furthermore, even when the reference price is unknown, firms can still employ this procedure, albeit with extended collaboration periods necessary to gather enough data for the reference price. However, when all firms need to estimate their sensitivity parameters, it is also worth mentioning that a total number of $\mathcal{O}(n)$ periods are required for executing the above cooperation mechanism. If the number of firms involved is large, these additional periods are not negligible in the final complexity bound.

Appendix B Discussion on Alternative Methods

As briefly mentioned in Section 3, our problem can also be translated into a standard $2n$-player online game or a dynamical system. In this appendix, we will discuss these alternative methods and explain why existing tools from the literature on online games and nonlinear dynamical systems cannot be applied.

B.1 Standard $2n$-player Online Game Formulation

The oligopoly competition in our study involves a varying underlying state, i.e., reference price, which depends on firms' price decisions and changes every period. To convert this problem to a standard game without the varying state, we can view the reference price $\mathbf{r} = (r_i)_{i\in N}$ as the decision variables of n additional virtual players with carefully designed objective functions. In each period, these virtual players update its decision variable, i.e., the reference price, using gradient ascent with fixed step-sizes. Specifically, for each virtual player $i \in N$, its objective function $R_i(p_i, r_i)$ and step-sizes $\{\eta_r^t\}_{t\geq 0}$ are defined as follows

$$\begin{aligned} R_i(p_i, r_i) &= -\frac{1}{2} r_i^2 + r_i p_i, \quad \forall i \in N; \\ \eta_r^t \equiv \eta_r &:= 1-\alpha, \quad \forall t \geq 0,\ \forall i \in N. \end{aligned} \tag{B.1}$$

To summarize, in this standard $2n$-player game, each real firm $i \in N$ has its log-revenue $\log\big(\Pi_i(\mathbf{p}, \mathbf{r})\big)$ as the objective function and updates its variable p_i via projected gradient ascent with step-size $\{\eta^t\}_{t\geq 0}$; each virtual firm $i \in N$ has the objective function $R_i(p_i, r_i)$ and updates its variable r_i using the standard gradient ascent with fixed step-size η_r. Below, we detail the update rule for this $2n$-player game in Algorithm 3, which essentially generates the same sequence $\{(\mathbf{p}^t, \mathbf{r}^t)\}_{t\geq 0}$ as Algorithm 1.

Algorithm 3: Standard $2n$-player Game with No Varying State

Input: Initial reference price $\mathbf{r}^0$, initial price $\mathbf{p}^0$, and step-sizes $\{\eta^t\}_{t\geq 0}$ and $\eta_r = 1-\alpha$.

for $t = 0, 1, 2, \ldots$ **do**

  for $i \in N$ **do**

    Update posted price: $p_i^{t+1} = \texttt{Proj}_{\mathcal{P}}\left(p_i^t + \eta^t \cdot \dfrac{\partial \log\big(\Pi_i(\mathbf{p}^t, \mathbf{r}^t)\big)}{\partial p_i}\right) = \texttt{Proj}_{\mathcal{P}}\left(p_i^t + \eta^t D_i^t\right)$.

    Update reference price: $r_i^{t+1} = r_i^t + \eta_r \cdot \dfrac{\partial R_i(p_i^t, r_i^t)}{\partial r_i} = \alpha r_i^t + (1-\alpha) p_i^t$.

  end

end

It can be easily seen that the pure strategy Nash equilibrium of this $2n$-player static game is equivalent to the SNE (see Definition 2) of the original n-player dynamic game. However, even after converting to the static game, no general convergence results are readily applicable in this problem, primarily for the following two reasons.

B.1.1 Lack of Variational Stability for $2n$-player Game. The first obstacle comes from the lack of critical properties in our $2n$-player game, such as monotonicity (Lin et al. 2020) or variational stability (Mertikopoulos and Zhou 2019), where the latter is a strictly weaker version of the former. Without loss of generality, we demonstrate the absence of the variational stability using a duopoly competition (i.e., $n=2$), which can be transformed to a standard 4-player game, with the price and reference price updating via Algorithm 3. To show this standard 4-player game is not variationally stable, it suffices to demonstrate that the second-order test for variational stability, outlined in Mertikopoulos and Zhou (2019, Table 1), is not consistently satisfied. This test essentially requires the negative definiteness of a 4×4 symmetric matrix $H^{\mathcal{G}}$, which will be formally defined below. First, let $\mathbf{x}=(\mathbf{p},\mathbf{r})$ be the 4-dimensional decision variable for all players, and let $f_k(\mathbf{x})$ be the objective functions where $k\in\{1,2,v_1,v_2\}$. Specifically, indices $1,2$ represent two real firms and index v_i represents the virtual firm that corresponds to the reference price of product $i\in\{1,2\}$. By construction, we have that

$$f_k(\mathbf{x})=\begin{cases}\log\big(\Pi_i(\mathbf{p},\mathbf{r})\big) & \text{if } k=i,\\ R_i(p_i,r_i) & \text{if } k=v_i,\end{cases}\qquad \text{where } i\in\{1,2\}. \tag{B.2}$$

Next, we define the 4×4 matrix $H^{\mathcal{G}}$, where we let $1,2,v_1,v_2$ correspond to matrix indices $1,2,3,4$, respectively. Then, the (m,l)-th entry of $H^{\mathcal{G}}$ are defined as follows

$$\left(H^{\mathcal{G}}\right)_{ml}:=\frac{\partial^2 f_m(\mathbf{x}^\star)}{\partial x_m\partial x_l}+\frac{\partial^2 f_l(\mathbf{x}^\star)}{\partial x_m\partial x_l},\quad \forall 1\le m,l\le 4, \tag{B.3}$$

where we denote $\mathbf{x}^\star=(\mathbf{p}^\star,\mathbf{p}^\star)$. Using direct computation and the optimality condition in Eq. (C.16), it holds that

$$H^{\mathcal{G}}=\begin{bmatrix}-2(\tilde{b}_1)^2(1-d_1^\star) & 2\tilde{b}_1\tilde{b}_2d_1^\star d_2^\star & 1+\tilde{b}_1c_1d_1^\star(1-d_1^\star) & -\tilde{b}_1c_2d_1^\star d_2^\star\\ 2\tilde{b}_1\tilde{b}_2d_1^\star d_2^\star & -2(\tilde{b}_2)^2(1-d_2^\star) & -\tilde{b}_2c_1d_1^\star d_2^\star & 1+\tilde{b}_2c_2d_2^\star(1-d_2^\star)\\ 1+\tilde{b}_1c_1d_1^\star(1-d_1^\star) & -\tilde{b}_2c_1d_1^\star d_2^\star & -2 & 0\\ -\tilde{b}_1c_2d_1^\star d_2^\star & 1+\tilde{b}_2c_2d_2^\star(1-d_2^\star) & 0 & -2\end{bmatrix}, \tag{B.4}$$

where we use the shorthand notations $\tilde{b}_i:=b_i+c_i$ and $d_i^\star=d_i(\mathbf{p}^\star,\mathbf{p}^\star)$. Consider the principal minor of $H^{\mathcal{G}}$ formed by removing the first row and the first column of $H^{\mathcal{G}}$. Its determinant can be computed as follows

$$\begin{aligned}&\mathtt{det}\left(\begin{bmatrix}-2(\tilde{b}_2)^2(1-d_2^\star) & -\tilde{b}_2c_1d_1^\star d_2^\star & 1+\tilde{b}_2c_2d_2^\star(1-d_2^\star)\\ -\tilde{b}_2c_1d_1^\star d_2^\star & -2 & 0\\ 1+\tilde{b}_2c_2d_2^\star(1-d_2^\star) & 0 & -2\end{bmatrix}\right)\\ &=-8(\tilde{b}_2)^2(1-d_2^\star)+2\left[1+\tilde{b}_2c_2d_2^\star(1-d_2^\star)\right]^2+2\left(\tilde{b}_2c_1d_1^\star d_2^\star\right)^2\\ &>2-8(\tilde{b}_2)^2.\end{aligned} \tag{B.5}$$

If $H^{\mathcal{G}}$ is negative definite, then the above determinant should be negative. Yet, from Eq. (B.5), it is evident that determinant must be positive when $\tilde{b}_2 = b_2 + c_2 \leq 1/2$. Consequently, this implies that the second-order test for variational stability does not always hold, which indicates that our $2n$-player game is not variationally stable.

B.1.2 Inflexible Step-sizes for Virtual Players. The other obstacle stems from the asynchronous updates for the real firms (price players) and the virtual firms (reference price players). While the real firms have the flexibility in adopting time-varying step-sizes, the virtual firms must stick to the constant step-size of $(1-\alpha)$. As a result, this inflexibility perplexes the analysis, as the typical convergence results of online games require the step-sizes of multiple players to have the same pattern (all diminishing or sufficiently small constant step-sizes) (see, e.g., Nagurney and Zhang (1995), Scutari et al. (2010), Bravo et al. (2018), Mertikopoulos and Zhou (2019)).

We are aware that Golrezaei et al. (2020) also have the same challenge. Below, we would like to elaborate on how Golrezaei et al. (2020) handle the issue of heterogeneous step-sizes and clarify why this approach is not applicable to the oligopoly competition studied in this paper. The central lemma in their analysis is Golrezaei et al. (2020, Lemma 9.1), which essentially shows that

$$\sum_{i=1,2} (p_i^\star - p_i) \cdot \frac{\partial \pi_i(\mathbf{p}, r)}{\partial p_i}\bigg|_{r=\theta_1 p_1 + \theta_2 p_2} > 0, \quad \forall \mathbf{p} \in \mathcal{P}^n, \tag{B.6}$$

where $\pi_i(\mathbf{p}, r)$ denotes their revenue function for firm i and $\mathbf{p}^\star$ denotes their SNE. Note that under their reference price update model, the condition $r = \theta_1 p_1 + \theta_2 p_2$ indicates that the reference price already converges to the price. The inequality in Eq. (B.6) basically demonstrates a similar property as variational stability for their duopoly competition, except for requiring $r = \theta_1 p_1 + \theta_2 p_2$. When the real firms adopt decreasing step-sizes, the reference price would gradually converge towards the price. Then, together with Eq. (B.6), Golrezaei et al. (2020) manage to derive the global convergence of their algorithm. It is worth mentioning that both two parts of the proof for Golrezaei et al. (2020, Theorem 5.1) rely on Eq. (B.6).

For our oligopoly game with logit demand, we are able to prove a property analogous to Eq. (B.6) but only holds locally around $\mathbf{p}^\star$. The proof of Lemma EC.1 is deferred to Appendix M.1.

LEMMA EC.1. *In the loss-neutral scenario, define function $\mathcal{H}(\mathbf{p})$ as follows:*

$$\mathcal{H}(\mathbf{p}) := \sum_{i \in N} (p_i^\star - p_i) \cdot \frac{\partial \log\big(\Pi(\mathbf{p}, \mathbf{r})\big)}{\partial p_i}\bigg|_{\mathbf{r}=\mathbf{p}} = \sum_{i \in N} \left[\frac{1}{p_i} + (b_i + c_i)\big(d_i(\mathbf{p}, \mathbf{p}) - 1\big)\right](p_i^\star - p_i), \tag{B.7}$$

where $\mathbf{p}^\star$ is the unique SNE. Then, there exist $\gamma > 0$ and a open set $U_\gamma \ni \mathbf{p}^\star$ such that

$$\mathcal{H}(\mathbf{p}) \geq \gamma \cdot \|\mathbf{p} - \mathbf{p}^\star\|^2, \quad \forall \mathbf{p} \in U_\gamma. \tag{B.8}$$

Leveraging Lemma EC.1, we proceed to establish local convergence in the subsequent proposition (see Appendix M.2 for its proof). It is important to note that Proposition EC.1 guarantees convergence only in the vicinity of $\mathbf{p}^\star$, since Lemma EC.1 is only applicable on a local scale.

PROPOSITION EC.1 **(Local Convergence of OPGA).** *In the loss-neutral scenario, let the step-sizes $\{\eta^t\}_{t\geq 0}$ be a non-increasing sequence such that $\lim_{t\to\infty}\eta^t = 0$ and $\sum_{t=0}^{\infty}\eta^t = \infty$ hold. Then, there exists some neighborhood $\mathcal{B}$ of $\mathbf{p}^\star$ such that when the price path $\{\mathbf{p}^t\}_{t\geq 0}$ enters $\mathcal{B}$ with a sufficiently small step-size, the price path will stay in $\mathcal{B}$ during subsequent periods.*

Furthermore, suppose the step-sizes satisfy $\eta^t = \frac{C_\eta}{t}$ for all $t\geq 1$, where C_η is some general constant. Then, the local convergence rate of $\{(\mathbf{p}^t,\mathbf{r}^t)\}_{t\geq 0}$ after the path stays in $\mathcal{B}$ satisfies that

$$\left\|\mathbf{p}^\star - \mathbf{p}^t\right\|^2 \leq \mathcal{O}\left(\frac{1}{t}\right), \quad \left\|\mathbf{p}^\star - \mathbf{r}^t\right\|^2 \leq \mathcal{O}\left(\frac{1}{t}\right). \tag{B.9}$$

Since Lemma EC.1 does not always hold for general $\mathbf{p}\in\mathcal{P}^n$, we are unable to derive the global convergence via a similar two-step analysis employed in Golrezaei et al. (2020), which makes it necessary for us to devise new techniques. In our proofs for Theorems 1 and 2, we introduce a weighted ℓ^1-distance (defined in Eq. (11)) to measure the convergence, and our analysis mainly leverages a structural property of the SNE under the MNL model, as shown in Lemma EC.3.

Moreover, it should be highlighted that the analyses based on the variational stability itself (Mertikopoulos and Zhou 2019) or its variant (Golrezaei et al. 2020) achieve, at their best, an $\mathcal{O}(1/t)$ convergence rate in the noise-free setting. In comparison, by exploiting characteristics of the MNL model, we manage to derive a faster rate of $\mathcal{O}(1/t^2)$ in Theorem 2. This improvement further sets apart our convergence results in the loss-neutral scenario from those reported by Mertikopoulos and Zhou (2019) and Golrezaei et al. (2020).

B.2 Nonlinear Dynamical System Formulation

The study of the limiting behavior of a competitive gradient-based learning algorithm is related to dynamical system theories (Mazumdar et al. 2020). In fact, the update of Algorithm 1 can be viewed as a nonlinear dynamical system. Assume a constant step-size is employed, i.e., $\eta^t\equiv\eta$, $\forall t\geq 0$. Then, Lines 5 and 7 in Algorithm 1 are equivalent to the dynamical system

$$(\mathbf{p}^{t+1},\mathbf{r}^{t+1}) = \mathbf{f}(\mathbf{p}^t,\mathbf{r}^t), \quad \forall t\geq 0, \tag{B.10}$$

where $\mathbf{f}(\cdot)$ is a vector-valued function defined as

$$\mathbf{f}(\mathbf{p},\mathbf{r}) := \begin{pmatrix} \mathtt{Proj}_{\mathcal{P}}\Big(p_1 + \eta\Big(1/p_1 + (b_1+c_1)\cdot d_1(\mathbf{p},\mathbf{r}) - (b_1+c_1)\Big)\Big) \\ \cdots \\ \mathtt{Proj}_{\mathcal{P}}\Big(p_n + \eta\Big(1/p_n + (b_n+c_n)\cdot d_n(\mathbf{p},\mathbf{r}) - (b_n+c_n)\Big)\Big) \\ \alpha r_1 + (1-\alpha)p_1 \\ \cdots \\ \alpha r_n + (1-\alpha)p_n \end{pmatrix}. \tag{B.11}$$

Under the assumption that $\mathbf{p}^\star \in \mathcal{P}^n$, it is evident that $\mathbf{p}^\star$ is the unique fixed point of the system in Eq. (B.10). Generally, fixed points can be categorized into three classes:

- *Asymptotically stable* when all nearby solutions converge to it.
- *Stable* when all nearby solutions remain in close proximity.
- *Unstable* when almost all nearby solutions diverge away from the fixed point.

Hence, if we can demonstrate the asymptotic stability of $\mathbf{p}^\star$, we can at least prove the local convergence of the price and reference price.

Standard dynamical systems theory (Arrowsmith et al. 1990) states that $\mathbf{p}^\star$ is asymptotically stable if the spectral radius of the Jacobian matrix $\nabla\mathbf{f}(\mathbf{p}^\star, \mathbf{p}^\star)$ is strictly less than one. Yet, computing the spectral radius is not straightforward. The primary challenge stems from the fact that the entries of $\nabla\mathbf{f}(\mathbf{p}^\star, \mathbf{p}^\star)$ contain $\mathbf{p}^\star$ and $d_i(\mathbf{p}^\star, \mathbf{p}^\star)$, but there is no closed-form expression for $\mathbf{p}^\star$.

Apart from the above issue, it is worth noting that the function $\mathbf{f}(\cdot,\cdot)$ is not globally smooth due to the presence of the projection operator. Furthermore, the function $\mathbf{f}(\cdot,\cdot)$ also depends on the step-size η. When the firms adopt time-varying step-sizes, the dynamical system in Eq. (B.10) becomes non-stationary, i.e., $(\mathbf{p}^{t+1}, \mathbf{r}^{t+1}) = \mathbf{f}^t(\mathbf{p}^t, \mathbf{r}^t)$. Although the sequence of functions $\{\mathbf{f}^t(\cdot,\cdot)\}_{t\geq 0}$ shares the same fixed point, verifying the convergence (stability) of the system requires examining the spectral radius of $\nabla\mathbf{f}^t(\mathbf{p}^\star, \mathbf{p}^\star)$ for all $t \geq 0$.

Most significantly, even if asymptotic stability holds, it can only guarantee local convergence of Algorithm 1. Our goal, however, is to prove global convergence, such that both the price and reference price converge to the SNE for arbitrary initializations.

Appendix C Proofs of Propositions

C.1 Proof of Proposition 1

Proposition 1 **(Restated)**. *Let $\mathcal{S}$ be the set of SNE(s). Then, the following statements hold:*

- *If there exists any gain-seeking product, an SNE never exists, i.e., $\mathcal{S}$ is empty.*
- *Otherwise, with only loss-averse and loss-neutral products, an SNE always exists, and $\mathcal{S}$ can be expressed as*

$$\mathcal{S} = \left\{ \mathbf{p}^\star \;\middle|\; p_i^\star \in \left[\frac{1}{\left(b_i + c_i^-\right)\cdot\left(1 - d_i(\mathbf{p}^\star, \mathbf{p}^\star)\right)}, \frac{1}{\left(b_i + c_i^+\right)\cdot\left(1 - d_i(\mathbf{p}^\star, \mathbf{p}^\star)\right)} \right],\ \forall i \in N \right\}. \tag{C.1}$$

Proof of Proposition 1. We prove the two parts of the proposition separately.

Part 1. In this part where there is one or more gain-seeking product(s), i.e., $\exists i \in N$ such that $c_i^+ > c_i^-$, we show the non-existence of SNE by contradiction-based arguments. Suppose that there exists an SNE $\mathbf{p}^\star$ under gain-seeking reference effects. By Definition 2, $\mathbf{p}^\star$ must satisfy $\mathbf{p}^E(\mathbf{p}^\star) = \mathbf{p}^\star$, i.e., the price at SNE is equal to the corresponding reference price. This implies that the revenue function is non-smooth at an SNE as a result of gain-seeking reference effects. For a non-smooth point to be a Nash equilibrium, its left-hand derivative must be non-negative, and its right-hand derivative must be non-positive, implying the left-hand derivative is no greater than the right-hand derivative. Below, we take the left-hand and right-hand derivatives of $\Pi_i(\mathbf{p},\mathbf{r})$ with respect to p_i at its SNE, i.e., $(\mathbf{p},\mathbf{r}) = (\mathbf{p}^\star,\mathbf{p}^\star)$

$$\lim_{\Delta p_i \to 0^-} \frac{\Pi_i\big((p_i^\star + \Delta p_i, \mathbf{p}_{-i}^\star), \mathbf{p}^\star\big) - \Pi_i\big(\mathbf{p}^\star, \mathbf{p}^\star\big)}{\Delta p_i} = d_i(\mathbf{p}^\star,\mathbf{p}^\star) \cdot \left[1 - p_i^\star(b_i + c_i^+)\big(1 - d_i(\mathbf{p}^\star,\mathbf{p}^\star)\big)\right], \quad \text{(C.2a)}$$

$$\lim_{\Delta p_i \to 0^+} \frac{\Pi_i\big((p_i^\star + \Delta p_i, \mathbf{p}_{-i}^\star), \mathbf{p}^\star\big) - \Pi_i\big(\mathbf{p}^\star, \mathbf{p}^\star\big)}{\Delta p_i} = d_i(\mathbf{p}^\star,\mathbf{p}^\star) \cdot \left[1 - p_i^\star(b_i + c_i^-)\big(1 - d_i(\mathbf{p}^\star,\mathbf{p}^\star)\big)\right], \quad \text{(C.2b)}$$

where the left-hand derivative Eq. (C.2a) has the effective reference price sensitivity c_i^+ because when p_i approaches $p_i^\star$ from left, it follows that $p_i \le r_i = p_i^\star$. For the similar reason, the right-hand derivative Eq. (C.2b) uses c_i^- as the effective reference price sensitivity. We notice that the left-hand derivative is smaller than the right-hand derivative since product i has the gain-seeking reference effect, i.e., $c_i^+ > c_i^-$. This conflicts with the necessary condition for $\mathbf{p}^\star$ to be an NE. We conclude that no SNE exists in the gain-seeking scenario; hence, the price and reference price paths are cyclic in the long run for any given initial reference price.

Part 2. In this part where there are only loss-averse and loss-neutral products, our **first step** is to show that any SNE price must satisfy the characterization in Eq. (C.1). In the **second step**, we demonstrate that for any given pseudo sensitivities $(\tilde{c}_i)_{i\in N}$ where $\tilde{c}_i \in [c_i^+, c_i^-]$, there exists a unique price vector $\mathbf{p}$ that satisfies

$$p_i = \frac{1}{(b_i + \tilde{c}_i)\cdot(1 - d_i(\mathbf{p},\mathbf{p}))}, \quad \forall i \in N, \qquad \text{(C.3)}$$

and $\mathbf{p}$ is also an SNE. Together, these two steps prove the existence of SNE and show that $\mathcal{S}$ admits the expression in Eq. (C.1).

We start with the first step. According to Definition 2, it holds that $\mathbf{p}^E(\mathbf{p}^\star) = \mathbf{p}^\star$ for any SNE $\mathbf{p}^\star \in \mathcal{S}$, i.e., the price output by the equilibrium pricing policy is the same as the input reference price. As the SNE is a special case of NE, each firm's revenue needs to satisfy the first-order condition. By expanding the derivative and incorporating the sub-gradient at non-smooth points, we find that equilibrium $\mathbf{p}^E(\mathbf{r}) = (p_1^E(\mathbf{r}),\dots,p_n^E(\mathbf{r}))$ for the given reference price $\mathbf{r}$ admits that

$$\left.\frac{\partial \Pi_i(\mathbf{p},\mathbf{r})}{\partial p_i}\right|_{(\mathbf{p}^E(\mathbf{r}),\mathbf{r})} = 0 \iff p_i^E(\mathbf{r})\cdot\big(1 - d_i(\mathbf{p}^E(\mathbf{r}),\mathbf{r})\big) - \frac{1}{b_i + c_i\big(p_i^E(\mathbf{r}), r_i\big)} = 0, \quad \forall i \in N, \qquad \text{(C.4)}$$

where $c_i(p_i, r_i) := \mathbb{1}\{p_i < r_i\}\cdot c_i^+ + \mathbb{1}\{p_i > r_i\}\cdot c_i^- + \mathbb{1}\{p_i = r_i\}\cdot \tilde{c}_i$ represents the effective reference price sensitivity for product i at (p_i, r_i), and $\tilde{c}_i$ take the unique value between c_i^+ and c_i^- that makes the equality in Eq. (C.4) holds. For any SNE $\mathbf{p}^\star \in \mathcal{S}$, since $\mathbf{p}^E(\mathbf{p}^\star) = \mathbf{p}^\star$, we evaluate Eq. (C.4) at $(\mathbf{p}^E(\mathbf{r}), \mathbf{r}) = (\mathbf{p}^\star, \mathbf{p}^\star)$ to obtain that

$$p_i^\star = \frac{1}{\big(b_i + c_i(p_i^\star, p_i^\star)\big) \cdot (1 - d_i(\mathbf{p}^\star, \mathbf{p}^\star))}, \quad \forall i \in N. \tag{C.5}$$

Since the $c_i(p_i^\star, p_i^\star) \in [c_i^+, c_i^-]$, this proves that $\mathcal{S}$ must be a subset of the set characterized by the right-hand side of Eq. (C.1). This completes the proof for the first step.

Now, we proceed to the second step and begin by showing that given any pseudo sensitivities $(\tilde{c}_i)_{i\in N}$ where $\tilde{c}_i \in [c_i^+, c_i^-]$, Eq. (C.3) produces a unique price vector $\mathbf{p}$. By definition of the market share, we have that

$$d_i(\mathbf{p}, \mathbf{p}) = \frac{\exp(a_i - b_i p_i)}{1 + \sum_{k\in N} \exp(a_k - b_k p_k)} = d_0(\mathbf{p}, \mathbf{p}) \cdot \exp\left(a_i - \frac{b_i}{(b_i + \tilde{c}_i)\big(1 - d_i(\mathbf{p}, \mathbf{p})\big)}\right), \tag{C.6}$$

where the last equality follows from substituting p_i with the right-hand side of Eq. (C.3), and $d_0(\mathbf{p}, \mathbf{p}) := \frac{1}{1+\sum_{k\in N}\exp(a_k - b_k p_k)}$, which is the no-purchase probability. Rearranging Eq. (C.6), we move all terms containing $d_i(\mathbf{p}, \mathbf{p})$ to the left-hand side to obtain that

$$d_i(\mathbf{p}, \mathbf{p}) \cdot \exp\left(\frac{b_i}{b_i + \tilde{c}_i} \cdot \frac{1}{1 - d_i(\mathbf{p}, \mathbf{p})}\right) = d_0(\mathbf{p}, \mathbf{p}) \cdot \exp(a_i). \tag{C.7}$$

Define function $V_{\tilde{c}_i}(x) : (0, \infty) \to (0, 1)$ as the unique real solution v to the following equation:

$$v \cdot \exp\left(\frac{b_i}{b_i + \tilde{c}_i} \cdot \frac{1}{1 - v}\right) = x. \tag{C.8}$$

Then, from Eq. (C.7), we can express $d_i(\mathbf{p}, \mathbf{p})$ in terms of $V_{\tilde{c}_i}(\cdot)$ as

$$d_i(\mathbf{p}, \mathbf{p}) = V_{\tilde{c}_i}\big(d_0(\mathbf{p}, \mathbf{p}) \cdot \exp(a_i)\big). \tag{C.9}$$

Since $d_0(\mathbf{p}, \mathbf{p}) + \sum_{i\in N} d_i(\mathbf{p}, \mathbf{p}) = 1$, together with Eq. (C.9), we have that

$$d_0(\mathbf{p}, \mathbf{p}) + \sum_{i\in N} V_{\tilde{c}_i}\big(d_0(\mathbf{p}, \mathbf{p}) \cdot \exp(a_i)\big) = 1. \tag{C.10}$$

We observe that function $V_{\tilde{c}_i}(x)$ is strictly increasing, i.e., v is monotone increasing in x in Eq. (C.8). Hence, the left-hand side of Eq. (C.10) is monotone increasing in $d_0(\mathbf{p}, \mathbf{p})$, and its range clearly contains one as $d_0(\mathbf{p}, \mathbf{p})$ increases from zero to one. So, there exist a unique solution $d_0(\mathbf{p}, \mathbf{p})$ that satisfies Eq. (C.10). Together with Eq. (C.9), we observe that the demand for every product i is uniquely determined from Eq. (C.3). Due to the one-to-one mapping between $\mathbf{p}$ and $\{d_i(\mathbf{p}, \mathbf{p})\}_{i\in N}$, we conclude that there must exist a unique price vector that satisfies both Eqs. (C.9) and (C.10), equivalently Eq. (C.3). Below, we denote this unique solution as $\mathbf{p}^s$.

Next, we show that $\mathbf{p}^s$ is an SNE. Since Eq. (C.5) arises from the first-order condition for NE, we know $\mathbf{p}^s$ is a stationary point. Then, to prove $\mathbf{p}^s$ is indeed an SNE, it suffices to show that for all $i \in N$

$$\lim_{\Delta p_i \to 0} \frac{\Pi_i\big((p_i, \mathbf{p}^s_{-i}), \mathbf{p}^s\big) - \Pi_i\big((p_i - \Delta p_i, , \mathbf{p}^s_{-i}), \mathbf{p}^s\big)}{\Delta p_i} \geq 0, \quad \forall p_i \leq p_i^s, \tag{C.11a}$$

$$\lim_{\Delta p_i \to 0} \frac{\Pi_i\big((p_i + \Delta p_i, \mathbf{p}^s_{-i}), \mathbf{p}^s\big) - \Pi_i\big((p_i, \mathbf{p}^s_{-i}), \mathbf{p}^s\big)}{\Delta p_i} \leq 0, \quad \forall p_i \geq p_i^s. \tag{C.11b}$$

If Eq. (C.11) holds, then for every firm $i \in N$, the revenue always increases in p_i when $p_i \leq p_i^s$ and decreases in p_i when $p_i \geq p_i^s$, assuming the prices for all other products remain at $\mathbf{p}^s_{-i}$. This implies that firm i can never achieve a higher revenue by deviating from the stationary price p_i^s, and thereby $\mathbf{p}^s$ is an SNE. We compute the left-hand derivative and observe that Eq. (C.11a) is equivalent to

$$p_i \cdot \Big[1 - d_i\big((p_i, \mathbf{p}^s_{-i}), \mathbf{p}^s\big)\Big] - \frac{1}{b_i + c_i^+} \leq 0, \quad \forall p_i \leq p_i^s, \tag{C.12}$$

where we use c_i^+ because the $r_i = p_i^s \geq p_i$. It is clear that the left-hand side of Eq. (C.12) is monotone increasing in p_i. Together with $c_i^+ \leq \tilde{c}_i$, we have for all $p_i \leq p_i^s$

$$p_i \cdot \Big[1 - d_i\big((p_i, \mathbf{p}^s_{-i}), \mathbf{p}^s\big)\Big] - \frac{1}{b_i + c_i^+} \leq p_i^s \cdot \Big[1 - d_i(\mathbf{p}^s, \mathbf{p}^s)\Big] - \frac{1}{b_i + \tilde{c}_i} = 0, \tag{C.13}$$

where the last equality stems from the fact that $\mathbf{p}^s$ is the unique solution to Eq. (C.3). Similarly, we validate Eq. (C.11b) by showing that

$$p_i \cdot \Big[1 - d_i\big((p_i, \mathbf{p}^s_{-i}), \mathbf{p}^s\big)\Big] - \frac{1}{b_i + c_i^-} \geq p_i^s \cdot \Big[1 - d_i(\mathbf{p}^s, \mathbf{p}^s)\Big] - \frac{1}{b_i + \tilde{c}_i} = 0, \quad \forall p_i \geq p_i^s, \tag{C.14}$$

as Eq. (C.11b) is equivalent to $p_i\Big[1 - d_i\big((p_i, \mathbf{p}^s_{-i}), \mathbf{p}^s\big)\Big] - 1/(b_i + c_i^-) \geq 0$. Since both conditions in (C.11) are satisfied, we conclude that $\mathbf{p}^s$ is an SNE.

Combining the results in both parts, we finally complete the proof of Proposition 1. □

C.2 Proof of Proposition 2

PROPOSITION 2 **(Restated)**. *In loss-averse and loss-neutral scenarios where SNE(s) always exists, its uniqueness depends on the presence of any loss-averse product. Specifically,*

- *The SNE is unique, i.e., $\mathcal{S}$ is a singleton, if and only if all products are loss-neutral.*
- *Otherwise, with any loss-averse product, there always exists a continuum of SNEs, and $\mathcal{S}$ can be a non-convex set.*

Furthermore, any SNE $\mathbf{p}^\star \in \mathcal{S}$ can be bounded as

$$\frac{1}{b_i + c_i^-} < p_i^\star < \frac{1}{b_i + c_i^+} + \frac{1}{b_i} W\left(\frac{b_i}{b_i + c_i^+} \exp\left(a_i - \frac{b_i}{b_i + c_i^+}\right)\right), \quad \forall i \in N, \tag{C.15}$$

where $W(\cdot)$ is the Lambert W function (see definition in Eq. (C.25)).

Proof of Proposition 2. First, the uniqueness of SNE when all products are loss-neutral directly follows from the characterization of SNE in Eq. (C.1). As $c_i^+ = c_i^- := c_i$ in the loss-neutral scenario, the interval in Eq. (C.1) reduces to a single value, and thus there only exists a unique price vector, denoted by $\mathbf{p}^\star$, such that

$$p_i^\star = \frac{1}{(b_i + c_i)\cdot(1 - d_i(\mathbf{p}^\star, \mathbf{p}^\star))}, \quad \forall i \in N, \tag{C.16}$$

where the uniqueness follows from the same reasoning as Eqs. (C.6) to Eq. (C.10).

Next, to prove the reverse direction ($\mathcal{S}$ is a singleton only if all products are loss-neutral), we show that in the presence of any loss-averse product, there always exist infinitely many SNEs that form a continuum. Without loss of generality, suppose consumers are loss-averse towards product $i_0 \in N$, i.e., $c_{i_0}^+ < c_{i_0}^-$. Then, going back to Eq. (C.3), it suffices to show that for two different $\tilde{c}_{i_0,1}, \tilde{c}_{i_0,2} \in [c_{i_0}^+, c_{i_0}^-]$, the pseudo sensitivities $(\tilde{c}_{i_0,1}, \tilde{\mathbf{c}}_{-i_0})$ and $(\tilde{c}_{i_0,2}, \tilde{\mathbf{c}}_{-i_0})$ produce two different SNEs. Note that we use $\tilde{\mathbf{c}}_{-i}$ to denote the vector $(\tilde{c}_j)_{j\neq i}$.

Let $\tilde{\mathbf{p}}^{\star,1}$ be the SNE that satisfies Eq. (C.3) with pseudo sensitivities $(\tilde{c}_{i_0,1}, \tilde{\mathbf{c}}_{-i_0})$, and $\tilde{\mathbf{p}}^{\star,2}$ be the SNE with pseudo sensitivities $(\tilde{c}_{i_0,2}, \tilde{\mathbf{c}}_{-i_0})$. Below, we show that $\tilde{\mathbf{p}}^{\star,1} \neq \tilde{\mathbf{p}}^{\star,2}$. Suppose by contradiction that $\tilde{\mathbf{p}}^{\star,1} = \tilde{\mathbf{p}}^{\star,2}$, which implies that $d_0(\tilde{\mathbf{p}}^{\star,1}, \tilde{\mathbf{p}}^{\star,1}) = d_0(\tilde{\mathbf{p}}^{\star,2}, \tilde{\mathbf{p}}^{\star,2})$. Since the two SNEs share the same pseudo sensitivities for all products except i_0, we deduce from Eq. (C.9) that for all $i \neq i_0$

$$d_i(\tilde{\mathbf{p}}^{\star,1}, \tilde{\mathbf{p}}^{\star,1}) = V_{\tilde{c}_i}(d_0(\tilde{\mathbf{p}}^{\star,1}, \tilde{\mathbf{p}}^{\star,1})\cdot\exp(a_i)) = V_{\tilde{c}_i}(d_0(\tilde{\mathbf{p}}^{\star,2}, \tilde{\mathbf{p}}^{\star,2})\cdot\exp(a_i)) = d_i(\tilde{\mathbf{p}}^{\star,2}, \tilde{\mathbf{p}}^{\star,2}). \tag{C.17}$$

Since $d_0(\mathbf{p},\mathbf{r}) + \sum_{i\in N} d_i(\mathbf{p},\mathbf{r}) = 1$, it also holds that

$$\begin{aligned} d_{i_0}(\tilde{\mathbf{p}}^{\star,1}, \tilde{\mathbf{p}}^{\star,1}) = 1 - d_0(\tilde{\mathbf{p}}^{\star,1}, \tilde{\mathbf{p}}^{\star,1}) - \sum_{i\neq i_0} d_i(\tilde{\mathbf{p}}^{\star,1}, \tilde{\mathbf{p}}^{\star,1}) &= 1 - d_0(\tilde{\mathbf{p}}^{\star,2}, \tilde{\mathbf{p}}^{\star,2}) - \sum_{i\neq i_0} d_i(\tilde{\mathbf{p}}^{\star,2}, \tilde{\mathbf{p}}^{\star,2}) \\ &= d_{i_0}(\tilde{\mathbf{p}}^{\star,2}, \tilde{\mathbf{p}}^{\star,2}). \end{aligned} \tag{C.18}$$

Together with Eq. (C.3), Eq. (C.18) indicates that

$$b_i + \tilde{c}_{i_0,1} = \frac{1}{\tilde{p}_{i_0}^{\star,1}\big(1 - d_{i_0}(\tilde{\mathbf{p}}^{\star,1}, \tilde{\mathbf{p}}^{\star,1})\big)} = \frac{1}{\tilde{p}_{i_0}^{\star,2}\big(1 - d_{i_0}(\tilde{\mathbf{p}}^{\star,2}, \tilde{\mathbf{p}}^{\star,2})\big)} = b_i + \tilde{c}_{i_0,2}, \tag{C.19}$$

which contradicts with the assumption that $\tilde{c}_{i_0,1} \neq \tilde{c}_{i_0,2}$. Therefore, we must have $\tilde{\mathbf{p}}^{\star,1} \neq \tilde{\mathbf{p}}^{\star,2}$. Since $[c_i^+, c_i^-]$ is a continuous interval, we conclude that there exist infinitely many SNEs in the presence of any loss-averse product. Finally, it is clear from Eqs. (C.3) to (C.10) that the dependency of the SNE price $\tilde{\mathbf{p}}^\star$ on the pseudo sensitivities $(\tilde{c}_i)_{i\in N}$ is continuous. Thus, the set of SNEs must form a continuous area, i.e., $\mathcal{S}$ is a continuum. The non-convexity of set $\mathcal{S}$ has already been confirmed by Figure 1.

Below, we show the boundedness of set $\mathcal{S}$. By performing a transformation on the relation in Eq. (C.1), we obtain the following inequalities for any $i \in N$ and $\mathbf{p} \in \mathcal{S}$:

$$p_i(b_i + c_i^-) \geq 1 + \frac{\exp(a_i - b_i p_i)}{1 + \sum_{k\neq i}\exp(a_k - b_k p_k)}, \quad p_i(b_i + c_i^+) \leq 1 + \frac{\exp(a_i - b_i p_i)}{1 + \sum_{k\neq i}\exp(a_k - b_k p_k)}. \tag{C.20}$$

Then, it immediately follows that

$$\frac{1}{b_i+c_i^-} < p_i < \frac{1+\exp(a_i-b_ip_i)}{b_i+c_i^+}, \quad \forall i\in N,\ \forall \mathbf{p}\in\mathcal{S}. \tag{C.21}$$

Now, we derive the upper bound in Eq. (C.15) from the second inequality in Eq. (C.21). Since the quantity on the right-hand side of Eq. (C.21) is monotone decreasing in p_i, any price that satisfies Eq. (C.21) must be upper-bounded by the unique solution y_i to the following equation

$$y_i = \frac{1+\exp(a_i-b_iy_i)}{b_i+c_i^+}. \tag{C.22}$$

Define $x_i := -b_i/(b_i+c_i^+) + b_iy_i$. Then, one can easily verify that Eq. (C.22) can be converted into

$$x_i\exp(x_i) = \frac{b_i}{b_i+c_i^+}\exp\left(a_i - \frac{b_i}{b_i+c_i^+}\right), \tag{C.23}$$

which implies that

$$x_i = W\left(\frac{b_i}{b_i+c_i^+}\exp\left(a_i - \frac{b_i}{b_i+c_i^+}\right)\right), \tag{C.24}$$

where $W(\cdot)$ is known as the Lambert W function (Weisstein 2002). For any value $z\geq 0$, $W(z)$ is defined to be the unique real solution w to the equation

$$w\cdot\exp\left(w\right) = z. \tag{C.25}$$

Hence, we have that

$$p_i < y_i = \frac{1}{b_i+c_i^+} + \frac{1}{b_i}W\left(\frac{b_i}{b_i+c_i^+}\exp\left(a_i - \frac{b_i}{b_i+c_i^+}\right)\right), \quad \forall i\in N,\ \forall \mathbf{p}\in\mathcal{S}. \tag{C.26}$$

Together with the lower bound provided in Eq. (C.21), this completes the proof. □

C.3 Proof of Proposition 3

Proposition 3 **(Restated)**. *The set of SNEs can be equivalently expressed as* $\mathcal{S} = \{\mathbf{p}\in\mathcal{P}^n \mid \widetilde{\kappa}(\mathbf{p},\mathbf{p}) = 0\}$, *where* $\widetilde{\kappa}(\cdot,\cdot)$ *is the metric defined in Eq. (17).*

Proof of Proposition 3. By Eq. (C.1) and the definition of functions $D_i^+(\mathbf{p},\mathbf{p}), D_i^-(\mathbf{p},\mathbf{p})$ in Eq. (18), we observe that the set of SNE(s) can be equivalently written as

$$\mathcal{S} = \left\{\mathbf{p}\in\mathcal{P}^n \;\middle|\; D_i^+(\mathbf{p},\mathbf{p})\geq 0,\ D_i^-(\mathbf{p},\mathbf{p})\leq 0,\ \forall i\in N\right\}. \tag{C.27}$$

Therefore, since $D_i^+(\mathbf{p},\mathbf{r})$ is consistently greater than $D_i^-(\mathbf{p},\mathbf{r})$ in the loss-averse scenario, we can derive that

$$\begin{aligned}
\widetilde{\kappa}(\mathbf{p},\mathbf{p}) = 0 \quad &\Leftrightarrow \quad \sum_{i\in N}\mathtt{dist}\Big(0, \mathtt{Hull}\big\{D_i^-(\mathbf{p},\mathbf{p}), D_i^+(\mathbf{p},\mathbf{p})\big\}\Big) = 0 \\
&\Leftrightarrow \quad \mathtt{dist}\big(0, I_i(\mathbf{p})\big) = 0, \quad \forall i\in N \\
&\Leftrightarrow \quad D_i^+(\mathbf{p},\mathbf{p})\geq 0,\ D_i^-(\mathbf{p},\mathbf{p})\leq 0, \quad \forall i\in N,
\end{aligned} \tag{C.28}$$

where we use the notation $I_i(\mathbf{p})$ to denote the interval $[D_i^-(\mathbf{p},\mathbf{p}), D_i^+(\mathbf{p},\mathbf{p})]$. This completes the proof of the proposition. □

Appendix D Proof of Theorem 1

THEOREM 1 **(Restated)**. *In the loss-neutral scenario, suppose all firms adopt Algorithm 1 with non-increasing step-sizes $\{\eta^t\}_{t\geq 0}$ such that $\lim_{t\to\infty}\eta^t = 0$ and $\sum_{t=0}^{\infty}\eta^t = \infty$. Then, their price and reference price paths converge to the unique stationary Nash equilibrium.*

Proof of Theorem 1. We will leverage the metrics $\kappa(\cdot)$ and $\kappa_\epsilon(\cdot)$ defined in Eq. (11). It is clear that

$$\kappa_\epsilon(\mathbf{p}) \leq \kappa(\mathbf{p}) \leq \kappa_\epsilon(\mathbf{p}) + n\epsilon, \quad \forall \mathbf{p} \in \mathcal{P}^n. \tag{D.1}$$

In our proof, we will show that for every $\epsilon > 0$, it holds that $\lim_{t\to\infty}\kappa(\mathbf{p}^t) \leq \mathcal{O}(\epsilon)$, thereby proving the convergence of the price path $\{\mathbf{p}^t\}_{t\geq 0}$. As the reference price is updated through the exponential smoothing scheme (see Eq. (4)), the convergence of $\{\mathbf{p}^t\}_{t\geq 0}$ also implies the convergence of the reference path $\{\mathbf{r}^t\}_{t\geq 0}$.

Before the proof, we introduce some helpful definitions. Let $G_i(\mathbf{p},\mathbf{r})$ be the scaled partial derivative of the log-revenue, defined as

$$G_i(\mathbf{p},\mathbf{r}) := \frac{1}{b_i + c_i}\cdot\frac{\partial \log\big(\Pi_i(\mathbf{p},\mathbf{r})\big)}{\partial p_i} = \frac{1}{(b_i+c_i)p_i} + d_i(\mathbf{p},\mathbf{r}) - 1, \quad \forall i \in N. \tag{D.2}$$

For the ease of notation, we denote $\mathcal{P}_i := \{p/(b_i+c_i) \mid p \in \mathcal{P}\}$ as the scaled price range. Then, the price update in Line 5 of Algorithm 1 is equivalent to

$$\frac{p_i^{t+1}}{b_i+c_i} = \mathtt{Proj}_{\mathcal{P}_i}\left(\frac{p_i^t}{b_i+c_i} + \eta^t\frac{D_i^t}{b_i+c_i}\right) = \mathtt{Proj}_{\mathcal{P}_i}\left(\frac{p_i^t}{b_i+c_i} + \eta^t G_i(\mathbf{p}^t,\mathbf{r}^t)\right). \tag{D.3}$$

Let $\mathtt{sign}(\cdot)$ be the sign function defined as

$$\mathtt{sign}(x) := \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases} \tag{D.4}$$

An essential observation from Eq. (D.3) is that: if $\mathtt{sign}(p_i^\star - p_i^t)\cdot G_i(\mathbf{p}^t,\mathbf{r}^t) > 0$, we have that $\mathtt{sign}(p_i^\star - p_i^t) = \mathtt{sign}\big(G_i(\mathbf{p}^t,\mathbf{r}^t)\big) = \mathtt{sign}(p_i^{t+1} - p_i^t)$, i.e., the update from p_i^t to p_i^{t+1} is toward the direction of the SNE price $p_i^\star$. Conversely, if $\mathtt{sign}(p_i^\star - p_i^t)\cdot G_i(\mathbf{p}^t,\mathbf{r}^t) < 0$, the update from p_i^t to p_i^{t+1} is deviating from $p_i^\star$. Finally, for every $t \geq 0$, we define

$$N_\epsilon^t := \left\{ i \in N \;\middle|\; \frac{|p_i^\star - p_i^t|}{b_i + c_i} < \epsilon \right\}, \quad \overline{N}_\epsilon^t := N \backslash N_\epsilon^t. \tag{D.5}$$

By definition, N_ϵ^t is the set of products whose prices are close to their SNE prices at period t, and $\overline{N}_\epsilon^t$ is its complement.

Now, we are ready to present the proof. By Lemma EC.2, when $\{\eta^t\}_{t\geq 0}$ is non-increasing and $\lim_{t\to\infty}\eta^t = 0$, the difference between reference price and price converges to zero as t goes to infinity, i.e., $\lim_{t\to\infty}\big(\mathbf{p}^t - \mathbf{r}^t\big) = 0$. Hence, for every $\epsilon > 0$, there exists $T_\epsilon > 0$ such that $\forall t \geq T_\epsilon$, it holds that

$\eta^t M_G < \epsilon$ and $\|\mathbf{p}^t - \mathbf{r}^t\| < \epsilon$, where M_G is an upper bound on $|G_i(\mathbf{p},\mathbf{r})|$ defined in Eq. (L.30). For every $t \geq T_\epsilon$, it follows from the definition of $\overline{N}_\epsilon^{t+1}$ in Eq. (D.5) that

$$\kappa_\epsilon(\mathbf{p}^{t+1}) = \sum_{i\in N} \max\left\{\frac{\left|p_i^\star - p_i^{t+1}\right|}{b_i+c_i} - \epsilon, 0\right\} = \sum_{i\in\overline{N}_\epsilon^{t+1}} \left(\frac{\left|p_i^\star - p_i^{t+1}\right|}{b_i+c_i} - \epsilon\right). \tag{D.6}$$

For every $i \in \overline{N}_\epsilon^{t+1}$, we have $\left|p_i^\star - p_i^{t+1}\right|/(b_i+c_i) \geq \epsilon$. Then, since

$$\frac{\left|p_i^{t+1} - p_i^t\right|}{b_i+c_i} \leq \frac{\eta^t D_i^t}{b_i+c_i} = \eta^t G_i(\mathbf{p}^t,\mathbf{r}^t) \leq \eta^t M_G < \epsilon, \tag{D.7}$$

it follows that $\mathtt{sign}\left(p_i^\star - p_i^{t+1}\right) = \mathtt{sign}\left(p_i^\star - p_i^t\right)$, and therefore

$$\begin{aligned}
\frac{\left|p_i^\star - p_i^{t+1}\right|}{b_i+c_i} &= \mathtt{sign}\left(p_i^\star - p_i^{t+1}\right)\frac{p_i^\star - p_i^{t+1}}{b_i+c_i} \\
&\leq \mathtt{sign}\left(p_i^\star - p_i^{t+1}\right)\frac{p_i^\star - p_i^t - \eta^t D_i^t}{b_i+c_i} \\
&= \mathtt{sign}\left(p_i^\star - p_i^t\right)\frac{p_i^\star - p_i^t - \eta^t D_i^t}{b_i+c_i} \\
&= \frac{|p_i^\star - p_i^t|}{b_i+c_i} - \eta^t \mathtt{sign}\left(p_i^\star - p_i^t\right) G_i(\mathbf{p}^t,\mathbf{r}^t),
\end{aligned} \tag{D.8}$$

where the inequality is due to the property of the projection operator. We substitute Eq. (D.8) into the right-hand side of Eq. (D.6) to derive that

$$\begin{aligned}
\kappa_\epsilon(\mathbf{p}^{t+1}) &\leq \sum_{i\in\overline{N}_\epsilon^{t+1}} \left[\frac{|p_i^\star - p_i^t|}{b_i+c_i} - \epsilon - \eta^t \mathtt{sign}\left(p_i^\star - p_i^t\right) G_i(\mathbf{p}^t,\mathbf{r}^t)\right] \\
&\leq \sum_{i\in N} \max\left\{\frac{|p_i^\star - p_i^t|}{b_i+c_i} - \epsilon, 0\right\} - \eta^t \sum_{i\in\overline{N}_\epsilon^{t+1}} \mathtt{sign}\left(p_i^\star - p_i^t\right) G_i(\mathbf{p}^t,\mathbf{r}^t) \\
&= \kappa_\epsilon(\mathbf{p}^t) - \eta^t \sum_{i\in\overline{N}_\epsilon^{t+1}} \mathtt{sign}\left(p_i^\star - p_i^t\right) G_i(\mathbf{p}^t,\mathbf{r}^t).
\end{aligned} \tag{D.9}$$

Thus, finding a lower bound for the summation term on the right-hand side of Eq. (D.9) is the key to the proof. Based on N_ϵ^{t+1}, we construct a price vector $\widehat{\mathbf{p}}^t$ as follows: $\widehat{p}_i^t = p_i^t$ if $i \in \overline{N}_\epsilon^{t+1}$, and

$\widehat{p}_i^t = p_i^\star$ if $i \in N_\epsilon^{t+1}$. Then, it holds that

$$\begin{aligned}
&\sum_{i\in\overline{N}_\epsilon^{t+1}} \mathtt{sign}\,(p_i^\star - p_i^t)\, G_i(\mathbf{p}^t,\mathbf{r}^t) \\
&\geq \sum_{i\in\overline{N}_\epsilon^{t+1}} \Big[\mathtt{sign}\,(p_i^\star - p_i^t)\, G_i(\mathbf{p}^t,\mathbf{p}^t) - \max_{\mathbf{r}\in\mathcal{P}^n}\left\{\left\|\nabla_{\mathbf{r}} G_i(\mathbf{p}^t,\mathbf{r})\right\|\right\}\left\|\mathbf{p}^t-\mathbf{r}^t\right\|\Big] \\
&\overset{(\Delta_1)}{\geq} \sum_{i\in\overline{N}_\epsilon^{t+1}} \Big[\mathtt{sign}\,(p_i^\star - \widehat{p}_i^t)\, G_i(\widehat{\mathbf{p}}^t,\widehat{\mathbf{p}}^t) - \max_{\mathbf{r}\in\mathcal{P}^n}\left\{\left\|\nabla_{\mathbf{r}} G_i(\mathbf{p}^t,\mathbf{r})\right\|\right\}\left\|\mathbf{p}^t-\mathbf{r}^t\right\| - \left|d_i(\mathbf{p}^t,\mathbf{p}^t) - d_i(\widehat{\mathbf{p}}^t,\widehat{\mathbf{p}}^t)\right|\Big] \\
&\geq \sum_{i\in\overline{N}_\epsilon^{t+1}} \Big[\mathtt{sign}\,(p_i^\star - \widehat{p}_i^t)\, G_i(\widehat{\mathbf{p}}^t,\widehat{\mathbf{p}}^t) - \max_{\mathbf{r}\in\mathcal{P}^n}\left\{\left\|\nabla_{\mathbf{r}} G_i(\mathbf{p}^t,\mathbf{r})\right\|\right\}\left\|\mathbf{p}^t-\mathbf{r}^t\right\| - \max_{\mathbf{p}\in\mathcal{P}^n}\left\{\left\|\nabla_{\mathbf{p}} d_i(\mathbf{p},\mathbf{p})\right\|\right\}\left\|\mathbf{p}^t-\widehat{\mathbf{p}}^t\right\|\Big] \\
&\overset{(\Delta_2)}{\geq} \sum_{i\in\overline{N}_\epsilon^{t+1}} \left[\mathtt{sign}\,(p_i^\star - \widehat{p}_i^t)\, G_i(\widehat{\mathbf{p}}^t,\widehat{\mathbf{p}}^t) - \ell_{r,i}\left\|\mathbf{p}^t-\mathbf{r}^t\right\| - \ell_{d,i}\left\|\mathbf{p}^t-\widehat{\mathbf{p}}^t\right\|\right] \\
&\geq \sum_{i\in N} \mathtt{sign}\,(p_i^\star - \widehat{p}_i^t)\, G_i(\widehat{\mathbf{p}}^t,\widehat{\mathbf{p}}^t) - \sum_{i\in N}\left[\ell_{r,i}\left\|\mathbf{p}^t-\mathbf{r}^t\right\| + \ell_{d,i}\left\|\mathbf{p}^t-\widehat{\mathbf{p}}^t\right\|\right],
\end{aligned} \tag{D.10}$$

where inequality (Δ_1) uses the definition of $G_i(\mathbf{p},\mathbf{r})$ in Eq. (D.2). Since $\widehat{p}_i^t = p_i^t$ when $i \in \overline{N}_\epsilon^{t+1}$, it follows that $|G_i(\widehat{\mathbf{p}}^t,\widehat{\mathbf{p}}^t) - G_i(\mathbf{p}^t,\mathbf{p}^t)| = |d_i(\mathbf{p}^t,\mathbf{p}^t) - d_i(\widehat{\mathbf{p}}^t,\widehat{\mathbf{p}}^t)|$. Note that this difference does not equal to zero in general, since the demand $d_i(\cdot,\cdot)$ depends on the prices and reference prices of all products. Then, step (Δ_2) in Eq. (D.10) applies the Lipschitz continuity of $G_i(\mathbf{p},\cdot)$ and $d_i(\mathbf{p},\mathbf{p})$ from Lemmas EC.4 and EC.5, respectively. Then, the last inequality holds because $\mathtt{sign}\,(p_i^\star - \widehat{p}_i^t) = \mathtt{sign}\,(0) = 0$ for all $i \in N_\epsilon^{t+1}$. Next, using Lemma EC.3, we have that

$$\sum_{i\in N} \mathtt{sign}\,(p_i^\star - \widehat{p}_i^t)\, G_i(\widehat{\mathbf{p}}^t,\widehat{\mathbf{p}}^t) = \mathcal{G}(\widehat{\mathbf{p}}^t) \geq \frac{\kappa(\widehat{\mathbf{p}}^t)}{\overline{p}\left\|\mathbf{p}^\star\right\|_\infty}. \tag{D.11}$$

To relate Eq. (D.11) with the original inequality in Eq. (D.9), we observe that

$$\begin{aligned}
\kappa(\widehat{\mathbf{p}}^t) &= \sum_{i\in N} \frac{|p_i^\star - \widehat{p}_i^t|}{b_i + c_i} \\
&\overset{(\Delta)}{=} \sum_{i\in\overline{N}_\epsilon^{t+1}} \frac{|p_i^\star - p_i^t|}{b_i + c_i} \\
&\geq \sum_{i\in\overline{N}_\epsilon^{t+1}} \max\left\{\frac{|p_i^\star - p_i^t|}{b_i + c_i} - \epsilon, 0\right\} \\
&\geq \sum_{i\in\overline{N}_\epsilon^{t}} \max\left\{\frac{|p_i^\star - p_i^t|}{b_i + c_i} - \epsilon, 0\right\} - \sum_{i\in\overline{N}_\epsilon^{t}\setminus\overline{N}_\epsilon^{t+1}} \max\left\{\frac{|p_i^\star - p_i^t|}{b_i + c_i} - \epsilon, 0\right\} \\
&= \kappa_\epsilon(\mathbf{p}^t) - \sum_{i\in\overline{N}_\epsilon^{t}\setminus\overline{N}_\epsilon^{t+1}} \max\left\{\frac{|p_i^\star - p_i^t|}{b_i + c_i} - \epsilon, 0\right\},
\end{aligned} \tag{D.12}$$

where equality (Δ) follows from the definition of $\widehat{\mathbf{p}}^t$. Since $i \in \overline{N}_\epsilon^t \backslash \overline{N}_\epsilon^{t+1}$ means that $|p_i^\star - p_i^t|/(b_i + c_i) \geq \epsilon$ and $\left|p_i^\star - p_i^{t+1}\right|/(b_i + c_i) < \epsilon$, we deduce from Eq. (D.7) that $|p_i^\star - p_i^t|/(b_i + c_i) \leq \epsilon + \eta^t G_i(\mathbf{p}^t, \mathbf{r}^t) \leq 2\epsilon$. Thus, Eq. (D.12) further implies that

$$\kappa(\widehat{\mathbf{p}}^t) \geq \kappa_\epsilon(\mathbf{p}^t) - \sum_{i \in \overline{N}_\epsilon^t \backslash \overline{N}_\epsilon^{t+1}} M_G \eta^t \geq \kappa_\epsilon(\mathbf{p}^t) - n\epsilon. \tag{D.13}$$

We use the shorthand notation $\lambda := 1/\left(\overline{p}\left\|\mathbf{p}^\star\right\|_\infty\right)$. Then, by substituting Eq. (D.11) and Eq. (D.13) back into Eq. (D.10), we derive that

$$\begin{aligned}\sum_{i \in \overline{N}_\epsilon^{t+1}} \mathtt{sign}\left(p_i^\star - p_i^t\right) G_i(\mathbf{p}^t, \mathbf{r}^t) &\geq \lambda\left(\kappa_\epsilon(\mathbf{p}^t) - n\epsilon\right) - \sum_{i \in N}\left[\ell_{r,i}\left\|\mathbf{p}^t - \mathbf{r}^t\right\| + \ell_{d,i}\left\|\mathbf{p}^t - \widehat{\mathbf{p}}^t\right\|\right] \\ &\overset{(\Delta)}{\geq} \lambda\kappa_\epsilon(\mathbf{p}^t) - \left(n\lambda + \sum_{i \in N}\ell_{r,i} + 2\sqrt{\sum_{i \in N}(b_i + c_i)^2} \cdot \sum_{i \in N}\ell_{d,i}\right)\epsilon,\end{aligned} \tag{D.14}$$

where inequality (Δ) is due to $\|\mathbf{p}^t - \mathbf{r}^t\| \leq \epsilon$ and

$$\left\|\mathbf{p}^t - \widehat{\mathbf{p}}^t\right\| = \sqrt{\sum_{i \in N_\epsilon^{t+1}}\left(p_i^\star - p_i^t\right)^2} \leq \sqrt{\sum_{i \in N_\epsilon^{t+1}}\left[2(b_i + c_i)\epsilon\right]^2} \leq 2\epsilon\sqrt{\sum_{i \in N}(b_i + c_i)^2}. \tag{D.15}$$

Let $C_\kappa := n\lambda + \sum_{i \in N}\ell_{r,i} + 2\sqrt{\sum_{i \in N}(b_i + c_i)^2} \cdot \sum_{i \in N}\ell_{d,i}$. By combining Eq. (D.9) with Eq. (D.14), we have that

$$\kappa_\epsilon(\mathbf{p}^{t+1}) \leq (1 - \lambda\eta^t)\kappa_\epsilon(\mathbf{p}^t) + C_\kappa\epsilon\eta^t, \quad \forall t \geq T_\epsilon. \tag{D.16}$$

Therefore, by unrolling the recursion in Eq. (D.16), we deduce that for all $t > T_\epsilon$

$$\begin{aligned}\kappa_\epsilon(\mathbf{p}^t) &\leq \kappa_\epsilon(\mathbf{p}^{T_\epsilon})\prod_{t'=T_\epsilon}^{t-1}(1 - \lambda\eta^{t'}) + C_\kappa\epsilon\sum_{t'=T_\epsilon}^{t-1}\eta^{t'}\prod_{t''=t'+1}^{t-1}(1 - \lambda\eta^{t''}) \\ &\overset{(\Delta)}{\leq} \kappa_\epsilon(\mathbf{p}^{T_\epsilon})\exp\left(-\lambda\sum_{t'=T_\epsilon}^{t-1}\eta^{t'}\right) + C_\kappa\epsilon\sum_{t'=T_\epsilon}^{t-1}\eta^{t'}\prod_{t''=t'+1}^{t-1}(1 - \lambda\eta^{t''}),\end{aligned} \tag{D.17}$$

where we apply the elementary inequality $1 - x \leq \exp(-x)$ in (Δ). We remark that $\prod_{t''=t}^{t-1}(1 - \lambda\eta^{t''}) = 1$ by default. Since $\sum_{t=0}^\infty \eta^t = \infty$, the exponential term in Eq. (D.17) clearly converges to zero as $t \to \infty$. Hence, the key is the second term, denoted as $X^{t-1} := \sum_{t'=T_\epsilon}^{t-1}\eta^{t'}\prod_{t''=t'+1}^{t-1}(1 - \lambda\eta^{t''})$. For every $t > T_\epsilon$, we observe that $X^t = \eta^t + (1 - \lambda\eta^t)X^{t-1}$, and thereby

$$\begin{aligned}X^t - \frac{1}{\lambda} &= \eta^t + (1 - \lambda\eta^t)X^{t-1} - \frac{1}{\lambda} \\ &= (1 - \lambda\eta^t)\left(X^{t-1} - \frac{1}{\lambda}\right) \\ &= \left(X^{T_\epsilon} - \frac{1}{\lambda}\right)\prod_{t'=T_\epsilon+1}^{t}(1 - \lambda\eta^{t'}),\end{aligned} \tag{D.18}$$

which implies $X^t - 1/\lambda$ always has the same sign as $X^{T_\epsilon} - 1/\lambda$ and converges to zero as $t \to \infty$. Thus, together with the relation $\kappa(\mathbf{p}) \le \kappa_\epsilon(\mathbf{p}) + n\epsilon$ in Eq. (D.1), Eqs. (D.17) and (D.18) imply that

$$\begin{aligned}\lim_{t\to\infty} \kappa(\mathbf{p}^t) &\le \lim_{t\to\infty} \kappa_\epsilon(\mathbf{p}^t) + n\epsilon \\ &\le \lim_{t\to\infty} \left[\kappa_\epsilon(\mathbf{p}^{T_\epsilon}) \exp\left(-\lambda \sum_{t'=T_\epsilon}^{t-1} \eta^{t'}\right) + C_\kappa \epsilon X^{t-1}\right] + n\epsilon \\ &= \left(\frac{C_\kappa}{\lambda} + n\right)\epsilon.\end{aligned} \tag{D.19}$$

Since ϵ can take any non-negative value, and both C_κ and λ depend only on the problem parameters, we conclude that $\lim_{t\to\infty} \kappa(\mathbf{p}^t) = 0$. Therefore, both the price and reference paths converge to $\mathbf{p}^\star$, which completes the proof of Theorem 1. □

Appendix E Proof of Theorem 2

Theorem 2 **(Restated)**. *In the loss-neutral scenario, suppose all firms adopt Algorithm 1 with step-sizes $\eta^t = \frac{C_\eta \log(t+1)}{t+1}$ for $t \ge 2$. Then, there exist constants T_1, C_p, and C_r such that when $C_\eta > 2\overline{p}^2/\log 2$, it holds for all $t > \max\{2T_1, 10\}$ that*

$$\left\|\mathbf{p}^\star - \mathbf{p}^t\right\|^2 \le C_p \left(\frac{\log t}{t}\right)^2 = \widetilde{\mathcal{O}}\left(\frac{1}{t^2}\right), \quad \left\|\mathbf{p}^\star - \mathbf{r}^t\right\|^2 \le C_r \left(\frac{\log t}{t}\right)^2 = \widetilde{\mathcal{O}}\left(\frac{1}{t^2}\right), \tag{E.1}$$

where constants T_1, C_p, and C_r are explicitly defined in Table EC.1.

Proof of Theorem 2. We note that the choice of $\eta^t = \frac{C_\eta \log(t+1)}{t+1}$ for $t \ge 2$ satisfies the step-size condition specified in Theorem 1. Hence, all analyses in the proof of Theorem 1 are applicable. We first prove the convergence rate in terms of the metric $\kappa(\mathbf{p})$ defined in Eq. (11).

By Lemma EC.2, we have $\|\mathbf{p}^t - \mathbf{r}^t\| \le \eta^t C_{rp}$ for all $t \ge T_1$, where the constants C_{rp} and T_1 are defined in Eq. (L.1). Hence, for every $\epsilon > 0$, we can take T_ϵ as the smallest integer greater than T_1 such that $\epsilon > \max\{\eta^{T_\epsilon} M_G, \eta^{T_\epsilon} C_{rp}\}$, so that we have $\eta^t M_G < \epsilon$ and $\|\mathbf{p}^t - \mathbf{r}^t\| < \epsilon$ for every $t \ge T_\epsilon$. Equivalently, T_ϵ is the smallest integer such that

$$\frac{\epsilon}{C_\eta \widehat{C}_{rp}} \ge \frac{\log(T_\epsilon + 1)}{T_\epsilon + 1} \text{ and } T_\epsilon \ge T_1, \tag{E.2}$$

where $\widehat{C}_{rp} := \max\{C_{rp}, M_G\}$. For every $t > T_\epsilon$, we have from Eq. (D.17) that

$$\kappa(\mathbf{p}^t) \le \kappa_\epsilon(\mathbf{p}^t) + n\epsilon \le \kappa_\epsilon(\mathbf{p}^{T_\epsilon}) \exp\left(-\lambda \sum_{t'=T_\epsilon}^{t-1} \eta^{t'}\right) + C_\kappa \epsilon X^{t-1} + n\epsilon, \tag{E.3}$$

where we recall that $X^{t-1} = \sum_{t'=T_\epsilon}^{t-1} \eta^{t'} \prod_{t''=t'+1}^{t-1} (1 - \lambda \eta^{t''})$. From the recursion in Eq. (D.18), we observe that if $X^{T_\epsilon} < 1/\lambda$, then $X^t \le 1/\lambda$ for every $t > T_\epsilon$. Otherwise, it still holds that

$$X^{t-1} = \frac{1}{\lambda} + \left(X^{T_\epsilon} - \frac{1}{\lambda}\right) \prod_{t'=T_\epsilon+1}^{t-1} (1 - \lambda \eta^{t'}) \le \frac{1}{\lambda} + \eta^{T_\epsilon} \le \frac{1}{\lambda} + C_\eta < 2C_\eta, \quad \forall t > T_\epsilon, \tag{E.4}$$

where the first inequality is because $X^{T_\epsilon} = \eta^{T_\epsilon}$ by definition, and the last inequality is due to the choice $C_\eta > 2\overline{p}^2/\log 2 > 2/(\lambda \log 2)$, as $\lambda = 1/\left(\overline{p}\,\|\mathbf{p}^\star\|_\infty\right)$. Thus, it remains to control the exponential term on the right-hand side of Eq. (E.3). Using the integration lower bound, we deduce that

$$\sum_{t'=T_\epsilon}^{t-1} \eta^{t'} \geq C_\eta \int_{T_\epsilon}^{t} \frac{\log(t'+1)}{t'+1} dt' = \frac{C_\eta}{2}\left[\log^2(t+1) - \log^2(T_\epsilon+1)\right]. \tag{E.5}$$

Therefore, it follows that for any $t > T_\epsilon$

$$\begin{aligned} \exp\left(-\lambda \sum_{t'=T_\epsilon}^{t-1} \eta^{t'}\right) &\leq \exp\left(-\frac{\lambda C_\eta}{2}\left[\log^2(t+1) - \log^2(T_\epsilon+1)\right]\right) \\ &= \left[\frac{\exp\left(\log^2(T_\epsilon+1)\right)}{\exp\left(\log^2(t+1)\right)}\right]^{\frac{\lambda C_\eta}{2}} \\ &= \left[\frac{(T_\epsilon+1)^{\log(T_\epsilon+1)}}{(t+1)^{\log(t+1)}}\right]^{\frac{\lambda C_\eta}{2}} \\ &= \left(\frac{T_\epsilon+1}{t+1}\right)^{\frac{\lambda C_\eta \log(T_\epsilon+1)}{2}} \cdot \left(\frac{1}{t+1}\right)^{\frac{\lambda C_\eta}{2}\cdot \log\left(\frac{t+1}{T_\epsilon+1}\right)}. \end{aligned} \tag{E.6}$$

Hence, when $t \geq 2T_\epsilon + 1$, it holds that

$$\begin{aligned} \exp\left(-\lambda \sum_{t'=T_\epsilon}^{t-1} \eta^{t'}\right) \;\leq \left(\frac{1}{t+1}\right)^{\frac{\lambda C_\eta}{2}\cdot \log 2} &\overset{(\Delta_1)}{\leq} \frac{1}{t+1} \overset{(\Delta_2)}{\leq} \frac{1}{2(T_\epsilon+1)} \cdot \frac{C_{rp} C_\eta \log(T_\epsilon+1)}{\sqrt{n}(\overline{p}-\underline{p})} \\ &\leq \frac{\epsilon}{2\sqrt{n}(\overline{p}-\underline{p})}, \end{aligned} \tag{E.7}$$

where step (Δ_1) is due to $C_\eta > 2\overline{p}^2/\log 2 > 2/(\lambda \log 2)$, and step (Δ_2) uses the premise that $t \geq 2T_\epsilon + 1$. We note that by the definition of C_{rp} in Eq. (L.1), it is easy to observe that the second fraction in (Δ_2) is clearly greater than one. Finally, the last inequality in Eq. (E.7) applies the definition of T_ϵ in Eq. (E.2). Together, Eqs. (E.3), (E.4), and (E.7) imply that

$$\kappa(\mathbf{p}^t) \leq \left(2C_\kappa C_\eta + n + \frac{\kappa_\epsilon(\mathbf{p}^{T_\epsilon})}{2\sqrt{n}(\overline{p}-\underline{p})}\right)\epsilon \leq \left(2C_\kappa C_\eta + n + \frac{M_\kappa}{2\sqrt{n}(\overline{p}-\underline{p})}\right)\epsilon, \tag{E.8}$$

where we replace $\kappa_\epsilon(\mathbf{p}^{T_\epsilon})$ by its universal upper bound $M_\kappa := \sum_{i\in N}(\overline{p}-\underline{p})/(b_i+c_i)$. By far, we have shown that $\kappa(\mathbf{p}^t) = \mathcal{O}(\epsilon)$ when $t \geq 2T_\epsilon + 1$. To obtain the convergence rate that explicitly depends on t, we consider the definition of T_ϵ in Eq. (E.2). We claim that it suffices to choose

$$T_\epsilon = \max\left\{T_1, \left\lceil \frac{2C_\eta \widehat{C}_{rp}}{\epsilon} \log\left(\frac{C_\eta \widehat{C}_{rp}}{\epsilon}\right) - 1\right\rceil\right\}. \tag{E.9}$$

To validate that such a choice satisfies the condition in Eq. (E.2), we compute that

$$\frac{\log(T_\epsilon+1)}{T_\epsilon+1} \leq \frac{\log\left(2C_\eta \widehat{C}_{rp}/\epsilon\right) + \log\left(\log\left(C_\eta \widehat{C}_{rp}/\epsilon\right)\right)}{\left(2C_\eta \widehat{C}_{rp}/\epsilon\right)\cdot \log\left(C_\eta \widehat{C}_{rp}/\epsilon\right)} \overset{(\Delta)}{<} \frac{\epsilon}{C_\eta \widehat{C}_{rp}}, \tag{E.10}$$

where step (Δ) uses the inequality $\max_{x>0}\big(\log 2x+\log(\log x)\big)/(2\log x)=1/2+1/e<1$. Thus, Eq. (E.8) holds true as long as

$$t\geq 2T_\epsilon+1\geq\max\left\{2T_1+1,\left\lceil\frac{4C_\eta\widehat{C}_{rp}}{\epsilon}\log\left(\frac{C_\eta\widehat{C}_{rp}}{\epsilon}\right)\right\rceil\right\}. \tag{E.11}$$

We observe the following equivalence

$$t=\left(4C_\eta\widehat{C}_{rp}/\epsilon\right)\cdot\log\left(C_\eta\widehat{C}_{rp}/\epsilon\right)\quad\Leftrightarrow\quad\log\left(C_\eta\widehat{C}_{rp}/\epsilon\right)=W(t/4), \tag{E.12}$$

where $W(\cdot)$ is the Lambert W function defined in Eq. (C.25). Using the lower bound of the Lambert function $W(x)\geq\log x-\log(\log x)$ for all $x\geq e$, we find that for all $t\geq 4e\approx 10.9$

$$\log\left(\frac{C_\eta\widehat{C}_{rp}}{\epsilon}\right)\geq\log\left(\frac{t}{4}\right)-\log\left(\log\left(\frac{t}{4}\right)\right), \tag{E.13}$$

which is equivalent to

$$\epsilon\leq\frac{C_\eta\widehat{C}_{rp}\log(t/4)}{t/4}\leq\frac{4C_\eta\widehat{C}_{rp}\log t}{t}. \tag{E.14}$$

Together with requirements $t\geq 2T_1+1$ and $t\geq 11$, we conclude from the bound in Eq. (E.8) that

$$\kappa(\mathbf{p}^t)\leq\left(2C_\kappa C_\eta+n+\frac{M_\kappa}{2\sqrt{n}(\overline{p}-\underline{p})}\right)\epsilon\leq\widehat{C}_\kappa\frac{\log t}{t},\quad\forall t>\{2T_1,10\}, \tag{E.15}$$

where we define $\widehat{C}_\kappa:=4C_\eta\widehat{C}_{rp}\left(2C_\kappa C_\eta+n+\frac{M_\kappa}{2\sqrt{n}(\overline{p}-\underline{p})}\right)$. Finally, to obtain the upper bounds in Eq. (13), we observe that for all $t>\{2T_1,10\}$

$$\left\|\mathbf{p}^\star-\mathbf{p}^t\right\|^2\leq\left(\sum_{i\in N}\max_{k\in N}\{b_k+c_k\}\cdot\frac{|p_i^\star-p_i^t|}{b_i+c_i}\right)^2\leq\max_{i\in N}\left\{(b_i+c_i)^2\right\}\cdot[\kappa(\mathbf{p}^t)]^2\leq C_p\left(\frac{\log t}{t}\right)^2, \tag{E.16}$$

where $C_p:=\max_{i\in N}\{(b_i+c_i)^2\}\cdot(\widehat{C}_\kappa)^2$. For the reference price path, we can similarly deduce that

$$\begin{aligned}\left\|\mathbf{p}^\star-\mathbf{r}^t\right\|^2&=\left\|\mathbf{p}^\star-\mathbf{p}^t+\mathbf{p}^t-\mathbf{r}^t\right\|^2\\&\leq 2\left\|\mathbf{p}^\star-\mathbf{p}^t\right\|^2+2\left\|\mathbf{p}^t-\mathbf{r}^t\right\|^2\\&\leq 2C_p\left(\frac{\log t}{t}\right)^2+2\left(\eta^tC_{rp}\right)^2\\&\leq 2\big(C_p+(C_\eta C_{rp})^2\big)\cdot\left(\frac{\log t}{t}\right)^2.\end{aligned} \tag{E.17}$$

The proof of Theorem 2 is completed by letting $C_r:=2\big(C_p+(C_\eta C_{rp})^2\big)$. □

Appendix F Proof of Theorem 3

Theorem 3 **(Restated)**. *In the loss-neutral scenario, if all firms adopt Algorithm 1 with step-sizes* $\{\eta^t\}_{t\geq 0}$ *satisfying* $\lim_{t\to\infty}\eta^t = 0$ *and* $\sum_{t=0}^{\infty}\eta^t = \infty$*, the dynamic regret of each firm grows in a sublinear rate, i.e.,*

$$\lim_{T\to\infty}\frac{1}{T}\times \text{D-Regret}_i(T) = 0, \quad \forall i\in N. \tag{F.1}$$

Furthermore, if the step-sizes are specified as $\eta^t = \frac{C_\eta \log(t+1)}{t+1}$ *for* $t\geq 2$*, there exist constants* T_1 *and* $C_{R,i}$ *such that when* $C_\eta > 2\overline{p}^2/\log 2$*, it holds that*

$$\text{D-Regret}_i(T) \leq \overline{p}\cdot\max\{2T_1, 10\} + 2C_{R,i} = \mathcal{O}(1), \quad \forall T\geq 1,\ \forall i\in N, \tag{F.2}$$

where constants T_1 *and* $C_{R,i}$ *are explicitly defined in Table EC.1.*

Proof of Theorem 3. We begin by demonstrating an auxiliary result on the smoothness of revenue function $\Pi_i(\mathbf{p},\mathbf{r})$ with respect to p_i, which would be useful in the following proof. Note that

$$\begin{aligned}\frac{\partial \Pi_i(\mathbf{p},\mathbf{r})}{\partial p_i} &= d_i(\mathbf{p},\mathbf{r}) - (b_i+c_i)p_i\cdot d_i(\mathbf{p},\mathbf{r})\big(1-d_i(\mathbf{p},\mathbf{r})\big)\\ &= d_i(\mathbf{p},\mathbf{r})\cdot\left[1-(b_i+c_i)p_i\cdot\big(1-d_i(\mathbf{p},\mathbf{r})\big)\right].\end{aligned} \tag{F.3}$$

Then, the second-order derivative of $\Pi_i(\mathbf{p},\mathbf{r})$ with respect to p_i can be computed as follows

$$\begin{aligned}\frac{\partial^2 \Pi_i(\mathbf{p},\mathbf{r})}{\partial p_i^2} =& -(b_i+c_i)\cdot d_i(\mathbf{p},\mathbf{r})\big(1-d_i(\mathbf{p},\mathbf{r})\big)\left[1-(b_i+c_i)p_i\cdot\big(1-d_i(\mathbf{p},\mathbf{r})\big)\right]\\ &+ d_i(\mathbf{p},\mathbf{r})\left[-(b_i+c_i)\big(1-d_i(\mathbf{p},\mathbf{r})\big)-(b_i+c_i)^2p_i\cdot d_i(\mathbf{p},\mathbf{r})\big(1-d_i(\mathbf{p},\mathbf{r})\big)\right]\\ =& -2(b_i+c_i)\cdot d_i(\mathbf{p},\mathbf{r})\big(1-d_i(\mathbf{p},\mathbf{r})\big) + (b_i+c_i)^2p_i\cdot d_i(\mathbf{p},\mathbf{r})\big(1-d_i(\mathbf{p},\mathbf{r})\big)^2\\ &-(b_i+c_i)^2p_i\cdot\big(d_i(\mathbf{p},\mathbf{r})\big)^2\big(1-d_i(\mathbf{p},\mathbf{r})\big)\\ =&(b_i+c_i)\cdot d_i(\mathbf{p},\mathbf{r})\big(1-d_i(\mathbf{p},\mathbf{r})\big)\cdot\left[-2+(b_i+c_i)p_i\cdot\big(1-2d_i(\mathbf{p},\mathbf{r})\big)\right],\end{aligned} \tag{F.4}$$

and thereby this second-order derivative can be bounded as

$$\left|\frac{\partial^2 \Pi_i(\mathbf{p},\mathbf{r})}{\partial p_i^2}\right| \leq \frac{1}{4}(b_i+c_i)\big(2+(b_i+c_i)\overline{p}\big) =: h_i, \quad \forall \mathbf{p}\in\mathcal{P}^n,\ \forall \mathbf{r}\in\mathcal{P}^n. \tag{F.5}$$

Hence, we have that $\Pi_i(\mathbf{p},\mathbf{r})$ is h_i-smooth with respect to p_i.

Now, we proceed to prove the theorem. For brevity, we denote the regret of firm i at period t as R_i^t, i.e., $R_i^t := \max_{p_i\in\mathcal{P}}\left\{\Pi_i\big((p_i,\mathbf{p}_{-i}^t),\mathbf{r}^t\big)\right\} - \Pi_i(\mathbf{p}^t,\mathbf{r}^t)$, and therefore the total regret over the entire T periods can be expressed as $\text{D-Regret}_i(T) = \sum_{t=1}^{T} R_i^t$. Let $p_i^B(\cdot,\cdot)$ be a function defined as $p_i^B\big(r_i,\mathbf{u}_{-i}(\mathbf{p}_{-i},\mathbf{r}_{-i})\big) := \arg\max_{p_i\in\mathcal{P}}\left\{\Pi_i\big((p_i,\mathbf{p}_{-i}),\mathbf{r}\big)\right\}$, where $\mathbf{u}_{-i}(\mathbf{p}_{-i},\mathbf{r}_{-i})$ is the vector of utilities for all firms other than i, as defined in Eq. (L.41). The function $p_i^B\big(r_i,\mathbf{u}_{-i}(\mathbf{p}_{-i},\mathbf{r}_{-i})\big)$ represents the best-response price for firm i that achieve the optimal single-period revenue, given the reference price $\mathbf{r}$ and the price of other products $\mathbf{p}_{-i}$.

We observe that R_i^t can be upper-bounded as follows

$$
\begin{aligned}
R_i^t &= \max_{p_i \in \mathcal{P}} \left\{ \Pi_i\big((p_i, \mathbf{p}_{-i}^t), \mathbf{r}^t\big) \right\} - \Pi_i(\mathbf{p}^t, \mathbf{r}^t) \\
&= \Pi_i\big((p_i^{B,t}, \mathbf{p}_{-i}^t), \mathbf{r}^t\big) - \Pi_i\left(\mathbf{p}^t, \mathbf{r}^t\right) \\
&\overset{(\Delta_1)}{\leq} \frac{\partial \Pi_i\big((p_i^{B,t}, \mathbf{p}_{-i}^t), \mathbf{r}^t\big)}{\partial p_i} \cdot \left(p_i^t - p_i^{B,t}\right) + \frac{h_i}{2}\left(p_i^{B,t} - p_i^t\right)^2 \\
&\overset{(\Delta_2)}{\leq} \frac{h_i}{2}\left(p_i^{B,t} - p_i^t\right)^2,
\end{aligned}
\tag{F.6}
$$

where we use the shorthand notation $p_i^{B,t} := p_i^B\big(r_i^t, \mathbf{u}_{-i}(\mathbf{p}_{-i}^t, \mathbf{r}_{-i}^t)\big)$ to denote the best-response price for firm i at period t. The step (Δ_1) utilizes the h_i-smoothness of $\Pi_i(\mathbf{p}, \mathbf{r})$ with respect to p_i, as shown in Eq. (F.5). In step (Δ_2), since $p_i^{B,t}$ is the best-response price, it holds that $\partial \Pi_i\big((p_i^{B,t}, \mathbf{p}_{-i}^t), \mathbf{r}^t\big)/\partial p_i = 0$ by the first-order condition.

In the following part, we evaluate term $\left(p_i^{B,t} - p_i^t\right)^2$ in Eq. (F.6), which can be decomposed as

$$
\left(p_i^{B,t} - p_i^t\right)^2 = \left(p_i^{B,t} - p_i^\star + p_i^\star - p_i^t\right)^2 \leq 2\left(p_i^{B,t} - p_i^\star\right)^2 + 2\left(p_i^\star - p_i^t\right)^2,
\tag{F.7}
$$

where the last step is due to the basic inequality $(x+y)^2 \leq 2x^2 + 2y^2$. Since the reference price at SNE is also equal to $\mathbf{p}^\star$, we can further upper-bound the first term on the right-hand side of Eq. (F.7) as follows

$$
\begin{aligned}
\left(p_i^{B,t} - p_i^\star\right)^2 &= \Big[p_i^B\left(r_i^t, \mathbf{u}_{-i}(\mathbf{p}_{-i}^t, \mathbf{r}_{-i}^t)\right) - p_i^B\left(p_i^\star, \mathbf{u}_{-i}(\mathbf{p}_{-i}^\star, \mathbf{p}_{-i}^\star)\right)\Big]^2 \\
&= \Big[p_i^B\left(r_i^t, \mathbf{u}_{-i}(\mathbf{p}_{-i}^t, \mathbf{r}_{-i}^t)\right) - p_i^B\left(p_i^\star, \mathbf{u}_{-i}(\mathbf{p}_{-i}^t, \mathbf{r}_{-i}^t)\right) \\
&\qquad + p_i^B\left(p_i^\star, \mathbf{u}_{-i}(\mathbf{p}_{-i}^t, \mathbf{r}_{-i}^t)\right) - p_i^B\left(p_i^\star, \mathbf{u}_{-i}(\mathbf{p}_{-i}^\star, \mathbf{p}_{-i}^\star)\right)\Big]^2 \\
&\leq 2\Big[p_i^B\left(r_i^t, \mathbf{u}_{-i}(\mathbf{p}_{-i}^t, \mathbf{r}_{-i}^t)\right) - p_i^B\left(p_i^\star, \mathbf{u}_{-i}(\mathbf{p}_{-i}^t, \mathbf{r}_{-i}^t)\right)\Big]^2 \\
&\qquad + 2\Big[p_i^B\left(p_i^\star, \mathbf{u}_{-i}(\mathbf{p}_{-i}^t, \mathbf{r}_{-i}^t)\right) - p_i^B\left(p_i^\star, \mathbf{u}_{-i}(\mathbf{p}_{-i}^\star, \mathbf{p}_{-i}^\star)\right)\Big]^2 \\
&\overset{(\Delta_1)}{\leq} 2\Big(\frac{c_i}{b_i + c_i}\Big)^2 (r_i^t - p_i^\star)^2 + 2\overline{p}^2 \left\|\mathbf{u}_{-i}\big(\mathbf{p}_{-i}^t, \mathbf{r}_{-i}^t\big) - \mathbf{u}_{-i}\big(\mathbf{p}_{-i}^\star, \mathbf{p}_{-i}^\star\big)\right\|^2 \\
&\overset{(\Delta_2)}{\leq} 2\Big(\frac{c_i}{b_i + c_i}\Big)^2 (r_i^t - p_i^\star)^2 + 4\overline{p}^2 \Big(\max_{j \neq i}\left\{(b_j + c_j)^2\right\}\left\|\mathbf{p}_{-i}^t - \mathbf{p}_{-i}^\star\right\|^2 + \max_{j \neq i}\left\{c_j^2\right\}\left\|\mathbf{r}_{-i}^t - \mathbf{p}_{-i}^\star\right\|^2\Big),
\end{aligned}
\tag{F.8}
$$

where in step (Δ_1), we use the $c_i/(b_i + c_i)$-Lipschitz continuity of $p_i^B\big(\cdot, \mathbf{u}_{-i}(\mathbf{p}_{-i}, \mathbf{r}_{-i})\big)$ and the $\overline{p}$-Lipschitz continuity of $p_i^B(r_i, \cdot)$ from Eq. (L.43) in Lemma EC.6. In step (Δ_2), by the definition of utility in Eq. (1), for all $j \in N \backslash \{i\}$, it holds that $u_j(p_j^t, r_j^t) - u_j(p_j^\star, p_j^\star) = -(b_j + c_j) \cdot (p_j^t - p_j^\star) + c_j \cdot (r_j^t - p_j^\star)$. Therefore, it holds that

$$
\left\|\mathbf{u}_{-i}\big(\mathbf{p}_{-i}^t, \mathbf{r}_{-i}^t\big) - \mathbf{u}_{-i}\big(\mathbf{p}_{-i}^\star, \mathbf{p}_{-i}^\star\big)\right\|^2 \leq 2\max_{j \neq i}\{(b_j + c_j)^2\}\left\|\mathbf{p}_{-i}^t - \mathbf{p}_{-i}^\star\right\|^2 + 2\max_{j \neq i}\{c_j^2\}\left\|\mathbf{r}_{-i}^t - \mathbf{p}_{-i}^\star\right\|^2.
$$

Combining Eqs. (F.6) and (F.7), we have that $R_i^t \leq h_i\left(p_i^{B,t} - p_i^\star\right)^2 + h_i\left(p_i^\star - p_i^t\right)^2$, where recall that h_i is a constant defined in Eq. (F.5). Then, substituting the term $\left(p_i^{B,t} - p_i^\star\right)^2$ with its bound in Eq. (F.8), R_i^t evolves as

$$
\begin{aligned}
R_i^t \leq & \frac{2h_i c_i^2 \cdot (r_i^t - p_i^\star)^2}{(b_i + c_i)^2} + 4h_i \overline{p}^2 \left(\max_{j\neq i}\left\{(b_j + c_j)^2\right\} \left\|\mathbf{p}_{-i}^t - \mathbf{p}_{-i}^\star\right\|^2 + \max_{j\neq i}\left\{c_j^2\right\}\left\|\mathbf{r}_{-i}^t - \mathbf{p}_{-i}^\star\right\|^2 \right) + h_i\left(p_i^\star - p_i^t\right)^2 \\
\leq & 2h_i \cdot \max\left\{ \left(\frac{c_i}{b_i + c_i}\right)^2,\ 2\overline{p}^2 \max_{j\neq i}\left\{c_j^2\right\} \right\} \left\|\mathbf{p}^\star - \mathbf{r}^t\right\|^2 + h_i \cdot \max\left\{4\overline{p}^2 \max_{j\neq i}\left\{(b_j + c_j)^2\right\}, 1\right\} \left\|\mathbf{p}^\star - \mathbf{p}^t\right\|^2,
\end{aligned}
\tag{F.9}
$$

where we apply the basic inequality $k_1 x^2 + k_2 y^2 \leq \max\{k_1, k_2\}(x^2 + y^2)$ in the last step.

When the step-sizes $\{\eta^t\}_{t\geq 0}$ are non-increasing with $\lim_{t\to\infty}\eta^t = 0$ and $\sum_{t=0}^\infty \eta^t = \infty$, we have from Theorem 1 that $\mathbf{p}^t \to \mathbf{p}^\star$ and $\mathbf{r}^t \to \mathbf{p}^\star$. Hence, the dynamic regret grows in a sublinear rate, which completes the proof of Eq. (15).

When the step-sizes are specified as $\eta^t = \frac{C_\eta \log(t+1)}{t+1}$ for $t \geq 2$, we can further quantify the regret using the convergence rate in Theorem 2. In the case of $t > \widehat{T}_1$, where $\widehat{T}_1 := \max\{2T_1, 10\}$ and T_1 can be found in Table EC.1, we can bound terms $\|\mathbf{p}^\star - \mathbf{p}^t\|$ and $\|\mathbf{p}^\star - \mathbf{r}^t\|$ by Eq. (13). Then, Eq. (F.9) becomes that for all $t > \widehat{T}_1$,

$$
R_i^t \leq \left[2h_i C_r \cdot \max\left\{ \left(\frac{c_i}{b_i + c_i}\right)^2,\ 2\overline{p}^2 \max_{j\neq i}\left\{c_j^2\right\} \right\} + h_i C_p \cdot \max\left\{4\overline{p}^2 \max_{j\neq i}\left\{(b_j + c_j)^2\right\}, 1\right\} \right] \left(\frac{\log t}{t}\right)^2.
\tag{F.10}
$$

We use $C_{R,i}$ to denote the multiple of $(\log t / t)^2$ in Eq. (F.10), i.e.,

$$
C_{R,i} := 2h_i C_r \cdot \max\left\{ \left(\frac{c_i}{b_i + c_i}\right)^2,\ 2\overline{p}^2 \max_{j\neq i}\left\{c_j^2\right\} \right\} + h_i C_p \cdot \max\left\{4\overline{p}^2 \max_{j\neq i}\left\{(b_j + c_j)^2\right\}, 1\right\}.
\tag{F.11}
$$

In the case of $t \leq \widehat{T}_1$, we use the plain bound on R_i^t, i.e.,

$$
R_i^t \leq \max_{p_i \in \mathcal{P}}\left\{\Pi_i\left((p_i, \mathbf{p}_{-i}^t), \mathbf{r}^t\right)\right\} \leq \overline{p}.
\tag{F.12}
$$

Finally, combining Eqs. (F.10) and (F.12), we are ready to derive the regret bound as follows

$$
\begin{aligned}
\text{D-Regret}_i(T) &= \sum_{t=1}^{\widehat{T}_1} R_i^t + \sum_{t=\widehat{T}_1+1}^{T} R_i^t \\
&\leq \overline{p}\cdot\widehat{T}_1 + C_{R,i} \sum_{t=\widehat{T}_1+1}^{T} \left(\frac{\log t}{t}\right)^2 \\
&\leq \overline{p}\cdot\widehat{T}_1 + C_{R,i} \int_1^\infty \left(\frac{\log t}{t}\right)^2 dt \\
&= \overline{p}\cdot\widehat{T}_1 + 2C_{R,i}, \qquad \forall T \geq \widehat{T}_1,\ \forall i \in N.
\end{aligned}
\tag{F.13}
$$

Since Eq. (F.13) has already upper-bounded the regrets during the first $\widehat{T}_1 = \max\{2T_1, 10\}$ periods, this upper bound also holds for all $1 \leq T < \widehat{T}_1$, which completes the proof of Theorem 3. $\square$

Appendix G Proof of Theorem 4

Theorem 4 **(Restated)**. *In the loss-averse scenario, let the step-sizes $\{\eta^t\}_{t\geq 0}$ be a non-increasing sequence such that $\lim_{t\to\infty}\eta^t = 0$ and $\sum_{t=0}^{\infty}\eta^t = \infty$. Then, for any reasonably small $\epsilon > 0$, the price and reference price paths generated by Algorithm 2 with the step-sizes $\{\eta^t\}_{t\geq 0}$ and threshold ϵ converge to a $\tilde{C}_\kappa\epsilon$-SNE, where constant $\tilde{C}_\kappa$ is explicitly defined in Table EC.2.*

Proof of Theorem 4. As we have mentioned, instead of directly working on $\widetilde{\kappa}(\mathbf{p},\mathbf{r})$, we will leverage the surrogate metric defined in Eq. (21), i.e., $\widetilde{\kappa}(\mathbf{p}) = \sum_{i\in N}\mathtt{dist}\big(0,\mathtt{Hull}\big\{G_i^-(\mathbf{p},\mathbf{p}),G_i^+(\mathbf{p},\mathbf{p})\big\}\big)$, where $G_i^-(\mathbf{p},\mathbf{r})$ and $G_i^+(\mathbf{p},\mathbf{r})$ are the scaled true/virtual derivatives defined as

$$G_i^-(\mathbf{p},\mathbf{r}) := \frac{1}{(b_i+c_i^-)p_i} + d_i(\mathbf{p},\mathbf{r}) - 1, \quad G_i^+(\mathbf{p},\mathbf{r}) := \frac{1}{(b_i+c_i^+)p_i} + d_i(\mathbf{p},\mathbf{r}) - 1. \tag{G.1}$$

In the loss-averse scenario, it is clear that $G_i^-(\mathbf{p},\mathbf{r}) \leq G_i^+(\mathbf{p},\mathbf{r})$ for all $i\in N$ with the equality only holds true if $c_i^- = c_i^+$, i.e., the consumer is loss-neutral towards this specific product. Below, we show that $\lim_{t\to\infty}\widetilde{\kappa}(\mathbf{p}^t) = \mathcal{O}(\epsilon)$, where ϵ is the pre-specified threshold in Algorithm 2.

Since $D_i^-(\mathbf{p},\mathbf{r}) = (b_i+c_i^-)\cdot G_i^-(\mathbf{p},\mathbf{r})$ and $D_i^+(\mathbf{p},\mathbf{r}) = (b_i+c_i^+)\cdot G_i^+(\mathbf{p},\mathbf{r})$, the pausing criteria in line 5 of Algorithm 2 is equivalent to $G_i^+(\mathbf{p}^t,\mathbf{r}^t) > -\epsilon/(b_i+c_i^+)$ and $G_i^-(\mathbf{p}^t,\mathbf{r}^t) < \epsilon/(b_i+c_i^-)$. Hence, we can classify the relation between p_i^{t+1} and p_i^t into the following three possibilities:

- If $G_i^+(\mathbf{p}^t,\mathbf{r}^t) \geq G_i^-(\mathbf{p}^t,\mathbf{r}^t) \geq \epsilon/(b_i+c_i^-)$, then

$$p_i^{t+1} = \mathtt{Proj}_{\mathcal{P}}\left(p_i^t + \eta^t\cdot\left(w_i^t\cdot D_i^{t,+} + (1-w_i^t)\cdot D_i^{t,-}\right)\right) \geq \mathtt{Proj}_{\mathcal{P}}\left(p_i^t + \eta^t\cdot D_i^{t,-}\right) \geq p_i^t, \tag{G.2}$$

where we recall that $D_i^{t,+} = D_i^+(\mathbf{p}^t,\mathbf{r}^t) \geq D_i^-(\mathbf{p}^t,\mathbf{r}^t) = D_i^{t,-}$ by Eq. (18).

- If $G_i^-(\mathbf{p}^t,\mathbf{r}^t) \leq G_i^+(\mathbf{p}^t,\mathbf{r}^t) \leq -\epsilon/(b_i+c_i^+)$, then

$$p_i^{t+1} = \mathtt{Proj}_{\mathcal{P}}\left(p_i^t + \eta^t\cdot\left(w_i^t\cdot D_i^{t,+} + (1-w_i^t)\cdot D_i^{t,-}\right)\right) \leq \mathtt{Proj}_{\mathcal{P}}\left(p_i^t + \eta^t\cdot D_i^{t,+}\right) \leq p_i^t. \tag{G.3}$$

- Otherwise, we must have $G_i^+(\mathbf{p}^t,\mathbf{r}^t) > -\epsilon/(b_i+c_i^+)$ and $G_i^-(\mathbf{p}^t,\mathbf{r}^t) < \epsilon/(b_i+c_i^-)$, and thereby the pausing criterion is triggered, i.e., $p_i^{t+1} = p_i^t$.

We remark that it is also possible for $p_i^{t+1} = p_i^t$ in the first two cases. This happens when p_i^t is on the boundary of $\mathcal{P} = [\underline{p},\overline{p}]$, and the price update is deprecated by the projection operation.

By Lemma EC.2, under the non-increasing step-sizes $\{\eta^t\}_{t\geq 0}$ with $\lim_{t\to\infty}\eta^t = 0$, there must exist T_ϵ such that

$$\max_{i\in N}\left\{(b_i+c_i^-)\cdot\tilde{\ell}_{r,i}\left\|\mathbf{p}^t-\mathbf{r}^t\right\|, (b_i+c_i^-)\cdot\eta^t\tilde{\ell}_{p,i}\tilde{M}_G\sqrt{\sum_{k\in N}(b_k+c_k^-)^2}\right\} \leq \frac{\epsilon}{2}, \quad \forall t\geq T_\epsilon, \tag{G.4}$$

where the definitions of $\tilde{\ell}_{r,i}$, $\tilde{\ell}_{p,i}$, and $\tilde{M}_G$ can be found in Table EC.2. Now, for any $t\geq T_\epsilon$, consider the following separation of N:

$$N_-^t := \left\{i\in N | p_i^t < p_i^{t+1}\right\}, \quad N_+^t := \left\{i\in N | p_i^t > p_i^{t+1}\right\}, \quad N_c^t := N\setminus\left(N_-^t\cup N_+^t\right). \tag{G.5}$$

We claim that if N_-^t is not empty, then for any $i \in N_-^t$, it holds that $G_i^-(\mathbf{p}^t, \mathbf{p}^t) \geq \frac{\epsilon}{2(b_i + c_i^-)}$ and $G_i^-(\mathbf{p}^{t+1}, \mathbf{p}^{t+1}) \geq 0$. Specifically, since $p_i^t < p_i^{t+1}$ for any $i \in N_-^t$, we have $G_i^-(\mathbf{p}^t, \mathbf{r}^t) \geq \epsilon/(b_i + c_i^-)$, and thereby

$$\begin{aligned} G_i^-(\mathbf{p}^t, \mathbf{p}^t) &= G_i^-(\mathbf{p}^t, \mathbf{r}^t) + \left[G_i^-(\mathbf{p}^t, \mathbf{p}^t) - G_i^-(\mathbf{p}^t, \mathbf{r}^t)\right] \\ &\geq \frac{\epsilon}{b_i + c_i^-} - \tilde{\ell}_{r,i} \left\|\mathbf{p}^t - \mathbf{r}^t\right\| \\ &\geq \frac{\epsilon}{2(b_i + c_i^-)}, \end{aligned} \tag{G.6}$$

where we use the Lipschitz continuity of $G_i^-(\mathbf{p}, \cdot)$ from Lemma EC.7 and the choice of T_ϵ in Eq. (G.4). In addition, using the Lipschitz continuity of $G_i^-(\mathbf{p}, \mathbf{p})$ with respect to $\mathbf{p}$ from Lemma EC.7, we derive that

$$\begin{aligned} G_i^-(\mathbf{p}^{t+1}, \mathbf{p}^{t+1}) &= G_i^-(\mathbf{p}^t, \mathbf{p}^t) + \left[G_i^-(\mathbf{p}^{t+1}, \mathbf{p}^{t+1}) - G_i^-(\mathbf{p}^t, \mathbf{p}^t)\right] \\ &\geq \frac{\epsilon}{2(b_i + c_i^-)} - \tilde{\ell}_{p,i} \left\|\mathbf{p}^{t+1} - \mathbf{p}^t\right\| \\ &\overset{(\Delta_1)}{\geq} \frac{\epsilon}{2(b_i + c_i^-)} - \eta^t \tilde{\ell}_{p,i} \sqrt{\sum_{k \in N} (D_k^t)^2} \\ &\overset{(\Delta_2)}{\geq} \frac{\epsilon}{2(b_i + c_i^-)} - \eta^t \tilde{\ell}_{p,i} \sqrt{\sum_{k \in N} (b_k + c_k^-)^2 (\tilde{M}_G)^2} \\ &\geq 0, \end{aligned} \tag{G.7}$$

where step (Δ_1) follows from the price update rule, and inequality (Δ_2) is because $|D_k^t| \leq \max\left\{(b_k + c_k^-)\left|G_k^-(\mathbf{p}^t, \mathbf{r}^t)\right|, (b_k + c_k^+)\left|G_k^+(\mathbf{p}^t, \mathbf{r}^t)\right|\right\} \leq (b_k + c_k^-)\tilde{M}_G$, using the upper bound $\left|G_i^-(\mathbf{p}, \mathbf{r})\right| \leq \tilde{M}_G$ from Lemma EC.7. Hence, combining Eqs. (G.6) and (G.7) with the fact $G_i^-(\mathbf{p}, \mathbf{r}) \leq G_i^+(\mathbf{p}, \mathbf{r})$, we have that

$$\mathtt{dist}\Big(0, \mathtt{Hull}\big\{G_i^-(\mathbf{p}^{t'}, \mathbf{p}^{t'}), G_i^+(\mathbf{p}^{t'}, \mathbf{p}^{t'})\big\}\Big) = G_i^-(\mathbf{p}^{t'}, \mathbf{p}^{t'}), \quad \forall t' \in \{t, t+1\},\ \forall i \in N_-^t. \tag{G.8}$$

Similarly, we can show that if $N_+^t \neq \emptyset$, then for any $i \in N_+^t$, it holds that $G_i^+(\mathbf{p}^t, \mathbf{p}^t) \leq -\frac{\epsilon}{2(b_i + c_i^+)}$ and $G_i^+(\mathbf{p}^{t+1}, \mathbf{p}^{t+1}) \leq 0$, and thus

$$\mathtt{dist}\Big(0, \mathtt{Hull}\big\{G_i^-(\mathbf{p}^{t'}, \mathbf{p}^{t'}), G_i^+(\mathbf{p}^{t'}, \mathbf{p}^{t'})\big\}\Big) = -G_i^+(\mathbf{p}^{t'}, \mathbf{p}^{t'}), \quad \forall t' \in \{t, t+1\},\ \forall i \in N_+^t. \tag{G.9}$$

Next, we assess the improvement $\widetilde{\kappa}(\mathbf{p}^{t+1})-\widetilde{\kappa}(\mathbf{p}^t)$ by breaking down the summation over N using the definitions of N_-^t, N_+^t, and N_c^t in Eq. (G.5).

$$\begin{aligned}
&\widetilde{\kappa}(\mathbf{p}^{t+1})-\widetilde{\kappa}(\mathbf{p}^t)\\
&=\sum_{i\in N}\Big[\mathtt{dist}\Big(0,\mathtt{Hull}\big\{G_i^-(\mathbf{p}^{t+1},\mathbf{p}^{t+1}),G_i^+(\mathbf{p}^{t+1},\mathbf{p}^{t+1})\big\}\Big)-\mathtt{dist}\Big(0,\mathtt{Hull}\big\{G_i^-(\mathbf{p}^{t},\mathbf{p}^{t}),G_i^+(\mathbf{p}^{t},\mathbf{p}^{t})\big\}\Big)\Big]\\
&\overset{(\Delta)}{=}\sum_{i\in N_-^t}\left[G_i^-(\mathbf{p}^{t+1},\mathbf{p}^{t+1})-G_i^-(\mathbf{p}^{t},\mathbf{p}^{t})\right]+\sum_{i\in N_+^t}\left[G_i^+(\mathbf{p}^{t},\mathbf{p}^{t})-G_i^+(\mathbf{p}^{t+1},\mathbf{p}^{t+1})\right]\\
&\quad+\sum_{i\in N_c^t}\Big[\mathtt{dist}\Big(0,\mathtt{Hull}\big\{G_i^-(\mathbf{p}^{t+1},\mathbf{p}^{t+1}),G_i^+(\mathbf{p}^{t+1},\mathbf{p}^{t+1})\big\}\Big)-\mathtt{dist}\Big(0,\mathtt{Hull}\big\{G_i^-(\mathbf{p}^{t},\mathbf{p}^{t}),G_i^+(\mathbf{p}^{t},\mathbf{p}^{t})\big\}\Big)\Big]\\
&=\sum_{i\in N_-^t}\left[\frac{1}{b_i+c_i^-}\left(\frac{1}{p_i^{t+1}}-\frac{1}{p_i^t}\right)+d_i(\mathbf{p}^{t+1},\mathbf{p}^{t+1})-d_i(\mathbf{p}^{t},\mathbf{p}^{t})\right]\\
&\quad+\sum_{i\in N_+^t}\left[\frac{1}{b_i+c_i^+}\left(\frac{1}{p_i^{t}}-\frac{1}{p_i^{t+1}}\right)+d_i(\mathbf{p}^{t},\mathbf{p}^{t})-d_i(\mathbf{p}^{t+1},\mathbf{p}^{t+1})\right]\\
&\quad+\sum_{i\in N_c^t}\Big[\mathtt{dist}\Big(0,\mathtt{Hull}\big\{G_i^-(\mathbf{p}^{t+1},\mathbf{p}^{t+1}),G_i^+(\mathbf{p}^{t+1},\mathbf{p}^{t+1})\big\}\Big)-\mathtt{dist}\Big(0,\mathtt{Hull}\big\{G_i^-(\mathbf{p}^{t},\mathbf{p}^{t}),G_i^+(\mathbf{p}^{t},\mathbf{p}^{t})\big\}\Big)\Big],
\end{aligned}\tag{G.10}$$

where step (Δ) uses the equalities in Eqs. (G.8) and (G.9). Although the right-hand side of Eq. (G.10) seems involved due to the presence of the summation over N_c^t, we make the following key observation: since $p_i^{t+1}=p_i^t$ for all $i\in N_c^t$, the difference between the two distance terms only arises from the change of the demand function (see the definitions in Eq. (G.1)). Based on the relative size of $d_i(\mathbf{p}^{t+1},\mathbf{p}^{t+1})$ and $d_i(\mathbf{p}^{t},\mathbf{p}^{t})$ for $i\in N_c^t$, we enlarge sets N_-^t and N_+^t as follows:

$$\begin{aligned}
\widehat{N}_-^t &:= N_-^t\cup\{i\in N_c^t|d_i(\mathbf{p}^{t+1},\mathbf{p}^{t+1})-d_i(\mathbf{p}^{t},\mathbf{p}^{t})\geq 0\},\\
\widehat{N}_+^t &:= N_+^t\cup\{i\in N_c^t|d_i(\mathbf{p}^{t+1},\mathbf{p}^{t+1})-d_i(\mathbf{p}^{t},\mathbf{p}^{t})< 0\}.
\end{aligned}\tag{G.11}$$

By definition, it is clear that $\widehat{N}_-^t\cup\widehat{N}_+^t=N_-^t\cup N_+^t\cup N_c^t=N$. Then, we can further deduce from Eq. (G.10) that

$$\begin{aligned}
&\widetilde{\kappa}(\mathbf{p}^{t+1})-\widetilde{\kappa}(\mathbf{p}^t)\\
&\leq\sum_{i\in N_-^t}\left[\frac{1}{b_i+c_i^-}\left(\frac{1}{p_i^{t+1}}-\frac{1}{p_i^t}\right)+d_i(\mathbf{p}^{t+1},\mathbf{p}^{t+1})-d_i(\mathbf{p}^{t},\mathbf{p}^{t})\right]\\
&\quad+\sum_{i\in N_+^t}\left[\frac{1}{b_i+c_i^+}\left(\frac{1}{p_i^{t}}-\frac{1}{p_i^{t+1}}\right)+d_i(\mathbf{p}^{t},\mathbf{p}^{t})-d_i(\mathbf{p}^{t+1},\mathbf{p}^{t+1})\right]+\sum_{i\in N_c^t}\left|d_i(\mathbf{p}^{t+1},\mathbf{p}^{t+1})-d_i(\mathbf{p}^{t},\mathbf{p}^{t})\right|\\
&=\sum_{i\in N_-^t}\frac{1}{b_i+c_i^-}\left(\frac{1}{p_i^{t+1}}-\frac{1}{p_i^t}\right)+\sum_{i\in N_+^t}\frac{1}{b_i+c_i^+}\left(\frac{1}{p_i^{t}}-\frac{1}{p_i^{t+1}}\right)+\sum_{i\in \widehat{N}_-^t}\left[d_i(\mathbf{p}^{t+1},\mathbf{p}^{t+1})-d_i(\mathbf{p}^{t},\mathbf{p}^{t})\right]\\
&\quad+\sum_{i\in \widehat{N}_+^t}\left[d_i(\mathbf{p}^{t},\mathbf{p}^{t})-d_i(\mathbf{p}^{t+1},\mathbf{p}^{t+1})\right].
\end{aligned}\tag{G.12}$$

Since $p_i^{t+1} \geq p_i^t$ for all $i \in \widehat{N}_-^t$ and $p_i^{t+1} \leq p_i^t$ for all $i \in \widehat{N}_+^t$, it holds that

$$\begin{aligned}\sum_{i\in\widehat{N}_-^t} d_i(\mathbf{p}^{t+1},\mathbf{p}^{t+1}) &= \frac{\sum_{i\in\widehat{N}_-^t}\exp\left(u_i(p_i^{t+1},p_i^{t+1})\right)}{1+\sum_{i\in\widehat{N}_-^t}\exp\left(u_i(p_i^{t+1},p_i^{t+1})\right)+\sum_{j\in\widehat{N}_+^t}\exp\left(u_j(p_j^{t+1},p_j^{t+1})\right)} \\ &\leq \frac{\sum_{i\in\widehat{N}_-^t}\exp\left(u_i(p_i^{t},p_i^{t})\right)}{1+\sum_{i\in\widehat{N}_-^t}\exp\left(u_i(p_i^{t},p_i^{t})\right)+\sum_{j\in\widehat{N}_+^t}\exp\left(u_j(p_j^{t+1},p_j^{t+1})\right)} \\ &\leq \frac{\sum_{i\in\widehat{N}_-^t}\exp\left(u_i(p_i^{t},p_i^{t})\right)}{1+\sum_{i\in\widehat{N}_-^t}\exp\left(u_i(p_i^{t},p_i^{t})\right)+\sum_{j\in\widehat{N}_+^t}\exp\left(u_j(p_j^{t},p_j^{t})\right)} = \sum_{i\in\widehat{N}_-^t} d_i(\mathbf{p}^t,\mathbf{p}^t),\end{aligned} \tag{G.13}$$

where we use the fact that $u_i(p_i,p_i) = a_i - b_i p_i$ is monotone decreasing p_i. Similarly, we have $\sum_{i\in\widehat{N}_+^t} d_i(\mathbf{p}^{t+1},\mathbf{p}^{t+1}) \geq \sum_{i\in\widehat{N}_+^t} d_i(\mathbf{p}^t,\mathbf{p}^t)$. These two relations, together with Eq. (G.12), imply that

$$\begin{aligned}\widetilde{\kappa}(\mathbf{p}^{t+1}) - \widetilde{\kappa}(\mathbf{p}^t) &\leq \sum_{i\in N_-^t}\frac{1}{b_i+c_i^-}\left(\frac{1}{p_i^{t+1}}-\frac{1}{p_i^t}\right) + \sum_{i\in N_+^t}\frac{1}{b_i+c_i^+}\left(\frac{1}{p_i^t}-\frac{1}{p_i^{t+1}}\right) \\ &\overset{(\Delta)}{=} -\sum_{i\in N_-^t}\frac{1}{p_i^t\cdot p_i^{t+1}}\frac{\left|p_i^{t+1}-p_i^t\right|}{b_i+c_i^-} - \sum_{i\in N_+^t}\frac{1}{p_i^t\cdot p_i^{t+1}}\frac{\left|p_i^{t+1}-p_i^t\right|}{b_i+c_i^-} \\ &\leq \frac{-1}{\overline{p}^2}\left(\sum_{i\in N_-^t}\frac{\left|p_i^{t+1}-p_i^t\right|}{b_i+c_i^-} + \sum_{i\in N_+^t}\frac{\left|p_i^{t+1}-p_i^t\right|}{b_i+c_i^-}\right) \\ &\leq \frac{-1}{\overline{p}^2}\max\left\{\max_{i\in N_-^t}\left\{\frac{\left|p_i^{t+1}-p_i^t\right|}{b_i+c_i^-}\right\},\max_{i\in N_+^t}\left\{\frac{\left|p_i^{t+1}-p_i^t\right|}{b_i+c_i^-}\right\}\right\},\end{aligned} \tag{G.14}$$

where we apply the definitions of N_-^t and N_+^t from Eq. (G.5) in (Δ). It is worth noting that, when both N_-^t and N_+^t are empty sets, i.e., $N_c^t = N$, then Eq. (G.14) reduces to $\widetilde{\kappa}(\mathbf{p}^{t+1}) - \widetilde{\kappa}(\mathbf{p}^t) = 0$, since $N_c^t = N$ implies $\mathbf{p}^{t+1} = \mathbf{p}^t$. Therefore, we find that the sequence $\{\widetilde{\kappa}(\mathbf{p}^t)\}_{t\geq T_\epsilon}$ is non-increasing.

We first consider the situation $N_c^t = N$ and demonstrate that when ϵ is reasonably small, $N_c^t = N$ implies that $\widetilde{\kappa}(\mathbf{p}^t) = \mathcal{O}(\epsilon)$. Note that $p_i^{t+1} = p_i^t$ can only happen for the following two reasons:

- The pausing criterion is triggered for product i, i.e., $G_i^+(\mathbf{p}^t,\mathbf{r}^t) > -\epsilon/(b_i+c_i^+)$ and $G_i^-(\mathbf{p}^t,\mathbf{r}^t) < \epsilon/(b_i+c_i^-)$, and thus no price update occurs.
- The price p_i^t is at the boundary of the feasible range $\mathcal{P}$, i.e., $p_i^t = \overline{p}$ or $p_i^t = \underline{p}$, and the price update is towards the outside direction, which is then deprecated by the projection operator.

Below, we show that for any $t \geq T_\epsilon$ and any reasonably small ϵ, the second scenario cannot happen when $\mathbf{p}^{t+1} = \mathbf{p}^t$. We argue from the reverse direction: if there exists $i_0 \in N$ with $p_{i_0}^t = \overline{p}$, then we must have $\mathbf{p}^{t+1} \neq \mathbf{p}^t$ (the case when $p_{i_0}^t = \underline{p}$ is equivalent). Denote $\xi_{i_0} := \min\left\{\overline{p} - p_{i_0}^\star | \mathbf{p}^\star \in \mathcal{S}\right\}$, i.e., the minimum distance between the SNE set $\mathcal{S}$ and the price upper bound in the i_0-th dimension. We consider the following new separation of N based on $\mathbf{p}^t$:

$$\tilde{N}_-^t := \left\{i\in N | G_i^-(\mathbf{p}^t,\mathbf{p}^t) > 0\right\}, \quad \tilde{N}_+^t := \left\{i\in N | G_i^+(\mathbf{p}^t,\mathbf{p}^t) < 0\right\}, \quad \tilde{N}_c^t = N\backslash\left(\tilde{N}_-^t\cup\tilde{N}_+^t\right). \tag{G.15}$$

Equivalently, we can write $\tilde{N}_c^t = \left\{i \in N | G_i^-(\mathbf{p}^t, \mathbf{p}^t) \le 0 \le G_i^+(\mathbf{p}^t, \mathbf{p}^t)\right\}$. Then, we define the pseudo sensitivities $(\tilde{c}_i)_{i\in N}$ as follows: for $i \in \tilde{N}_-^t$, let $\tilde{c}_i = c_i^-$; for $i \in \tilde{N}_+^t$, let $\tilde{c}_i = c_i^+$; for $i \in \tilde{N}_c^t$, let $\tilde{c}_i$ be the unique value that satisfies

$$\frac{1}{(b_i + \tilde{c}_i)p_i^t} + d_i(\mathbf{p}^t, \mathbf{p}^t) - 1 = 0. \tag{G.16}$$

By the definitions of scaled derivatives in Eq. (G.1), since $G_i^-(\mathbf{p}^t, \mathbf{p}^t) \le 0 \le G_i^+(\mathbf{p}^t, \mathbf{p}^t)$ for all $i \in \tilde{N}_c^t$, such a $\tilde{c}_i$ must exist and $\tilde{c}_i \in [c_i^+, c_i^-]$. Given $(\tilde{c}_i)_{i\in N}$, let $\tilde{\mathbf{p}}^\star \in \mathcal{S}$ be the unique SNE that satisfies

$$\tilde{p}_i^\star = \frac{1}{(b_i + \tilde{c}_i)\cdot(1 - d_i(\tilde{\mathbf{p}}^\star, \tilde{\mathbf{p}}^\star))}, \quad \forall i \in N, \tag{G.17}$$

whose existence is guaranteed by the expression of $\mathcal{S}$ (see Eq. (6) and the proofs below Eq. (C.3)). Next, for every $i \in N$, we further introduce that

$$\tilde{G}_i(\mathbf{p}, \mathbf{r}) := \frac{1}{(b_i + \tilde{c}_i)p_i} + d_i(\mathbf{p}, \mathbf{p}) - 1, \quad \tilde{\mathcal{G}}(\mathbf{p}) := \sum_{i\in N} \mathtt{sign}\left(\tilde{p}_i^\star - p_i\right)\tilde{G}_i(\mathbf{p}, \mathbf{p}). \tag{G.18}$$

We note that, since $\tilde{c}_i \in [c_i^+, c_i^-]$, it always holds $G_i^-(\mathbf{p}, \mathbf{r}) \le \tilde{G}_i(\mathbf{p}, \mathbf{r}) \le G_i^+(\mathbf{p}, \mathbf{r})$. By Lemma EC.8, $\tilde{\mathcal{G}}(\mathbf{p}^t)$ satisfies that

$$\tilde{\mathcal{G}}(\mathbf{p}^t) = \sum_{i\in N} \mathtt{sign}\left(\tilde{p}_i^\star - p_i^t\right)\tilde{G}_i(\mathbf{p}^t, \mathbf{p}^t) \ge \frac{1}{\overline{p}\,\|\tilde{\mathbf{p}}^\star\|_\infty} \cdot \sum_{i\in N} \frac{|\tilde{p}_i^\star - p_i^t|}{b_i + \tilde{c}_i} \ge \frac{\xi_{i_0}}{(b_{i_0} + \tilde{c}_{i_0})\overline{p}\,\|\tilde{\mathbf{p}}^\star\|_\infty}, \tag{G.19}$$

where the last inequality holds because $\left|\tilde{p}_{i_0}^\star - p_{i_0}^t\right| = \left|\tilde{p}_{i_0}^\star - \overline{p}\right| \ge \min\left\{\overline{p} - p_{i_0}^\star | \mathbf{p}^\star \in \mathcal{S}\right\} = \xi_{i_0}$. Hence, by the definition of $\tilde{\mathcal{G}}(\mathbf{p})$, if

$$\xi_{i_0} \ge \frac{\left[n(b_{i_0} + \tilde{c}_{i_0})\overline{p}\,\|\tilde{\mathbf{p}}^\star\|_\infty\right]}{\min_{i\in N}\left\{b_i + c_i^+\right\}} \cdot \frac{3\epsilon}{2}, \tag{G.20}$$

we can deduce from the lower bound in Eq. (G.19) that

$$\max_{i\in N}\left\{\mathtt{sign}\left(\tilde{p}_i^\star - p_i^t\right)\tilde{G}_i(\mathbf{p}^t, \mathbf{p}^t)\right\} \ge \frac{1}{n} \cdot \frac{\xi_{i_0}}{(b_{i_0} + \tilde{c}_{i_0})\overline{p}\,\|\tilde{\mathbf{p}}^\star\|_\infty} \ge \frac{3\epsilon}{2\min_{i\in N}\left\{b_i + c_i^+\right\}}. \tag{G.21}$$

Denote $i_1 := \arg\max_{i\in N}\left\{\mathtt{sign}\left(\tilde{p}_i^\star - p_i^t\right)\tilde{G}_i(\mathbf{p}^t, \mathbf{p}^t)\right\}$. Then, Eq. (G.21) implies that

$$\left|\tilde{G}_{i_1}(\mathbf{p}^t, \mathbf{p}^t)\right| \ge \frac{3\epsilon}{2\min_{i\in N}\left\{b_i + c_i^+\right\}} \ge \frac{3\epsilon}{2(b_{i_1} + c_{i_1}^+)}. \tag{G.22}$$

Since $\tilde{G}_i(\mathbf{p}^t, \mathbf{p}^t) = 0$ for all $i \in \tilde{N}_c^t$ by Eq. (G.16), we must have $i_1 \in \tilde{N}_-^t \cup \tilde{N}_+^t$. Now, we prove that $p_{i_1}^{t+1} \neq p_{i_1}^t$. Without loss of generality, suppose $i_1 \in \tilde{N}_-^t$, i.e., $G_{i_1}^-(\mathbf{p}^t, \mathbf{p}^t) > 0$. Then, by the definition of $(\tilde{c}_i)_{i\in N}$ above Eq. (G.16), it follows that $\tilde{G}_{i_1}(\mathbf{p}^t, \mathbf{p}^t) = G_{i_1}^-(\mathbf{p}^t, \mathbf{p}^t) > 0$. Hence, we must have $\mathtt{sign}\left(\tilde{p}_{i_1}^\star - p_{i_1}^t\right) > 0$, i.e., $p_{i_1}^t < \tilde{p}_{i_1}^\star$, which implies that $G_{i_1}^+(\mathbf{p}^t, \mathbf{p}^t) \ge G_{i_1}^-(\mathbf{p}^t, \mathbf{p}^t) \ge \frac{3\epsilon}{2(b_{i_1} + c_{i_1}^+)}$. In the meantime, since $t \ge T_\epsilon$, we deduce that

$$\begin{aligned} G_{i_1}^-(\mathbf{p}^t, \mathbf{r}^t) &= G_{i_1}^-(\mathbf{p}^t, \mathbf{p}^t) + \left[G_{i_1}^-(\mathbf{p}^t, \mathbf{r}^t) - G_{i_1}^-(\mathbf{p}^t, \mathbf{p}^t)\right] \\ &\ge G_{i_1}^-(\mathbf{p}^t, \mathbf{p}^t) - \tilde{\ell}_{r,i_1}\left\|\mathbf{p}^t - \mathbf{r}^t\right\| \\ &\ge \frac{3\epsilon}{2(b_{i_1} + c_{i_1}^+)} - \frac{\epsilon}{2(b_{i_1} + c_{i_1}^-)} \\ &\ge \frac{\epsilon}{b_{i_1} + c_{i_1}^-}, \end{aligned} \tag{G.23}$$

where the inequalities follow from the Lipschitz continuity of $G_{i_1}^-(\mathbf{p},\cdot)$ in Lemma EC.7 and the definition of T_ϵ in Eq. (G.4). Therefore, we conclude that the update $p_{i_1}^{t+1} \leftarrow \texttt{Proj}_{\mathcal{P}}\left(p_{i_1}^t + \eta^t \cdot D_{i_1}^t\right)$ is towards the SNE price $\tilde{p}_{i_1}^\star$, i.e., $\texttt{sign}\left(\tilde{p}_{i_1}^\star - p_{i_1}^t\right) = \texttt{sign}\left(p_{i_1}^{t+1} - p_{i_1}^t\right)$, and thereby $p_{i_1}^{t+1} \neq p_i^t$.

Based on the arguments above, we can provide a sufficient condition for the size of ϵ such that $N_c^t = N$ always implies that the pausing criteria are triggered for all products. We define that

$$\xi_i = \min\left\{\overline{p} - p_i, p_i - \underline{p} | \mathbf{p} \in \mathcal{S}\right\}, \quad \forall i \in N, \tag{G.24}$$

which stands for the minimum distance from $\mathcal{S}$ to the boundaries of the feasible region $\mathcal{P}$ in the i-th dimension. Then, the derivations in Eqs. (G.19) to (G.23) imply that it is sufficient to have

$$\epsilon \leq \frac{2\min_{i\in N}\left\{b_i + c_i^+\right\}}{3n\overline{p}\cdot \max_{\mathbf{p}\in\mathcal{S}}\left\{\|\mathbf{p}\|_\infty\right\}} \cdot \min_{i\in N}\left\{\frac{\xi_i}{b_i + c_i^-}\right\}. \tag{G.25}$$

When ϵ satisfies Eq. (G.25), then if there exists any product $i_0 \in N$ with $p_{i_0}^t = \overline{p}$ or $p_{i_0}^t = \underline{p}$, we can always follow the derivations in Eqs. (G.19) to (G.23) to show that $\mathbf{p}^{t+1}$ is different from $\mathbf{p}^t$ for at least one product. We remark that according to Proposition 1, the value of $\max_{\mathbf{p}\in\mathcal{S}}\left\{\|\mathbf{p}\|_\infty\right\}$ has roughly linear (or inversely linear) dependence on problem parameters. Hence, even when $\overline{p}$ is excessively large, the condition in Eq. (G.25) would not be restrictive because $\lim_{\overline{p}\to\infty} \xi_i/\overline{p} = 1$.

By far, from Eqs. (G.15) to (G.25), we have demonstrated that for reasonably small ϵ and $t \geq T_\epsilon$, $N_c^t = N$ will happen only if the pausing criteria are triggered for all products. Below, suppose $N_c^{t_0} = N$ for some $t_0 \geq T_\epsilon$, we show that $\mathbf{p}^{t_0}$ is already close to the set of SNEs. By the definition of $\widetilde{\kappa}(\cdot)$ from Eq. (21), it follows that

$$\begin{aligned}
\widetilde{\kappa}(\mathbf{p}^{t_0}) &= \sum_{i\in N} \texttt{dist}\Big(0, \texttt{Hull}\big\{G_i^-(\mathbf{p}^{t_0}, \mathbf{p}^{t_0}), G_i^+(\mathbf{p}^{t_0}, \mathbf{p}^{t_0})\big\}\Big) \\
&\overset{(\Delta_1)}{\leq} \sum_{i\in N}\left[\texttt{dist}\Big(0, \texttt{Hull}\big\{G_i^-(\mathbf{p}^{t_0}, \mathbf{r}^{t_0}), G_i^+(\mathbf{p}^{t_0}, \mathbf{r}^{t_0})\big\}\Big) + \frac{\epsilon}{2(b_i + c_i^-)}\right] \\
&\overset{(\Delta_2)}{\leq} \sum_{i\in N}\left(\frac{\epsilon}{b_i + c_i^+} + \frac{\epsilon}{2(b_i + c_i^-)}\right) \\
&\leq \left[\sum_{i\in N}\frac{3}{2(b_i + c_i^+)}\right]\epsilon,
\end{aligned} \tag{G.26}$$

where inequality (Δ_1) applies the Lipschitz properties from Lemma EC.7 and follows the same derivations as Eqs. (G.6) and (G.23). Step (Δ_2) leverages the presumption that pausing criteria are triggered for all products, i.e., $G_i^-(\mathbf{p}^{t_0}, \mathbf{r}^{t_0}) < \epsilon/(b_i + c_i^-)$ and $G_i^+(\mathbf{p}^{t_0}, \mathbf{r}^{t_0}) > -\epsilon/(b_i + c_i^+)$, and since $c_i^- \geq c_i^+$, we have

$$\texttt{dist}\Big(0, \texttt{Hull}\big\{G_i^-(\mathbf{p}^{t_0}, \mathbf{r}^{t_0}), G_i^+(\mathbf{p}^{t_0}, \mathbf{r}^{t_0})\big\}\Big) \leq \frac{\epsilon}{b_i + c_i^+}. \tag{G.27}$$

Therefore, since $\{\widetilde{\kappa}(\mathbf{p}^t)\}_{t\geq T_\epsilon}$ is non-increasing and $\lim_{t\to\infty}\|\mathbf{p}^t-\mathbf{r}^t\|=0$, we conclude that

$$\begin{aligned}
\lim_{t\to\infty}\widetilde{\kappa}(\mathbf{p}^t,\mathbf{r}^t) &= \lim_{t\to\infty}\left[\|\mathbf{p}^t-\mathbf{r}^t\|+\sum_{i\in N}\mathtt{dist}\Big(0,\mathtt{Hull}\big\{D_i^-(\mathbf{p}^t,\mathbf{r}^t),D_i^+(\mathbf{p}^t,\mathbf{r}^t)\big\}\Big)\right]\\
&= \lim_{t\to\infty}\left[\sum_{i\in N}\mathtt{dist}\Big(0,\mathtt{Hull}\big\{D_i^-(\mathbf{p}^t,\mathbf{p}^t),D_i^+(\mathbf{p}^t,\mathbf{p}^t)\big\}\Big)\right]\\
&\leq \lim_{t\to\infty}\left[\max_{k\in N}\left\{b_k+c_k^-\right\}\cdot\sum_{i\in N}\mathtt{dist}\Big(0,\mathtt{Hull}\big\{G_i^-(\mathbf{p}^t,\mathbf{p}^t),G_i^+(\mathbf{p}^t,\mathbf{p}^t)\big\}\Big)\right]\\
&= \max_{k\in N}\left\{b_k+c_k^-\right\}\cdot\lim_{t\to\infty}\widetilde{\kappa}(\mathbf{p}^t)\\
&\leq \underbrace{\left[\sum_{i\in N}\frac{3\max_{k\in N}\left\{b_k+c_k^-\right\}}{2(b_i+c_i^+)}\right]}_{=:\tilde{C}_\kappa}\epsilon,
\end{aligned}\tag{G.28}$$

where we use the fact that $D_i^\diamond(\mathbf{p},\mathbf{r})=(b_i+c_i^\diamond)G_i^\diamond(\mathbf{p},\mathbf{r})\leq\max_{k\in N}\left\{b_k+c_k^-\right\}\cdot G_i^\diamond(\mathbf{p},\mathbf{r})$ for every $\diamond\in\{+,-\}$ and all $i\in N$. Eq. (G.28) indicates that Algorithm 2 converges to a $\tilde{C}_\kappa\epsilon$-SNE, where we define $\tilde{C}_\kappa:=\sum_{i\in N}3\max_{k\in N}\left\{b_k+c_k^-\right\}/\left[2(b_i+c_i^+)\right]$.

Now, we show by contradiction that there always exists such a period $t_0\geq T_\epsilon$ with $N_c^{t_0}=N$. Suppose $N_-^t\cup N_+^t\neq\emptyset$ for all $t\geq T_\epsilon$. Then, Eq. (G.14) non-trivially holds true throughout the entire horizon of $t\geq T_\epsilon$. For the similar reasoning as Eqs. (G.19) to (G.23), we observe that for every $t\geq T_\epsilon$, there must exist a product $i^t\in N_-^t\cup N_+^t$ such that $\left|p_{i^t}^{t+1}-p_{i^t}^t\right|=\eta^t\left|D_{i^t}^t\right|$, i.e., the projection operation is not in effect. Without loss of generality, assume that $i^t\in N_-^t$. Then, we further deduce from Eq. (G.14) that

$$\begin{aligned}
\widetilde{\kappa}(\mathbf{p}^{t+1})-\widetilde{\kappa}(\mathbf{p}^t) &\leq \frac{-1}{\overline{p}^2}\cdot\frac{\eta^t\left|D_{i^t}^t\right|}{b_{i^t}+c_{i^t}^-}\\
&= \frac{-1}{\overline{p}^2}\cdot\frac{\eta^t\left(w_{i^t}^t\cdot D_{i^t}^{t,+}+(1-w_{i^t}^t)\cdot D_{i^t}^{t,-}\right)}{b_{i^t}+c_{i^t}^-}\\
&\leq \frac{-1}{\overline{p}^2}\cdot\frac{\eta^t D_{i^t}^{t,-}}{b_{i^t}+c_{i^t}^-}\\
&\leq \frac{-\eta^t\epsilon}{\overline{p}^2\cdot(b_{i^t}+c_{i^t}^-)},
\end{aligned}\tag{G.29}$$

where we use the fact that $D_i^{t,+}\geq D_i^{t,-}=(b_i+c_i^-)G_i^-(\mathbf{p}^t,\mathbf{r}^t)\geq\epsilon$ for any $i\in N_-^t$, due to the definition of N_-^t in Eq. (G.5). The same argument applies to the situation where $i^t\in N_+^t$. Thus, by applying a telescoping sum to Eq. (G.29), it follows that for any $t>T_\epsilon$

$$\widetilde{\kappa}(\mathbf{p}^t)\leq\widetilde{\kappa}(\mathbf{p}^{T_\epsilon})-\frac{\left(\sum_{t'=T_\epsilon}^{t-1}\eta^{t'}\right)\epsilon}{\overline{p}^2\cdot\max_{i\in N}\left\{b_i+c_i^-\right\}}.\tag{G.30}$$

Since the step-sizes satisfy $\sum_{t=0}^\infty\eta^t=\infty$, we deduce that $\lim_{t\to\infty}\widetilde{\kappa}(\mathbf{p}^t)=-\infty$, which contradicts with the definition of $\widetilde{\kappa}(\cdot)$. Therefore, there must exist $t_0\geq T_\epsilon$ such that $N_c^{t_0}=N$, which, together with Eq. (G.28), completes the proof of Theorem 4. □

Appendix H Proof of Theorem 5

Theorem 5 **(Restated).** *In the loss-averse scenario, suppose all firms adopt Algorithm 2 with step-sizes $\eta^t = \frac{C_\eta}{\sqrt{t+1}}$ and a reasonably small threshold ϵ, where C_η is some general constant. Then, there exists $\tilde{T} = \mathcal{O}(1/\epsilon^2)$ such that*

$$\widetilde{\kappa}(\mathbf{p}^t, \mathbf{r}^t) \le \left(\frac{1}{2\max_{i\in N}\left\{(b_i + c_i^-)\tilde{\ell}_{r,i}\right\}} + \sum_{i\in N} \frac{2\max_{k\in N}\left\{b_k + c_k^-\right\}}{b_i + c_i^+} \right)\epsilon, \quad \forall t \ge \tilde{T}, \tag{H.1}$$

where $\widetilde{\kappa}(\cdot)$ is defined in Eq. (17), and constants $\tilde{T}$ and $\tilde{\ell}_{r,i}$ are explicitly defined in Table EC.2.

Proof of Theorem 5. Consider the decreasing step-sizes of the form $\eta^t = C_\eta(t+1)^{-\beta}$ with $\beta \in (0,1]$, which satisfy the conditions $\lim_{t\to\infty}\eta^t = 0$ and $\sum_{t=0}^{\infty}\eta^t = \infty$. Hence, all analyses in the proof of Theorem 4 are applicable, and we will use them as the basis for the proof of the convergence rate.

By Lemma EC.2, we have $\|\mathbf{p}^t - \mathbf{r}^t\| \le \eta^t C_{rp,\beta}$ for all $t \ge T_\beta$, where the constants $C_{rp,\beta}$ and T_β are defined in Eq. (L.2). Hence, to satisfy Eq. (G.4), it suffices to choose T_ϵ as the smallest integer greater than T_β such that

$$\eta^{T_\epsilon} \cdot \max_{i\in N}\left\{ (b_i + c_i^-)\cdot\tilde{\ell}_{r,i}C_{rp,\beta},\ (b_i + c_i^-)\cdot\tilde{\ell}_{G,i}\tilde{M}_G\sqrt{\sum_{k\in N}\left(b_k + c_k^-\right)^2} \right\} \le \frac{\epsilon}{2}. \tag{H.2}$$

Denote $\tilde{C}_\beta := \max_{i\in N}\left\{ (b_i + c_i^-)\cdot\tilde{\ell}_{r,i}C_{rp,\beta},\ (b_i + c_i^-)\cdot\tilde{\ell}_{G,i}\tilde{M}_G\sqrt{\sum_{k\in N}\left(b_k + c_k^-\right)^2} \right\}$. Then, since $\eta^{T_\epsilon} = C_\eta(T_\epsilon + 1)^{-\beta}$, we observe from Eq. (H.2) that T_ϵ can be expressed as

$$T_\epsilon := \max\left\{ T_\beta,\ \left\lceil \left(\frac{2C_\eta\tilde{C}_\beta}{\epsilon} \right)^{\frac{1}{\beta}} \right\rceil - 1 \right\}. \tag{H.3}$$

Next, recall the separation introduced in Eq. (G.5). According to Eq. (G.26), if there exists $t_0 \ge T_\epsilon$ with $N_c^{t_0} = N$, i.e., the pausing criteria are triggered for all products, it follows that for all $t \ge t_0$

$$\begin{aligned}
\widetilde{\kappa}(\mathbf{p}^t, \mathbf{r}^t) &= \|\mathbf{p}^t - \mathbf{r}^t\| + \sum_{i\in N}\mathtt{dist}\Big(0, \mathtt{Hull}\big\{D_i^-(\mathbf{p}^t, \mathbf{r}^t), D_i^+(\mathbf{p}^t, \mathbf{r}^t)\big\}\Big) \\
&\overset{(\Delta_1)}{\le} \eta^t C_{rp,\beta} + \max_{k\in N}\left\{b_k + c_k^-\right\}\cdot\sum_{i\in N}\mathtt{dist}\Big(0, \mathtt{Hull}\big\{G_i^-(\mathbf{p}^t, \mathbf{r}^t), G_i^+(\mathbf{p}^t, \mathbf{r}^t)\big\}\Big) \\
&\overset{(\Delta_2)}{\le} \eta^t C_{rp,\beta} + \max_{k\in N}\left\{b_k + c_k^-\right\}\cdot\sum_{i\in N}\left[\mathtt{dist}\Big(0, \mathtt{Hull}\big\{G_i^-(\mathbf{p}^t), G_i^+(\mathbf{p}^t)\big\}\Big) + \frac{\epsilon}{2(b_i + c_i^-)}\right] \\
&= \eta^t C_{rp,\beta} + \max_{k\in N}\left\{b_k + c_k^-\right\}\left(\widetilde{\kappa}(\mathbf{p}^t) + \sum_{i\in N}\frac{\epsilon}{2(b_i + c_i^-)} \right) \\
&\overset{(\Delta_3)}{\le} \frac{\epsilon}{2\max_{i\in N}\left\{(b_i + c_i^-)\tilde{\ell}_{r,i}\right\}} + \left[\sum_{i\in N}\frac{2\max_{k\in N}\left\{b_k + c_k^-\right\}}{b_i + c_i^+}\right]\epsilon \\
&= \left(\frac{1}{2\max_{i\in N}\left\{(b_i + c_i^-)\tilde{\ell}_{r,i}\right\}} + \sum_{i\in N}\frac{2\max_{k\in N}\left\{b_k + c_k^-\right\}}{b_i + c_i^+} \right)\epsilon,
\end{aligned} \tag{H.4}$$

where in step (Δ_1), we leverage the bound $\|\mathbf{p}^t - \mathbf{r}^t\| \le \eta^t C_{rp,\beta}$ for all $t \ge T_\beta$ and use the relation $D_i^\diamond(\mathbf{p},\mathbf{r}) = (b_i + c_i^\diamond) G_i^\diamond(\mathbf{p},\mathbf{r}) \le \max_{k\in N}\left\{b_k + c_k^-\right\} \cdot G_i^\diamond(\mathbf{p},\mathbf{r})$ for every $\diamond \in \{+,-\}$. Next, step (Δ_2) uses the Lipschitz continuity of $G_i^\diamond(\mathbf{p},\cdot)$ in a similar manner as Eq. (G.6). In step (Δ_3), we first apply the upper bound $\eta^t C_{rp,\beta} \le \eta^{T_\epsilon} C_{rp,\beta} \le \epsilon/(2\max_{i\in N}\{(b_i + c_i^-)\tilde{\ell}_{r,i}\})$ from Eq. (H.2). Then, since $\{\widetilde{\kappa}(\mathbf{p}^t)\}_{t\ge T_\epsilon}$ is non-increasing from Eq. (G.14), we have

$$\widetilde{\kappa}(\mathbf{p}^t) \le \widetilde{\kappa}(\mathbf{p}^{t_0}) \le \left[\sum_{i\in N} \frac{3}{2(b_i + c_i^+)}\right]\epsilon, \tag{H.5}$$

where the last inequality follows from Eq. (G.26). Thus, Eq. (H.4) shows that Algorithm 2 converges to an $\left(\frac{1}{2\max_{i\in N}\left\{(b_i+c_i^-)\tilde{\ell}_{r,i}\right\}} + \sum_{i\in N}\frac{2\max_{k\in N}\left\{b_k+c_k^-\right\}}{b_i+c_i^+}\right)\epsilon$-SNE after t_0 iterations.

Therefore, it remains to determine when would such a period t_0 occur. Let $\tilde{T}$ be some integer greater than T_ϵ such that

$$\frac{\epsilon}{\overline{p}^2 \cdot \max_{i\in N}\left\{b_i + c_i^-\right\}}\left(\sum_{t=T_\epsilon}^{\tilde{T}-1}\eta^t\right) > n\tilde{M}_G. \tag{H.6}$$

Since $\sum_{t=0}^\infty \eta^t = \infty$, the existence of $\tilde{T}$ is guaranteed. We observe that $t_0 \le \tilde{T}$ must hold true, otherwise, we can deduce from Eq. (G.30) that

$$\widetilde{\kappa}(\mathbf{p}^{\tilde{T}}) \le \widetilde{\kappa}(\mathbf{p}^{T_\epsilon}) - \frac{\epsilon}{\overline{p}^2 \cdot \max_{i\in N}\left\{b_i + c_i^-\right\}}\left(\sum_{t=T_\epsilon}^{\tilde{T}-1}\eta^t\right) \overset{(\Delta)}{\le} n\tilde{M}_G - \frac{\epsilon}{\overline{p}^2 \cdot \max_{i\in N}\left\{b_i + c_i^-\right\}}\left(\sum_{t=T_\epsilon}^{\tilde{T}-1}\eta^t\right) < 0, \tag{H.7}$$

where we apply the upper bound $|G_i^\diamond(\mathbf{p},\mathbf{r})| \le \tilde{M}_G$ for $\diamond \in \{+,-\}$ from Lemma EC.7 in step (Δ) to derive that $\widetilde{\kappa}(\mathbf{p}) = \sum_{i\in N}\mathtt{dist}\Big(0, \mathtt{Hull}\big\{G_i^-(\mathbf{p},\mathbf{p}), G_i^+(\mathbf{p},\mathbf{p})\big\}\Big) \le n\tilde{M}_G$. Since $\widetilde{\kappa}(\cdot)$ is a non-negative metric, Eq. (H.7) is a clear contradiction.

Next, we compute $\tilde{T}$ under the step-size choice $\eta^t = C_\eta(t+1)^{-\beta}$ and determine the optimal value of β. Using the integration lower bound, we have that

$$\sum_{t=T_\epsilon}^{\tilde{T}-1}\eta^t \ge C_\eta \int_{T_\epsilon}^{\tilde{T}}(t+1)^{-\beta}dt = \frac{C_\eta}{1-\beta}\left[(\tilde{T}+1)^{1-\beta} - (T_\epsilon+1)^{1-\beta}\right]. \tag{H.8}$$

Hence, by Eqs. (H.6) and (H.8), we can choose $\tilde{T}$ to be any positive integer such that

$$\frac{C_\eta}{1-\beta}\left[(\tilde{T}+1)^{1-\beta} - (T_\epsilon+1)^{1-\beta}\right] > \frac{n\tilde{M}_G\overline{p}^2 \cdot \max_{i\in N}\left\{b_i + c_i^-\right\}}{\epsilon}, \tag{H.9}$$

which is further equivalent to

$$\begin{aligned}\tilde{T} &\ge \left[\frac{(1-\beta)n\tilde{M}_G\overline{p}^2 \cdot \max_{i\in N}\left\{b_i + c_i^-\right\}}{C_\eta\epsilon} + (T_\epsilon+1)^{1-\beta}\right]^{\frac{1}{1-\beta}} \\ &= \left[\frac{(1-\beta)n\tilde{M}_G\overline{p}^2 \cdot \max_{i\in N}\left\{b_i + c_i^-\right\}}{C_\eta\epsilon} + \left(\max\left\{T_\beta + 1, \left\lceil\left(\frac{2C_\eta\tilde{C}_\beta}{\epsilon}\right)^{\frac{1}{\beta}}\right\rceil\right\}\right)^{1-\beta}\right]^{\frac{1}{1-\beta}},\end{aligned} \tag{H.10}$$

where we substitute in the expression of T_ϵ from Eq. (H.3). We observe that the quantity on the right-hand side of Eq. (H.10) has the order $\mathcal{O}\Big(\epsilon^{\frac{-1}{1-\beta}}+\epsilon^{\frac{-1}{\beta}}\Big)$ and attains its minimum when $\beta=1/2$. Therefore, combining Eqs. (H.4) and (H.10), we conclude that under the step-size choice $\eta^t=C_\eta/\sqrt{t+1}$, Algorithm 2 achieves an $\left(\frac{1}{2\max_{i\in N}\{(b_i+c_i^-)\tilde{\ell}_{r,i}\}}+\sum_{i\in N}\frac{2\max_{k\in N}\{b_k+c_k^-\}}{b_i+c_i^+}\right)\epsilon$-SNE in $\tilde{T}=\mathcal{O}(1/\epsilon^2)$ iterations, where

$$\tilde{T}:=\left\lceil\frac{n\tilde{M}_G\overline{p}^2\cdot\max_{i\in N}\{b_i+c_i^-\}}{2C_\eta\epsilon}+\sqrt{\max\left\{T_{1/2}+1,\left\lceil\left(\frac{2C_\eta\tilde{C}_{1/2}}{\epsilon}\right)^2\right\rceil\right\}}\,\right\rceil^2. \tag{H.11}$$

In particular, we use $T_{1/2}$ and $\tilde{C}_{1/2}$ to denote the previously defined constants T_β and $\tilde{C}_\beta$ in the special case of $\beta=1/2$. By Eq. (L.2) and the definition of $\tilde{C}_\beta$ below Eq. (H.2), we have that

$$\begin{aligned}
T_{1/2}&=\left\lceil\frac{2\cdot(3+\alpha^2)-4}{4-(3+\alpha^2)}\right\rceil=\left\lceil\frac{2+2\alpha^2}{1-\alpha^2}\right\rceil,\\
\tilde{C}_{1/2}&=\max_{i\in N}\left\{(b_i+c_i^-)\cdot\tilde{\ell}_{r,i}\tilde{C}_{rp},\ (b_i+c_i^-)\cdot\tilde{\ell}_{G,i}\tilde{M}_G\sqrt{\sum_{i\in N}\left(b_i+c_i^-\right)^2}\right\},\\
\tilde{C}_{rp}&=\max\left\{\frac{2\tilde{M}_G\sqrt{(1+\alpha^2)\sum_{i\in N}(b_i+c_i^-)^2}}{1-\alpha^2},\frac{\sqrt{n}(\overline{p}-\underline{p})\sqrt{T_{1/2}+1}}{C_\eta}\right\}.
\end{aligned} \tag{H.12}$$

This completes the proof of Theorem 5. □

Appendix I Proof of Theorem 6

THEOREM 6 **(Restated)**. *Suppose that each firm $i\in N$ takes its own non-increasing step-sizes $\{\eta_i^t\}_{t\geq 0}$ such that $\lim_{t\to\infty}\eta_i^t=0$ and $\sum_{t=0}^\infty\eta_i^t=\infty$. Then, it follows that:*

- *In the loss-neutral scenario, the price and reference price paths generated by Algorithm 1 converge to the unique SNE, where the convergence rate is determined by the slowest decay rate among the step-size sequences.*
- *In the loss-averse scenario, the price and reference price paths generated by Algorithm 2 with threshold ϵ converge to an $\mathcal{O}(\epsilon)$-SNE, where the convergence rate is determined by both the slowest and fastest decay rates among the step-size sequences.*

Proof of Theorem 6. The proof is built upon the current proofs for Theorems 1, 2, 4, and 5.

We first consider the loss-neutral scenario. Since all the step-size sequences $\{\eta_i^t\}_{t\geq 0}$ are non-increasing, there exist a non-increasing sequence $\{\eta^t\}_{t\geq 0}$ and non-decreasing sequences $\{C_{\eta,i}^t\}_{t\geq 0}$ for every $i\in N$ such that $\eta_i^t=\eta^t C_{\eta,i}^t$ for every $t\geq 0$ and $i\in N$. The sequence $\{\eta^t\}_{t\geq 0}$ approximately measures the smallest step-sizes among all firms, i.e., the sequence with the fastest decay rates. For example, in a two-firm setting where $\eta_1^t=1/(t+1)$ and $\eta_2^t=1/(t+1)^2$, we can take $\eta^t=\eta_2^t$,

$C^t_{\eta,1} = t+1$, and $C^t_{\eta,2} = 1$. We note that our proofs for Theorems 1 and 2 are built upon two key results, the inequality in Eq. (D.9) and Lemma EC.3. To accommodate the firm-differentiated step-sizes, we first modify the distance metrics $\kappa(\cdot)$ and $\kappa_\epsilon(\cdot)$, defined in Eq. (11), to the following non-stationary metrics

$$\kappa^t(\mathbf{p}) := \sum_{i\in N} \frac{|p_i^\star - p_i|}{C^t_{\eta,i}(b_i+c_i)}, \quad \kappa^t_\epsilon(\mathbf{p}) := \sum_{i\in N} \max\left\{\frac{|p_i^\star - p_i|}{C^t_{\eta,i}(b_i+c_i)} - \epsilon, 0\right\}. \tag{I.1}$$

Then, it is easy to verify that Eq. (D.9) holds under the new metric $\kappa_\epsilon(\cdot)$. For Lemma EC.3, we also observe that

$$\mathcal{G}(\mathbf{p}) \geq \frac{1}{\overline{p}\,\|\mathbf{p}^\star\|_\infty} \cdot \sum_{i\in N} \frac{|p_i^\star - p_i|}{b_i+c_i} \geq \frac{\min_{i\in N}\left\{C^t_{\eta,i}\right\}}{\overline{p}\,\|\mathbf{p}^\star\|_\infty} \sum_{i\in N} \frac{|p_i^\star - p_i|}{C^t_{\eta,i}(b_i+c_i)} = \frac{\min_{i\in N}\left\{C^t_{\eta,i}\right\}}{\overline{p}\,\|\mathbf{p}^\star\|_\infty} \kappa^t(\mathbf{p}). \tag{I.2}$$

Using Eq. (I.2) and following the derivations from Eqs. (D.6) to (D.16), we can deduce that

$$\kappa^t_\epsilon(\mathbf{p}^{t+1}) \leq \left(1 - \min_{i\in N}\left\{C^t_{\eta,i}\right\}\lambda\eta^t\right)\kappa^t_\epsilon(\mathbf{p}^t) + C_\kappa \epsilon \eta^t, \tag{I.3}$$

where $\lambda = 1/\left(\overline{p}\,\|\mathbf{p}^\star\|_\infty\right)$ and C_κ is defined above Eq. (D.16). In the special scenario where $C^t_{\eta,i} \equiv C_{\eta,i}$ for all $t \geq 0$ and $i \in N$, i.e., the step-sizes for any two firms i and j only differ by a fixed multiplier $C_{\eta,i}/C_{\eta,j}$, Eq. (I.3) demonstrates a similar contraction property as Eq. (D.16), and thus the entire proofs for Theorems 1 and 2 can be adapted. We conclude that the price and reference price paths must converge to the unique SNE, where the convergence rate has the same order as the scenario of uniform step-sizes.

For the more general scenario described in Theorem 6, Eq. (I.3) is not a recursion yet, because we have $\kappa^t_\epsilon(\cdot)$ on both sides of the inequality. Nevertheless, since $C^{t+1}_{\eta,i} \geq C^t_{\eta,i}$, it naturally holds from Eq. (I.3) that

$$\kappa^{t+1}_\epsilon(\mathbf{p}^{t+1}) \leq \kappa^t_\epsilon(\mathbf{p}^{t+1}) \leq \left(1 - \min_{i\in N}\left\{C^t_{\eta,i}\right\}\lambda\eta^t\right)\kappa^t_\epsilon(\mathbf{p}^t) + C_\kappa \epsilon \eta^t, \tag{I.4}$$

which implies a similar convergence behavior for $\{\kappa^t(\mathbf{p}^t)\}_{t\geq 0}$ as Eqs. (D.19) and (E.8). The difference lies in the conversion from $\kappa^t(\mathbf{p}^t)$ to $\|\mathbf{p}^\star - \mathbf{p}^t\|$. Similar as Eq. (E.16), we have that

$$\left\|\mathbf{p}^\star - \mathbf{p}^t\right\|^2 \leq \left(\max_{i\in N}\left\{C^t_{\eta,i}(b_i+c_i)\right\}\right)^2 \cdot \left[\kappa^t(\mathbf{p}^t)\right]^2. \tag{I.5}$$

Hence, instead of only differing by a constant multiplier as that in Eq. (E.16), the bound in Eq. (I.5) depends on the non-decreasing sequences $\left\{C^t_{\eta,i}\right\}_{t\geq 0}$. We observe that, although it might hold $\lim_{t\to\infty} C^t_{\eta,i} = \infty$ for some $i \in N$, Eq. (I.5) still implies the convergence of $\{\|\mathbf{p}^\star - \mathbf{p}^t\|\}_{t\geq 0}$ to zero. In fact, since ϵ can be an arbitrary positive number, one consequence of Eq. (I.4) is that $\kappa^t(\mathbf{p}^t) = \mathcal{O}(\eta^t)$ for reasonably large t. This can be seen from the proof of Theorem 2 (from Eqs. (E.3) to (E.15)). Therefore, since $\lim_{t\to\infty} \eta^t_i = \lim_{t\to\infty} C^t_{\eta,i}\eta^t = 0$ for all $i \in N$, we conclude that $\|\mathbf{p}^\star - \mathbf{p}^t\| \to 0$ as

$t \to \infty$ and the convergence rate is dominated by slowest decay rate among the step-size sequences of all firms. This completes the proof of the first part of the theorem.

Now, for the loss-averse scenario, since the convergence is measured by the stationary metrics $\widetilde{\kappa}(\mathbf{p},\mathbf{r})$ in Eq. (17) and $\widetilde{\kappa}(\mathbf{p})$ in Eq. (21), the extension to the firm-differentiated step-sizes is more straightforward. Firstly, since the step-sizes for all firms decrease to zero as $t \to \infty$, there exists T_ϵ such that

$$\max_{i\in N}\left\{(b_i+c_i^-)\cdot\tilde{\ell}_{r,i}\left\|\mathbf{p}^t-\mathbf{r}^t\right\|,(b_i+c_i^-)\cdot\eta_i^t\tilde{\ell}_{p,i}\tilde{M}_G\sqrt{\sum_{k\in N}(b_k+c_k^-)^2}\right\}\leq\frac{\epsilon}{2},\quad\forall t\geq T_\epsilon. \tag{I.6}$$

By definition, the size of T_ϵ is determined by the slowest decay rate among the step-size sequences. Following the derivations in the proof of Theorem 4 (see Appendix G), we find that for all $t \geq T_\epsilon$, either $\mathbf{p}^t$ is already an $\mathcal{O}(\epsilon)$-SNE, or it holds that

$$\widetilde{\kappa}(\mathbf{p}^{t+1})-\widetilde{\kappa}(\mathbf{p}^t)\leq-\min_{i\in N}\left\{\frac{\eta_i^t}{b_i+c_i^-}\right\}\cdot\frac{\epsilon}{\overline{p}^2}, \tag{I.7}$$

Therefore, the decrease speed of $\{\widetilde{\kappa}(\mathbf{p}^t)\}_{t\geq T_\epsilon}$ is dominated by the fastest decay rate among the step-size sequences. Together, we conclude that Algorithm 2 still converges to an $\mathcal{O}(\epsilon)$-SNE, and the convergence rate is dominated by both the slowest and fastest decay rates among the step-size sequences. This completes the proof of Theorem 6. □

Appendix J Proof of Theorem 7

THEOREM 7 **(Restated)**. *Suppose that the firms can only access an inexact first-order oracle such that the errors are uniformly bounded by some $\delta > 0$. Let the step-sizes $\{\eta^t\}_{t\geq 0}$ be a non-increasing sequence such that $\lim_{t\to\infty}\eta^t = 0$ and $\sum_{t=0}^{\infty}\eta^t = \infty$. Then, the price and reference price paths generated by Algorithm 1 (or Algorithm 2 with threshold ϵ) converge to an $\mathcal{O}(\delta)$-neighborhood of the unique SNE in the loss-neutral scenario (or an $\mathcal{O}(\delta+\epsilon)$-SNE in the loss-averse scenario), where the convergence rate has the same order as the setting of exact first-order oracle.*

Proof of Theorem 7. The proof is based on the existing proofs of Theorems 1, 2, 4, and 5. The extensions for the loss-neutral scenario and the loss-averse scenario essentially follow the same idea. Below, we consider the loss-neutral scenario (Algorithm 1) for illustration.

In the inexact setting, each firm updates its price by a noisy derivative $\widehat{D}_i^t$, where we assume the difference between $\widehat{D}_i^t$ and the true derivative D_i^t is bounded by δ, i.e., $\left|\widehat{D}_i^t - D_i^t\right| < \delta$. Since $D_i^t = (b_i + c_i)G_i(\mathbf{p}^t,\mathbf{r}^t)$ by the definition in Eq. (D.2), we follow the derivations from Eqs. (D.3) to (D.9) to show that

$$\kappa_\epsilon(\mathbf{p}^{t+1})\leq\kappa_\epsilon(\mathbf{p}^t)-\eta^t\sum_{i\in\overline{N}_\epsilon^{t+1}}\mathtt{sign}\,(p_i^\star-p_i^t)\,G_i(\mathbf{p}^t,\mathbf{r}^t)+\eta^t\sum_{i\in N}\frac{\delta}{b_i+c_i}, \tag{J.1}$$

where $\kappa_\epsilon(\cdot)$ is the metric defined in Eq. (11). Then, by applying Eqs. (D.9) to (D.13), we have

$$\kappa_\epsilon(\mathbf{p}^{t+1}) \leq (1-\lambda\eta^t)\kappa_\epsilon(\mathbf{p}^t) + \left(C_\kappa \epsilon + \sum_{i\in N}\frac{\delta}{b_i+c_i}\right)\eta^t, \quad \forall t \geq T_\epsilon, \tag{J.2}$$

where the definitions of λ and C_κ can be found in Table EC.1, and T_ϵ is the break point defined above Eq. (D.6). Hence, by unrolling Eq. (J.2) in a similar manner as Eqs. (D.17) to (D.19), we deduce that

$$\lim_{t\to\infty} \kappa(\mathbf{p}^t) \leq \left(\frac{C_\kappa}{\lambda} + n\right)\epsilon + \sum_{i\in N}\frac{\delta}{\lambda(b_i+c_i)}. \tag{J.3}$$

Since Eq. (J.3) holds for any $\epsilon > 0$, we conclude that the price and reference price paths converge to an $\mathcal{O}(\delta)$-neighborhood of the unique SNE.

Now, in the loss-neutral scenario, suppose all firms adopt the step-sizes $\eta^t = \frac{C_\eta \log(t+1)}{t+1}$ for $t \geq 2$ with $C_\eta > 2\overline{p}^2/\log 2$. Similar as Eq. (E.3) in the proof of Theorem 2, we have from Eq. (J.2) that

$$\kappa(\mathbf{p}^t) \leq \kappa_\epsilon(\mathbf{p}^{T_\epsilon}) \exp\left(-\lambda \sum_{t'=T_\epsilon}^{t-1} \eta^{t'}\right) + \left(C_\kappa \epsilon + \sum_{i\in N}\frac{\delta}{b_i+c_i}\right) X^{t-1} + n\epsilon, \tag{J.4}$$

where $X^{t-1} = \sum_{t'=T_\epsilon}^{t-1} \eta^{t'} \prod_{t''=t'+1}^{t-1}(1-\lambda\eta^{t''})$. Then, following the same process as Eqs. (E.4) to (E.8), we first find that

$$\kappa(\mathbf{p}^t) \leq \left(2C_\kappa C_\eta + n + \frac{M_\kappa}{2\sqrt{n}(\overline{p}-\underline{p})}\right)\epsilon + \sum_{i\in N}\frac{2C_\eta\delta}{b_i+c_i}, \quad \forall t \geq 2T_\epsilon + 1, \tag{J.5}$$

where $M_\kappa = \sum_{i\in N}(\overline{p}-\underline{p})/(b_i+c_i)$ is the universal upper bound on $\kappa(\cdot)$ and T_ϵ is specified by Eq. (E.2). Next, to convert the upper bound in Eq. (J.5) to a bound that explicitly depends on t, we follow Eqs. (E.9) to (E.15) to conclude that

$$\kappa(\mathbf{p}^t) \leq \widehat{C}_\kappa \frac{\log t}{t} + \sum_{i\in N}\frac{2C_\eta\delta}{b_i+c_i}, \quad \forall t > \{2T_1, 10\}, \tag{J.6}$$

where $\widehat{C}_\kappa$ and T_1 are defined in Table EC.1. Hence, with inexact first-order oracles, the convergence rate of Algorithm 1 remains the same as the exact setting.

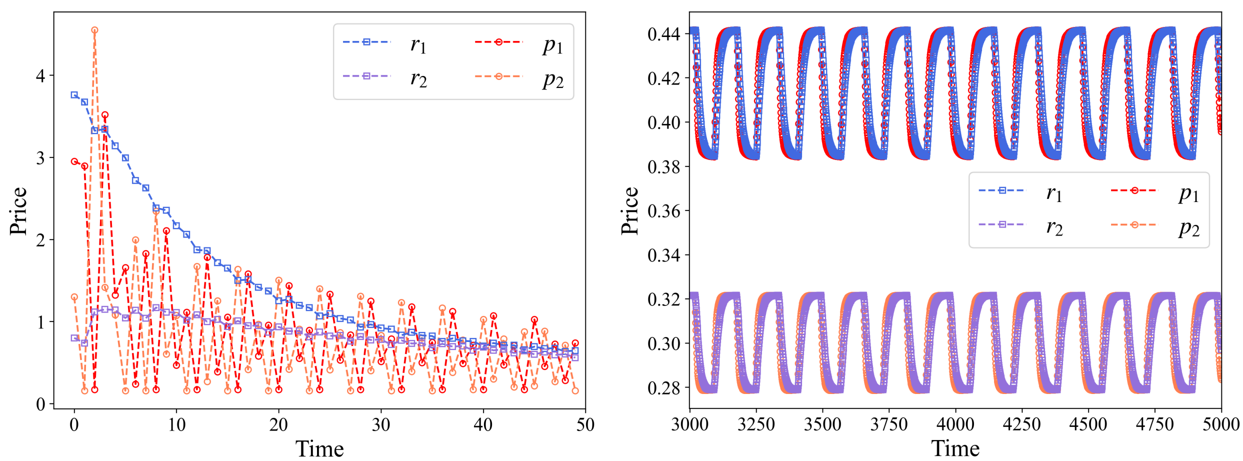
Finally, for loss-averse scenarios, we note that although the inexact first-order oracle affects the evaluation of the true/virtual derivatives, i.e., $D_i^{t,+}$ and $D_i^{t,+}$, a similar analysis as Eqs. (G.2) to (G.29) still holds true if there exists some product i_0 with $D_{i_0}^{t,-} = D_{i_0}^-(\mathbf{p}^t,\mathbf{r}^t) > (b_{i_0} + c_{i_0}^-)\epsilon + \delta$ or $D_{i_0}^{t,+} = D_{i_0}^+(\mathbf{p}^t,\mathbf{r}^t) < -(b_{i_0} + c_{i_0}^+)\epsilon - \delta$, i.e., being at least δ "away" from incurring the pausing criterion. If there does not exist such a product, then we can show that the algorithm has already arrived at an $\mathcal{O}(\delta+\epsilon)$-SNE. $\square$

Appendix K Illustration of OPGA in the Gain-seeking Scenario

In this section, we illustrate the convergence behavior of OPGA (Algorithm 1) in the gain-seeking scenario. Note that with the non-smooth revenue function under gain-seekingness, OPGA is equivalent to the online projected sub-gradient ascent. The reason that we do not plot C-OPGA (Algorithm 2) is due to its similarity with OPGA in gain-seeking scenarios. Recall the true and virtual derivatives $D_i^-(\mathbf{p},\mathbf{r})$ and $D_i^+(\mathbf{p},\mathbf{r})$ defined in Eq. (18). In contrast to the loss-averse scenario, it always holds that $D_i^+(\mathbf{p},\mathbf{r}) < D_i^-(\mathbf{p},\mathbf{r})$ for any gain-seeking product i. Hence, the pausing criterion can be triggered only if $-\epsilon < D_i^+(\mathbf{p}^t,\mathbf{r}^t) < D_i^-(\mathbf{p}^t,\mathbf{r}^t) < \epsilon$, which is unlikely to happen for small ϵ.

Figure EC.1 Cyclic Pattern of OPGA in Gain-seeking Scenario

(Parameters: $(a_1, b_1, c_1^+, c_1^-) = (4.70, 1.55, 4.25, 3.38)$, $(a_2, b_2, c_2^+, c_2^-) = (4.20, 1.15, 5.25, 4.50)$, $(r_1^0, r_2^0) = (3.76, 0.80)$, $(p_1^0, p_2^0) = (2.95, 1.30)$, $\alpha = 0.90$, and $\eta^t = 1/\sqrt{t+1}$.)

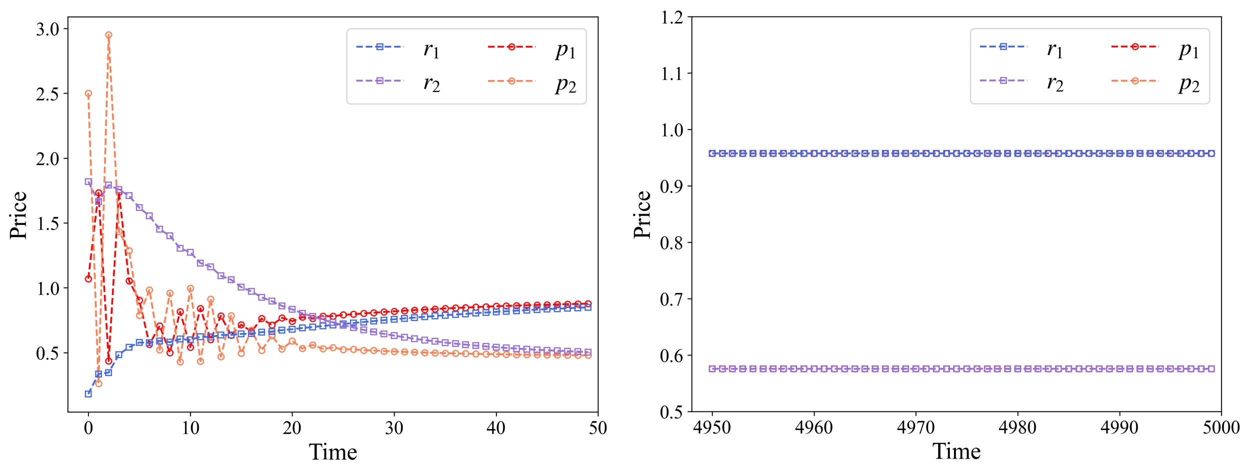


(a) Initial stage. (b) Long-run behavior.

Figures EC.1, EC.2, and EC.3 show the price and reference price paths of OPGA in three gain-seeking scenarios. In Figure EC.1, we observe that the paths oscillate indefinitely without admitting limiting points. Figures EC.2 and EC.3 share the same model parameters and only differ in initial reference prices and prices. Both figures show a convergent pattern in the long run, but have different limiting points. However, we highlight that neither limiting point represents an equilibrium, and both firms can achieve a higher revenue by unilaterally deviating from the limiting point. Indeed, such a convergence results from the monotonicity of price paths when approaching the limiting points. This ensures the effective reference sensitivity stays unchanged during the learning process, thereby leading the paths to converge to the SNE in the loss-neutral scenario. We can verify that the limiting point in Figure EC.2 is the same as the SNE in the loss-neutral scenario with parameters $(a_1, b_1, c_1) = (4.80, 1.30, 1.12)$ and $(a_2, b_2, c_2) = (4.22, 1.70, 1.27)$. The limiting point in Figure EC.3

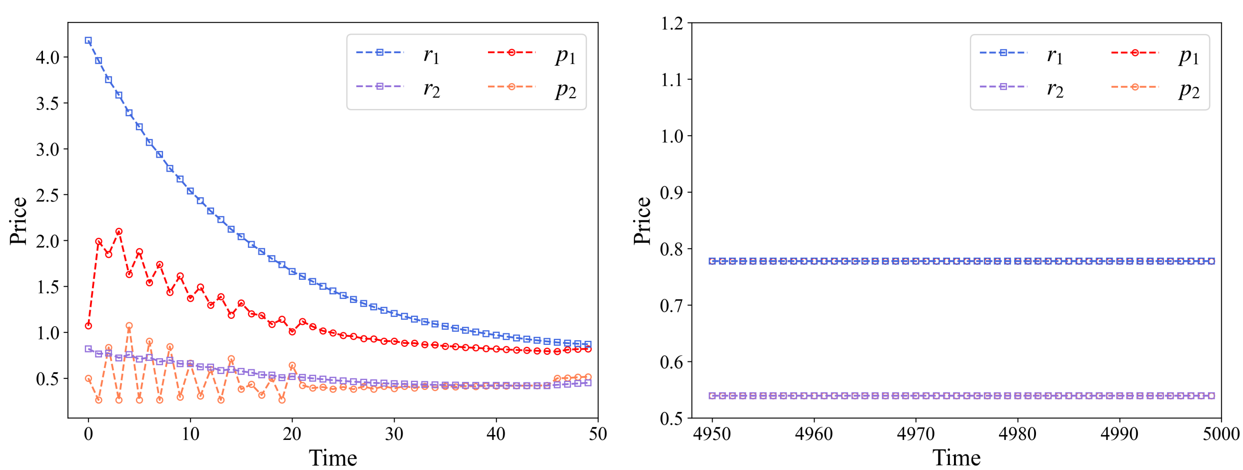
corresponds to the SNE under parameters $(a_1, b_1, c_1) = (4.80, 1.30, 2)$, $(a_2, b_2, c_2) = (4.22, 1.70, 1.27)$. Together, the experiments in Figures EC.1, EC.2, and EC.3 demonstrate that equilibrium and market stability cannot be simultaneously achieved in gain-seeking scenarios.

Figure EC.2 **Convergent Pattern of `OPGA` in Gain-seeking Scenario**
(Parameters: $(a_1, b_1, c_1^+, c_1^-) = (4.80, 1.30, 2.00, 1.12)$, $(a_2, b_2, c_2^+, c_2^-) = (4.22, 1.70, 2.12, 1.27)$, $(r_1^0, r_2^0) = (0.18, 1.82)$ and $(p_1^0, p_2^0) = (1.07, 2.50)$, $\alpha = 0.90$, and $\eta^t = 1/\sqrt{t+1}$.)



(a) Initial stage. (b) Long-run behavior.

Figure EC.3 **Convergent Pattern of `OPGA` in Gain-seeking Scenario with Different Initialization**
(Parameters are the same as Figure EC.2 except for the initializations, which are at $(r_1^0, r_2^0) = (4.18, 0.82)$ and $(p_1^0, p_2^0) = (1.07, 0.50)$.)



(a) Initial stage. (b) Long-run behavior.

Appendix L Supporting Lemmas

L.1 Lemma EC.2

LEMMA EC.2 **(Convergence of Price to Reference Price)**. *Let $\{\mathbf{p}^t\}_{t\geq 0}$ and $\{\mathbf{r}^t\}_{t\geq 0}$ be the price path and reference path generated by Algorithms 1 or 2 with non-increasing step-sizes $\{\eta^t\}_{t\geq 0}$ such that $\lim_{t\to\infty}\eta^t = 0$. Then, it holds that $\lim_{t\to\infty}\|\mathbf{p}^t - \mathbf{r}^t\| = 0$. In particular:*

1. *In the loss-neutral scenario, if $\eta^t = \frac{C_\eta \log(t+1)}{t+1}$ for $t \geq 2$, then there exist $C_{rp}, T_1 > 0$, such that $\|\mathbf{p}^t - \mathbf{r}^t\| \leq \eta^t C_{rp}$ for all $t \geq T_1$, where*

$$
\begin{aligned}
C_{rp} &= \max\left\{\frac{2M_G\sqrt{(1+\alpha^2)\sum_{i\in N}(b_i+c_i)^2}}{1-\alpha^2}, \frac{\sqrt{n}(\overline{p}-\underline{p})(T_1+1)}{C_\eta \log(T_1+1)}\right\}, \\
T_1 &= \left\lceil \frac{2\sqrt{3+\alpha^2}-2}{2-\sqrt{3+\alpha^2}} \right\rceil .
\end{aligned} \tag{L.1}
$$

2. *In the loss-averse scenario, if $\eta^t = C_\eta (t+1)^{-\beta}$ where $\beta \in (0,1]$, then there exist $C_{rp,\beta}, T_\beta > 0$, such that $\|\mathbf{p}^t - \mathbf{r}^t\| \leq \eta^t C_{rp,\beta}$ for all $t \geq T_\beta$, where*

$$
\begin{aligned}
C_{rp,\beta} &= \max\left\{\frac{2\tilde{M}_G\sqrt{(1+\alpha^2)\sum_{i\in N}(b_i+c_i^-)^2}}{1-\alpha^2}, \frac{\sqrt{n}(\overline{p}-\underline{p})(T_\beta+1)^\beta}{C_\eta}\right\}, \\
T_\beta &= \left\lceil \frac{2(3+\alpha^2)^{\frac{1}{2\beta}}-2^{\frac{1}{\beta}}}{2^{\frac{1}{\beta}}-(3+\alpha^2)^{\frac{1}{2\beta}}} \right\rceil .
\end{aligned} \tag{L.2}
$$

It is worth mentioning that the reason we limit the scope of Lemma EC.2 to the above two special cases is merely because they are sufficient for the proof of our main results.

Proof of Lemma EC.2. We first prove the general convergence result under non-increasing step-sizes $\{\eta^t\}_{t\geq 0}$ such that $\lim_{t\to\infty}\eta^t = 0$. Without loss of generality, we focus on the loss-neutral case and consider a trajectory $\{(\mathbf{p}^t, \mathbf{r}^t)\}_{t\geq 0}$ generated by Algorithm 1. The proof is the same for the loss-averse scenario.

First, recall that $D_i^t = (b_i + c_i)\cdot G_i(\mathbf{p}^t, \mathbf{r}^t)$, where $G_i(\mathbf{p}, \mathbf{r})$ is the scaled partial derivative of the log-revenue defined in Eq. (D.2). Then, it follows from Eq. (L.30) in Lemma EC.4 that $|D_i^t| \leq (b_i + c_i)M_G$. Since $\{\eta^t\}_{t\geq 0}$ is a non-increasing sequence with $\lim_{t\to\infty}\eta^t = 0$, for any constant $\eta > 0$, there exists $T_\eta \in \mathbb{N}$ such that $|\eta^t D_i^t| \leq \eta$ for every $t \geq T_\eta$ and for all $i \in N$. Thus, it holds that

$$
\left|p_i^{t+1} - p_i^t\right| = \left|\mathtt{Proj}_{\mathcal{P}}\left(p_i^t + \eta^t D_i^t\right) - p_i^t\right| \leq \left|\eta^t D_i^t\right| \leq \eta, \quad \forall t \geq T_\eta,\ \forall i \in N, \tag{L.3}
$$

where the first inequality is due to the property of the projection operator. Then, by the reference price update rule in Eq. (4), it follows that

$$\begin{aligned}\left|p_i^{t+1}-r_i^{t+1}\right| &= \left|p_i^{t+1}-\alpha r_i^t-(1-\alpha)p_i^t\right| \\ &= \left|\left(p_i^{t+1}-p_i^t\right)+\alpha\left(r_i^t-p_i^t\right)\right| \\ &\le \left|p_i^{t+1}-p_i^t\right|+\alpha\left|p_i^t-r_i^t\right| \\ &\le \eta+\alpha\left|p_i^t-r_i^t\right|, \quad \forall t\le T_\eta,\ \forall i\in N,\end{aligned} \tag{L.4}$$

where the last line results from the upper bound in Eq. (L.3). Applying Eq. (L.4) recursively from period t to period T_η, we further derive that

$$\begin{aligned}\left|p_i^{t+1}-r_i^{t+1}\right| &\le \eta\Big(\sum_{\tau=T_\eta}^{t}\alpha^{\tau-T_\eta}\Big)+\alpha^{t+1-T_\eta}\cdot\left|p_i^{T_\eta}-r_i^{T_\eta}\right| \\ &\le \frac{\eta}{1-\alpha}+\alpha^{t+1-T_\eta}\cdot(\overline{p}-\underline{p}), \quad \forall i\in N,\end{aligned} \tag{L.5}$$

which implies that

$$\left\|\mathbf{p}^{t+1}-\mathbf{r}^{t+1}\right\|\le\sqrt{n}\left(\frac{\eta}{1-\alpha}+\alpha^{t+1-T_\eta}\cdot(\overline{p}-\underline{p})\right). \tag{L.6}$$

Since η can be arbitrarily close to 0, we have that $\|\mathbf{p}^t-\mathbf{r}^t\|\to 0$ as $t\to\infty$, which completes the proof for the general convergence of price to reference price with non-increasing step-sizes.

In the next part, we consider two specific choices of step-sizes and explicitly quantify the convergence rate. This part of the proof relies on an important recursion, shown as follows

$$\begin{aligned}\left\|\mathbf{p}^{t+1}-\mathbf{r}^{t+1}\right\|^2 &= \left\|\mathbf{p}^{t+1}-\alpha\mathbf{r}^t-(1-\alpha)\mathbf{p}^t\right\|^2 \\ &= \left\|\alpha(\mathbf{p}^t-\mathbf{r}^t)+(\mathbf{p}^{t+1}-\mathbf{p}^t)\right\|^2 \\ &= \alpha^2\left\|\mathbf{p}^t-\mathbf{r}^t\right\|^2+\left\|\mathbf{p}^{t+1}-\mathbf{p}^t\right\|^2+2\alpha(\mathbf{p}^t-\mathbf{r}^t)^\top(\mathbf{p}^{t+1}-\mathbf{p}^t) \\ &\overset{(\Delta_1)}{\le} \alpha^2\left\|\mathbf{p}^t-\mathbf{r}^t\right\|^2+\left\|\eta^t\mathbf{D}^t\right\|^2+2\alpha\left\|\mathbf{p}^t-\mathbf{r}^t\right\|\left\|\eta^t\mathbf{D}^t\right\| \\ &\overset{(\Delta_2)}{\le} \alpha^2\left\|\mathbf{p}^t-\mathbf{r}^t\right\|^2+\left\|\eta^t\mathbf{D}^t\right\|^2+\frac{1-\alpha^2}{2}\left\|\mathbf{p}^t-\mathbf{r}^t\right\|^2+\frac{2\alpha^2}{1-\alpha^2}\left\|\eta^t\mathbf{D}^t\right\|^2 \\ &= \frac{1+\alpha^2}{2}\left\|\mathbf{p}^t-\mathbf{r}^t\right\|^2+\frac{1+\alpha^2}{1-\alpha^2}\left\|\eta^t\mathbf{D}^t\right\|^2,\end{aligned} \tag{L.7}$$

where $\mathbf{D}^t:=\left(D_1^t,\dots D_n^t\right)$ in (Δ_1) and the inequality holds due to the Cauchy-Schwarz inequality and the property of the projection operator. Then, step (Δ_2) stems from the inequality of arithmetic and geometric means.

1. **Loss-neutral Scenario.** We first focus on the loss-neutral scenario, where the step-size is specified as $\eta^t=\frac{C_\eta\log(t+1)}{t+1}$ for $t\ge 2$. We adopt an induction-based argument. At some period t, suppose that there exists a constant C_{rp} such that

$$\left\|\mathbf{p}^t-\mathbf{r}^t\right\|^2\le\left(\eta^tC_{rp}\right)^2=C_{rp}^2C_\eta^2\left(\frac{\log(t+1)}{t+1}\right)^2. \tag{L.8}$$

Together with Eq. (L.7), we have that at period $t+1$

$$\begin{aligned}
\left\|\mathbf{p}^{t+1}-\mathbf{r}^{t+1}\right\|^2 &\leq \frac{1+\alpha^2}{2}\left\|\mathbf{p}^t-\mathbf{r}^t\right\|^2+\frac{1+\alpha^2}{1-\alpha^2}\left\|\eta^t\mathbf{D}^t\right\|^2 \\
&\leq \frac{1+\alpha^2}{2}\left(\eta^t C_{rp}\right)^2+\frac{1+\alpha^2}{1-\alpha^2}\left(\sum_{i\in N}(b_i+c_i)^2\left[G_i(\mathbf{p}^t,\mathbf{r}^t)\right]^2\right)\cdot(\eta^t)^2 \\
&\overset{(\Delta)}{\leq} \frac{1+\alpha^2}{2}\left(\eta^t C_{rp}\right)^2+\underbrace{\left(\frac{1+\alpha^2}{1-\alpha^2}\cdot M_G^2\sum_{i\in N}(b_i+c_i)^2\right)}_{=:C_D}\cdot(\eta^t)^2 \\
&\leq C_\eta^2\left(\frac{\log(t+1)}{t+1}\right)^2\cdot\left[\frac{1+\alpha^2}{2}C_{rp}^2+C_D\right] \qquad \text{(L.9)}\\
&= C_\eta^2\left(\frac{\log(t+2)}{t+2}\right)^2\cdot\left[\frac{1+\alpha^2}{2}C_{rp}^2+C_D\right]\cdot\left(\frac{t+2}{t+1}\right)^2\cdot\left(\frac{\log(t+1)}{\log(t+2)}\right)^2 \\
&\leq C_\eta^2\left(\frac{\log(t+2)}{t+2}\right)^2\cdot\left[\frac{1+\alpha^2}{2}C_{rp}^2+C_D\right]\cdot\left(\frac{t+2}{t+1}\right)^2 \\
&\leq (\eta^{t+1})^2\cdot\left[\frac{1+\alpha^2}{2}C_{rp}^2+C_D\right]\left(\frac{t+2}{t+1}\right)^2,
\end{aligned}$$

where step (Δ) applies the upper bound on function $G_i(\cdot,\cdot)$ in Eq. (L.30) of Lemma EC.4. For the simplicity of notation, we denote the coefficient of $(\eta^t)^2$ in line (Δ) as C_D.

Since our goal is to have $\|\mathbf{p}^{t+1}-\mathbf{r}^{t+1}\|^2 \leq (\eta^{t+1}\cdot C_{rp})^2$, based on the inequality in Eq. (L.9), we only need to ensure that

$$\left[\frac{1+\alpha^2}{2}C_{rp}^2+C_D\right]\left(\frac{t+2}{t+1}\right)^2 \leq C_{rp}^2 \quad\Leftrightarrow\quad C_{rp}^2\left[\left(\frac{t+1}{t+2}\right)^2-\frac{1+\alpha^2}{2}\right]\geq C_D. \qquad \text{(L.10)}$$

As $(t+1)/(t+2)$ increases with respect to t, we choose T_1 to be the smallest integer that satisfies

$$\begin{aligned}
\left(\frac{T_1+1}{T_1+2}\right)^2-\frac{1+\alpha^2}{2}\geq\frac{1-\alpha^2}{4} \quad&\Leftrightarrow\quad \left(\frac{T_1+1}{T_1+2}\right)^2\geq\frac{3+\alpha^2}{4} \\
&\Leftrightarrow\quad \left(1+\frac{1}{T_1+1}\right)^2\leq\frac{4}{3+\alpha^2} \qquad \text{(L.11)}\\
&\Leftrightarrow\quad T_1=\left\lceil\frac{2\sqrt{3+\alpha^2}-2}{2-\sqrt{3+\alpha^2}}\right\rceil.
\end{aligned}$$

Given T_1 as specified above, it follows that for all $t\geq T_1$

$$\left(\frac{t+1}{t+2}\right)^2-\frac{1+\alpha^2}{2}\geq\left(\frac{T_1+1}{T_1+2}\right)^2-\frac{1+\alpha^2}{2}\geq\frac{1-\alpha^2}{4}. \qquad \text{(L.12)}$$

Hence, to ensure Eq. (L.10) hold true for all $t\geq T_1$, it suffices to choose C_{rp} that satisfies

$$C_{rp}\geq 2\sqrt{\frac{C_D}{1-\alpha^2}}, \qquad \text{(L.13)}$$

where C_D is defined in Eq. (L.7).

Lastly, the base case of the induction requires that $\|\mathbf{p}^{T_1}-\mathbf{r}^{T_1}\| \le C_{rp}C_\eta \frac{\log(T_1+1)}{T_1+1}$. Since $\|\mathbf{p}^{T_1}-\mathbf{r}^{T_1}\| \le \sqrt{n}(\overline{p}-\underline{p})$, it suffices to choose

$$\sqrt{n}(\overline{p}-\underline{p}) \le C_{rp}C_\eta \frac{\log(T_1+1)}{T_1+1} \quad\Leftrightarrow\quad C_{rp} \ge \frac{\sqrt{n}(\overline{p}-\underline{p})(T_1+1)}{C_\eta \log(T_1+1)}. \tag{L.14}$$

Combining the requirements for the base case in Eq. (L.14) and for the induction step in Eq. (L.13), we conclude that it is sufficient to choose

$$C_{rp} = \max\left\{ \frac{2M_G\sqrt{(1+\alpha^2)\sum_{i\in N}(b_i+c_i)^2}}{1-\alpha^2}, \frac{\sqrt{n}(\overline{p}-\underline{p})(T_1+1)}{C_\eta \log(T_1+1)} \right\}, \tag{L.15}$$

which completes the proof for the loss-neutral scenario.

2. **Loss-averse Scenario.** We use a similar induction method to prove the loss-averse case, where the step-size is $\eta^t = C_\eta(t+1)^{-\beta}$ with $\beta\in(0,1]$. At some period t, suppose that there exists a constant $C_{rp,\beta}$ such that

$$\left\|\mathbf{p}^t-\mathbf{r}^t\right\|^2 \le (\eta^t C_{rp,\beta})^2 = C_{rp,\beta}^2 C_\eta^2\cdot(t+1)^{-2\beta}. \tag{L.16}$$

Then, from the recursion in Eq. (L.7), we derive that

$$\begin{aligned}
\left\|\mathbf{p}^{t+1}-\mathbf{r}^{t+1}\right\|^2 &\le \frac{1+\alpha^2}{2}\left\|\mathbf{p}^t-\mathbf{r}^t\right\|^2 + \frac{1+\alpha^2}{1-\alpha^2}\left\|\eta^t\mathbf{D}^t\right\|^2 \\
&\overset{(\Delta)}{\le} \frac{1+\alpha^2}{2}(\eta^t C_{rp,\beta})^2 + \underbrace{\left(\frac{1+\alpha^2}{1-\alpha^2}\cdot \tilde{M}_G^2\sum_{i\in N}(b_i+c_i^-)^2\right)}_{=:\tilde{C}_D}\cdot(\eta^t)^2 \\
&\le \left(\frac{C_\eta}{(t+1)^\beta}\right)^2\cdot\left[\frac{1+\alpha^2}{2}C_{rp,\beta}^2+\tilde{C}_D\right] \\
&= \left(\frac{C_\eta}{(t+2)^\beta}\right)^2\cdot\left(\frac{t+2}{t+1}\right)^{2\beta}\cdot\left[\frac{1+\alpha^2}{2}C_{rp,\beta}^2+\tilde{C}_D\right] \\
&= (\eta^{t+1})^2\cdot\left[\frac{1+\alpha^2}{2}C_{rp,\beta}^2+\tilde{C}_D\right]\left(\frac{t+2}{t+1}\right)^{2\beta},
\end{aligned} \tag{L.17}$$

where we apply the upper bound $|D_i^t| \le \max_{\diamond\in\{+,-\}}\{(b_i+c_i^\diamond)G_i^\diamond(\mathbf{p}^t,\mathbf{r}^t)\} \le (b_i+c_i^-)\tilde{M}_G$ in step (Δ) with the constant $\tilde{M}_G$ coming from Eq. (L.51) of Lemma EC.7. Then, to ensure $\|\mathbf{p}^{t+1}-\mathbf{r}^{t+1}\|^2 \le (\eta^{t+1}\cdot C_{rp,\beta})^2$, we need the following condition to be satisfied

$$\left[\frac{1+\alpha^2}{2}C_{rp,\beta}^2+\tilde{C}_D\right]\left(\frac{t+2}{t+1}\right)^{2\beta} \le C_{rp,\beta}^2 \quad\Leftrightarrow\quad C_{rp,\beta}^2\left[\left(\frac{t+1}{t+2}\right)^{2\beta}-\frac{1+\alpha^2}{2}\right] \ge \tilde{C}_D. \tag{L.18}$$

Adopting the same approach used for the loss-neutral case described in Eq. (L.11), we choose T_β as the smallest integer such that

$$\left(\frac{T_\beta+1}{T_\beta+2}\right)^{2\beta} - \frac{1+\alpha^2}{2} \ge \frac{1-\alpha^2}{4} \quad\Leftrightarrow\quad T_\beta = \left\lceil \frac{2(3+\alpha^2)^{\frac{1}{2\beta}}-2^{\frac{1}{\beta}}}{2^{\frac{1}{\beta}}-(3+\alpha^2)^{\frac{1}{2\beta}}} \right\rceil. \tag{L.19}$$

With T_β as specified in Eq. (L.19) and $(t+1)/(t+2)$ increasing in t, it follows that for all $t \geq T_\beta$

$$\left(\frac{t+1}{t+2}\right)^{2\beta} - \frac{1+\alpha^2}{2} \geq \left(\frac{T_\beta+1}{T_\beta+2}\right)^2 - \frac{1+\alpha^2}{2} \geq \frac{1-\alpha^2}{4}. \tag{L.20}$$

The above inequality implies that the condition in Eq. (L.18) is held when $C_{rp,\beta}$ satisfies

$$C_{rp,\beta} \geq 2\sqrt{\frac{\tilde{C}_D}{1-\alpha^2}}, \tag{L.21}$$

where $\tilde{C}_D$ is defined in Eq. (L.17).

Finally, the base case of this induction requires that $\left\|\mathbf{p}^{T_\beta} - \mathbf{r}^{T_\beta}\right\| \leq \frac{C_{rp,\beta}C_\eta}{(T_\beta+1)^\beta}$. Since $\left\|\mathbf{p}^{T_\beta} - \mathbf{r}^{T_\beta}\right\| \leq \sqrt{n}(\overline{p} - \underline{p})$, it suffices to choose

$$\sqrt{n}(\overline{p} - \underline{p}) \leq \frac{C_{rp,\beta}C_\eta}{(T_\beta+1)^\beta} \quad \Leftrightarrow \quad C_{rp,\beta} \geq \frac{\sqrt{n}(\overline{p} - \underline{p})(T_\beta+1)^\beta}{C_\eta}. \tag{L.22}$$

Merging the requirements for $C_{rp,\beta}$ in both Eqs. (L.21) and (L.22), we complete the proof by showing the sufficient condition as follows

$$C_{rp,\beta} = \max\left\{\frac{2\tilde{M}_G\sqrt{(1+\alpha^2)\sum_{i\in N}(b_i + c_i^-)^2}}{1-\alpha^2}, \frac{\sqrt{n}(\overline{p} - \underline{p})(T_\beta+1)^\beta}{C_\eta}\right\}. \tag{L.23}$$

□

L.2 Lemma EC.3

Lemma EC.3. *In the loss-neutral scenario, define the function $\mathcal{G}(\mathbf{p})$ as*

$$\mathcal{G}(\mathbf{p}) := \sum_{i\in N} \texttt{sign}(p_i^\star - p_i)\, G_i(\mathbf{p},\mathbf{p}) = \sum_{i\in N} \texttt{sign}(p_i^\star - p_i)\left[\frac{1}{(b_i+c_i)p_i} + d_i(\mathbf{p},\mathbf{p}) - 1\right], \tag{L.24}$$

where $\mathbf{p}^\star$ is the unique SNE, $G_i(\mathbf{p},\mathbf{r})$ is the scaled derivative defined in Eq. (D.2), and function $\texttt{sign}(\cdot)$ is defined in Eq. (D.4). Then, it holds that

$$\mathcal{G}(\mathbf{p}) \geq \frac{\kappa(\mathbf{p})}{\overline{p}\,\|\mathbf{p}^\star\|_\infty} = \frac{1}{\overline{p}\,\|\mathbf{p}^\star\|_\infty} \cdot \sum_{i\in N} \frac{|p_i^\star - p_i|}{b_i+c_i}, \quad \forall \mathbf{p} \in \mathcal{P}^n, \tag{L.25}$$

where $\kappa(\cdot)$ is the weighted ℓ^1-metric function defined in Eq. (11).

Proof of Lemma EC.3. We first consider the following separation of N based on relative size between $\mathbf{p}$ and $\mathbf{p}^\star$:

$$N_1(\mathbf{p}) := \{i \in N | p_i > p_i^\star\}, \quad N_2(\mathbf{p}) := \{i \in N | p_i < p_i^\star\}, \quad N_c(\mathbf{p}) := \{i \in N | p_i = p_i^\star\}. \tag{L.26}$$

Then, since $\texttt{sign}(p_i^\star - p_i) = \texttt{sign}(0) = 0$ for all $i \in N_c(\mathbf{p})$, we rewrite $\mathcal{G}(\mathbf{p})$ to deduce that

$$
\begin{aligned}
\mathcal{G}(\mathbf{p}) &= \sum_{i\in N} \texttt{sign}(p_i^\star - p_i)\, G_i(\mathbf{p},\mathbf{p}) \\
&= \sum_{i\in N_1(\mathbf{p})} \left[1 - d_i(\mathbf{p},\mathbf{p}) - \frac{1}{(b_i+c_i)p_i}\right] + \sum_{i\in N_2(\mathbf{p})} \left[\frac{1}{(b_i+c_i)p_i} + d_i(\mathbf{p},\mathbf{p}) - 1\right] \\
&\overset{(\Delta_1)}{=} \sum_{i\in N_1(\mathbf{p})} \left\{\left[1 - d_i(\mathbf{p},\mathbf{p}) - \frac{1}{(b_i+c_i)p_i}\right] - \left[1 - d_i(\mathbf{p}^\star,\mathbf{p}^\star) - \frac{1}{(b_i+c_i)p_i^\star}\right]\right\} \\
&\quad + \sum_{i\in N_2(\mathbf{p})} \left\{\left[\frac{1}{(b_i+c_i)p_i} + d_i(\mathbf{p},\mathbf{p}) - 1\right] - \left[\frac{1}{(b_i+c_i)p_i^\star} + d_i(\mathbf{p}^\star,\mathbf{p}^\star) - 1\right]\right\} \\
&\overset{(\Delta_2)}{=} \sum_{i\in N_1(\mathbf{p})} \frac{1}{b_i+c_i}\left(\frac{1}{p_i^\star} - \frac{1}{p_i}\right) + \sum_{i\in N_2(\mathbf{p})} \frac{1}{b_i+c_i}\left(\frac{1}{p_i} - \frac{1}{p_i^\star}\right) \\
&\quad + \underbrace{\sum_{i\in N_1(\mathbf{p})} d_i(\mathbf{p}^\star,\mathbf{p}^\star) - \sum_{i\in N_1(\mathbf{p})} d_i(\mathbf{p},\mathbf{p})}_{\geq 0} + \underbrace{\sum_{i\in N_2(\mathbf{p})} d_i(\mathbf{p},\mathbf{p}) - \sum_{i\in N_2(\mathbf{p})} d_i(\mathbf{p}^\star,\mathbf{p}^\star)}_{\geq 0} \\
&\geq \sum_{i\in N} \frac{1}{b_i+c_i}\cdot\frac{|p_i - p_i^\star|}{p_i^\star p_i} \geq \frac{1}{\overline{p}\,\|\mathbf{p}^\star\|_\infty} \sum_{i\in N} \frac{|p_i - p_i^\star|}{b_i+c_i} = \frac{\kappa(\mathbf{p})}{\overline{p}\,\|\mathbf{p}^\star\|_\infty}.
\end{aligned}
\tag{L.27}
$$

In step (Δ_1), we introduce two dummy terms, which are equal to zero by Eq. (C.16). To derive step (Δ_2), using the facts that $p_i > p_i^\star$ for $i \in N_1$ and $p_i < p_i^\star$ for $i \in N_2$, we have that

$$
\begin{aligned}
\sum_{i\in N_1(\mathbf{p})} d_i(\mathbf{p}^\star,\mathbf{p}^\star) &= \frac{\sum_{i\in N_1(\mathbf{p})} \exp(a_i - b_i p_i^\star)}{1 + \sum_{i\in N_1(\mathbf{p})} \exp(a_i - b_i p_i^\star) + \sum_{i\in N_2(\mathbf{p})} \exp(a_i - b_i p_i^\star) + \sum_{i\in N_c(\mathbf{p})} \exp(a_i - b_i p_i^\star)} \\
&\geq \frac{\sum_{i\in N_1(\mathbf{p})} \exp(a_i - b_i p_i)}{1 + \sum_{i\in N_1(\mathbf{p})} \exp(a_i - b_i p_i) + \sum_{i\in N_2(\mathbf{p})} \exp(a_i - b_i p_i) + \sum_{i\in N_c(\mathbf{p})} \exp(a_i - b_i p_i)} \\
&= \sum_{i\in N_1(\mathbf{p})} d_i(\mathbf{p},\mathbf{p}).
\end{aligned}
\tag{L.28}
$$

By similar analysis, we show that the second under-brace term in Eq. (L.27) is also no less than zero, i.e., $\sum_{i\in N_2(\mathbf{p})} d_i(\mathbf{p},\mathbf{p}) > \sum_{i\in N_2(\mathbf{p})} d_i(\mathbf{p}^\star,\mathbf{p}^\star)$. This completes the proof of Lemma EC.3. $\square$

L.3 Lemma EC.4

LEMMA EC.4. *In the loss-neutral scenario, let $G_i(\mathbf{p},\mathbf{r})$ be the scaled partial derivative defined in Eq. (D.2). Then, it holds that*

$$
\frac{\partial G_i(\mathbf{p},\mathbf{r})}{\partial p_j} = \begin{cases} -\dfrac{1}{(b_i+c_i)p_i^2} - (b_i+c_i)\cdot d_i(\mathbf{p},\mathbf{r})\cdot\big(1 - d_i(\mathbf{p},\mathbf{r})\big) & \textit{if } j = i, \\ (b_j+c_j)\cdot d_i(\mathbf{p},\mathbf{r})\cdot d_j(\mathbf{p},\mathbf{r}) & \textit{if } j \neq i. \end{cases}
\tag{L.29a}
$$

$$\frac{\partial G_i(\mathbf{p},\mathbf{r})}{\partial r_j} = \frac{\partial d_i(\mathbf{p},\mathbf{r})}{\partial r_j} = \begin{cases} c_i \cdot d_i(\mathbf{p},\mathbf{r}) \cdot \big(1 - d_i(\mathbf{p},\mathbf{r})\big) & \text{if } j = i, \\ -c_j \cdot d_i(\mathbf{p},\mathbf{r}) \cdot d_j(\mathbf{p},\mathbf{r}) & \text{if } j \neq i. \end{cases} \tag{L.29b}$$

Meanwhile, $G_i(\mathbf{p},\mathbf{r})$ and its gradient are bounded as follows:

$$\big|G_i(\mathbf{p},\mathbf{r})\big| \leq M_G, \quad \big\|\nabla_{\mathbf{r}} G_i(\mathbf{p},\mathbf{r})\big\| \leq \ell_{r,i}, \quad \forall \mathbf{p},\mathbf{r} \in \mathcal{P}^n,\ \forall i \in N, \tag{L.30}$$

where the upper bound M_G and the Lipschitz constant $\ell_{r,i}$ are defined as

$$M_G := \max_{i\in N} \left\{ \frac{1}{(b_i + c_i)\underline{p}} \right\} + 1, \quad \ell_{r,i} := \frac{1}{4}\sqrt{c_i^2 + \max_{j\neq i}\left\{c_j^2\right\}}. \tag{L.31}$$

Proof of Lemma EC.4. We first verify the partial derivatives in Eqs. (L.29a) and (L.29b):

$$\begin{aligned} \frac{\partial G_i(\mathbf{p},\mathbf{r})}{\partial p_i} &= -\frac{1}{(b_i+c_i)p_i^2} + \frac{\partial d_i(\mathbf{p},\mathbf{r})}{\partial p_i} \\ &= -\frac{1}{(b_i+c_i)p_i^2} - \frac{(b_i+c_i)\cdot \exp\big(u_i(p_i,r_i)\big)\cdot\Big(1+\sum_{j\neq i}\exp\big(u_j(p_j,r_j)\big)\Big)}{\Big(1+\sum_{k\in N}\exp\big(u_k(p_k,r_k)\big)\Big)^2} \\ &= -\frac{1}{(b_i+c_i)p_i^2} - (b_i+c_i)\cdot d_i(\mathbf{p},\mathbf{r})\cdot\big(1-d_i(\mathbf{p},\mathbf{r})\big). \end{aligned} \tag{L.32}$$

For product $j \neq i$, its partial derivative can be computed as

$$\begin{aligned} \frac{\partial G_i(\mathbf{p},\mathbf{r})}{\partial p_j} &= \frac{\partial d_i(\mathbf{p},\mathbf{r})}{\partial p_j} \\ &= \frac{(b_j+c_j)\cdot\exp\big(u_i(p_i,r_i)\big)\cdot\exp\big(u_j(p_j,r_j)\big)}{\Big(1+\sum_{k\in N}\exp\big(u_k(p_k,r_k)\big)\Big)^2} \\ &= (b_j+c_j)\cdot d_i(\mathbf{p},\mathbf{r})\cdot d_j(\mathbf{p},\mathbf{r}). \end{aligned} \tag{L.33}$$

Then, the partial derivatives with respect to $\mathbf{r}$, as shown in Eq. (L.29b), can be similarly computed.

In the next part, we show that $G_i(\mathbf{p},\mathbf{r})$ (see its definition in Eq. (D.2)) is bounded for all $\mathbf{p},\mathbf{r}\in\mathcal{P}^n$ and all product $i \in N$.

$$\begin{aligned} \big|G_i(\mathbf{p},\mathbf{r})\big| &= \left|\frac{1}{(b_i+c_i)p_i} + d_i(\mathbf{p},\mathbf{r}) - 1\right| \\ &\leq \left|\frac{1}{(b_i+c_i)p_i}\right| + \big|d_i(\mathbf{p},\mathbf{r}) - 1\big| \\ &\leq \frac{1}{(b_i+c_i)\underline{p}} + 1 \\ &\leq \max_{k\in N}\left\{\frac{1}{(b_k+c_k)\underline{p}}\right\} + 1 =: M_G, \end{aligned} \tag{L.34}$$

where the maximum operation in the last line is to ensure the validity of the bound for all $i \in N$.

Finally, we demonstrate that $\left\|\nabla_{\mathbf{r}} G_i(\mathbf{p},\mathbf{r})\right\|$ is also bounded for all $\mathbf{p},\mathbf{r}\in\mathcal{P}^n$ and all product $i\in N$. From Eq. (L.29b), we have that

$$\begin{aligned}\left\|\nabla_{\mathbf{r}} G_i(\mathbf{p},\mathbf{r})\right\|^2 &= \Big(c_i\cdot d_i(\mathbf{p},\mathbf{r})\cdot\big(1-d_i(\mathbf{p},\mathbf{r})\big)\Big)^2+\sum_{j\neq i}\Big(-c_j\cdot d_i(\mathbf{p},\mathbf{r})\cdot d_j(\mathbf{p},\mathbf{r})\Big)^2\\ &\le c_i^2\cdot\Big(d_i(\mathbf{p},\mathbf{r})\cdot\big(1-d_i(\mathbf{p},\mathbf{r})\big)\Big)^2+\max_{j\neq i}\{c_j^2\}\cdot\big(d_i(\mathbf{p},\mathbf{r})\big)^2\sum_{j\neq i}\big(d_j(\mathbf{p},\mathbf{r})\big)^2\\ &\overset{(\Delta_1)}{\le} c_i^2\cdot\Big(d_i(\mathbf{p},\mathbf{r})\cdot\big(1-d_i(\mathbf{p},\mathbf{r})\big)\Big)^2+\max_{j\neq i}\{c_j^2\}\cdot\Big(d_i(\mathbf{p},\mathbf{r})\cdot\big(1-d_i(\mathbf{p},\mathbf{r})\big)\Big)^2\\ &\overset{(\Delta_2)}{\le}\frac{1}{16}\Big(c_i^2+\max_{j\neq i}\{c_j^2\}\Big),\end{aligned}\tag{L.35}$$

where step (Δ_1) results from the fact that $\sum_{j\neq i}\big(d_j(\mathbf{p},\mathbf{r})\big)^2\le\big(\sum_{j\neq i}d_j(\mathbf{p},\mathbf{r})\big)^2\le\big(1-d_i(\mathbf{p},\mathbf{r})\big)^2$. The inequality (Δ_2) follows from the fact that $x\cdot y\le 1/4$ for any two numbers such that $x,y>0$ and $x+y\le 1$. Therefore, it follows that $\left\|\nabla_{\mathbf{r}} G_i(\mathbf{p},\mathbf{r})\right\|_2\le(1/4)\sqrt{c_i^2+\max_{j\neq i}\{c_j^2\}}=:\ell_{r,i}$. □

L.4 Lemma EC.5

Lemma EC.5. *In the loss-neutral scenario, the revenue and demand function satisfy that*

$$\|\nabla_{\mathbf{p}}\Pi_i(\mathbf{p},\mathbf{r})\|\le\ell_{p,i},\quad\|\nabla_{\mathbf{r}}\Pi_i(\mathbf{p},\mathbf{r})\|\le\overline{p}\cdot\ell_{r,i},\quad\|\nabla_{\mathbf{p}}d_i(\mathbf{p})\|\le\ell_{d,i},\quad\forall\mathbf{p},\mathbf{r}\in\mathcal{P}^n,\ \forall i\in N,\tag{L.36}$$

where $d_i(\mathbf{p}):=d_i(\mathbf{p},\mathbf{p})$*, constant* $\ell_{r,i}$ *is defined in Eq.* (L.31)*, and the Lipschitz constants* $\ell_{p,i},\ell_{d,i}$ *are defined as*

$$\ell_{p,i}:=\frac{1}{4}\sqrt{16+\overline{p}^2\left[(b_i+c_i)^2+\max_{j\neq i}\{(b_j+c_j)^2\}\right]},\quad\ell_{d,i}:=\frac{1}{4}\sqrt{b_i^2+\max_{j\neq i}\{b_j^2\}}.\tag{L.37}$$

Proof of Lemma EC.5. We begin with showing the first bound in Eq. (L.36). Since we have that

$$\frac{\partial\Pi_i(\mathbf{p},\mathbf{r})}{\partial p_j}=\begin{cases}d_i(\mathbf{p},\mathbf{r})-p_i(b_i+c_i)\cdot d_i(\mathbf{p},\mathbf{r})\cdot\big(1-d_i(\mathbf{p},\mathbf{r})\big) & \text{if } j=i,\\ p_i(b_j+c_j)\cdot d_i(\mathbf{p},\mathbf{r})\cdot d_j(\mathbf{p},\mathbf{r}) & \text{if } j\neq i.\end{cases}\tag{L.38}$$

Using the partial derivatives in Eq. (L.38), we compute that

$$\begin{aligned}\|\nabla_{\mathbf{p}}\Pi_i(\mathbf{p},\mathbf{r})\|^2&\le 1+\overline{p}^2\left\{\Big[(b_i+c_i)\cdot d_i(\mathbf{p},\mathbf{r})\cdot\big(1-d_i(\mathbf{p},\mathbf{r})\big)\Big]^2+\sum_{j\neq i}\Big[(b_j+c_j)\cdot d_i(\mathbf{p},\mathbf{r})\cdot d_j(\mathbf{p},\mathbf{r})\Big]^2\right\}\\ &\le 1+\frac{\overline{p}^2}{16}\left[(b_i+c_i)^2+\max_{j\neq i}\{(b_j+c_j)^2\}\right],\quad\forall\mathbf{p},\mathbf{r}\in\mathcal{P}^n,\ \forall i\in N,\end{aligned}$$

where $1/16$ in the last line follows from the same reasoning as Eq. (L.35). Taking the square root on both sides of the above inequality yields the desired upper bound for $\|\nabla_{\mathbf{p}}\Pi_i(\mathbf{p},\mathbf{r})\|$.

Next, we show the bound for $\|\nabla_{\mathbf{r}}\Pi_i(\mathbf{p},\mathbf{r})\|$. According to the definition of $G_i(\mathbf{p},\mathbf{r})$ in Eq. (D.2), it holds that $\|\nabla_{\mathbf{r}}G_i(\mathbf{p},\mathbf{r})\|=\|\nabla_{\mathbf{r}}d_i(\mathbf{p},\mathbf{r})\|\le\ell_{r,i}$, where the last inequality stems from Eq. (L.30). Since $\Pi_i(\mathbf{p},\mathbf{r})=p_i\cdot d_i(\mathbf{p},\mathbf{r})$, it follows that $\|\nabla_{\mathbf{r}}\Pi_i(\mathbf{p},\mathbf{r})\|=p_i\,\|\nabla_{\mathbf{r}}d_i(\mathbf{p},\mathbf{r})\|\le\overline{p}\cdot\ell_{r,i}$.

Finally, we derive the last bound in Eq. (L.36). Recall that $d_i(\mathbf{p}) := d_i(\mathbf{p}, \mathbf{p})$.

$$\frac{\partial d_i(\mathbf{p})}{\partial p_j} = \begin{cases} -b_i \cdot d_i(\mathbf{p}) \cdot \big(1 - d_i(\mathbf{p})\big) & \text{if } j = i, \\ b_j \cdot d_i(\mathbf{p}) \cdot d_j(\mathbf{p}) & \text{if } j \neq i. \end{cases} \tag{L.39}$$

Then, the above partial derivatives indicate that

$$\begin{aligned} \|\nabla_{\mathbf{p}} d_i(\mathbf{p})\|^2 &\leq b_i^2 \Big(d_i(\mathbf{p}) \cdot \big(1 - d_i(\mathbf{p})\big)\Big)^2 + \max_{j \neq i}\{b_j^2\} \cdot \big(d_i(\mathbf{p})\big)^2 \sum_{j \neq i} \big(d_j(\mathbf{p})\big)^2 \\ &\leq \frac{1}{16}\big(b_i^2 + \max_{j \neq i}\{b_j^2\}\big), \quad \forall \mathbf{p} \in \mathcal{P}^n, \ \forall i \in N, \end{aligned} \tag{L.40}$$

where the last inequality uses a similar reasoning in Eq. (L.35) again. Lastly, we take the square root of both sides to obtain $\|\nabla_{\mathbf{p}} d_i(\mathbf{p})\| \leq (1/4)\sqrt{b_i^2 + \max_{j \neq i}\{b_j^2\}} =: \ell_{d,i}$. □

L.5 Lemma EC.6

We observe from Eqs. (2) and (3) that the revenue $\Pi_i(\mathbf{p}, \mathbf{r})$ depends on $\mathbf{p}_{-i}$ and $\mathbf{r}_{-i}$ through their utility functions. Recall that $\mathbf{p}_{-i} := \big(p_j\big)_{j \in N \setminus \{i\}}$ and $\mathbf{r}_{-i} := \big(r_j\big)_{j \in N \setminus \{i\}}$. We use $\mathbf{u}_{-i}(\mathbf{p}_{-i}, \mathbf{r}_{-i})$ to denote the vector of utilities for all products except i, i.e.,

$$\mathbf{u}_{-i}(\mathbf{p}_{-i}, \mathbf{r}_{-i}) := \big(u_1(p_1, r_1), \ldots, u_{i-1}(p_{i-1}, r_{i-1}), u_{i+1}(p_{i+1}, r_{i+1}), \ldots, u_n(p_n, r_n)\big). \tag{L.41}$$

Given $\mathbf{p}_{-i}$ and $\mathbf{r} = (r_i, \mathbf{r}_{-i})$, we use $p_i^B\big(r_i, \mathbf{u}_{-i}(\mathbf{p}_{-i}, \mathbf{r}_{-i})\big)$ to denote the best-response price that achieve the optimal single-period revenue for product i, defined as

$$p_i^B\big(r_i, \mathbf{u}_{-i}(\mathbf{p}_{-i}, \mathbf{r}_{-i})\big) := \operatorname*{arg\,max}_{p_i \in \mathcal{P}} \big\{\Pi_i\big((p_i, \mathbf{p}_{-i}), \mathbf{r}\big)\big\} = \operatorname*{arg\,max}_{p_i \in \mathcal{P}} \big\{p_i \cdot d_i\big((p_i, \mathbf{p}_{-i}), \mathbf{r}\big)\big\}. \tag{L.42}$$

In the following lemma, we demonstrate the Lipschitz continuity of $p_i^B\big(r_i, \mathbf{u}_{-i}(\mathbf{p}_{-i}, \mathbf{r}_{-i})\big)$.

LEMMA EC.6. *In the loss-neutral scenario, let $p_i^B\big(r_i, \mathbf{u}_{-i}(\mathbf{p}_{-i}, \mathbf{r}_{-i})\big)$ be the best-response price for product i given $\mathbf{p}_{-i}$ and $\mathbf{r}$, as defined in Eq. (L.42). Then, it holds that*

$$\left|\frac{\partial p_i^B\big(r_i, \mathbf{u}_{-i}(\mathbf{p}_{-i}, \mathbf{r}_{-i})\big)}{\partial r_i}\right| \leq \frac{c_i}{b_i + c_i}, \quad \big\|\nabla_{\mathbf{u}_{-i}} p_i^B\big(r_i, \mathbf{u}_{-i}(\mathbf{p}_{-i}, \mathbf{r}_{-i})\big)\big\| \leq \overline{p}, \tag{L.43}$$

for all $\mathbf{p}_{-i} \in \mathcal{P}^{n-1}$ and $\mathbf{r} \in \mathcal{P}^n$.

Proof of Lemma EC.6. Given $\mathbf{p}_{-i}$ and $\mathbf{r}$, we use $\Pi_i^B\left(r_i, \mathbf{u}_{-i}\left(\mathbf{p}_{-i}, \mathbf{r}_{-i}\right)\right)$ to denote the optimal single-period revenue for product i, defined as $\Pi_i^B\left(r_i, \mathbf{u}_{-i}\left(\mathbf{p}_{-i}, \mathbf{r}_{-i}\right)\right) := \max_{p_i \in \mathcal{P}} \big\{\Pi_i\big((p_i, \mathbf{p}_{-i}), \mathbf{r}\big)\big\}$.

Similar as the Part 2 proof of Proposition 1, we can actually show that the first order-condition (i.e., $\partial \Pi_i(\mathbf{p}, \mathbf{r})/\partial p_i = 0$) is necessary and sufficient for the best-response price (see Eqs. (C.11) to (C.14) in Appendix C.1). We refer the readers to Theorems 1 and 2 in Guo et al. (2022) for a more detailed discussion. Hence, $\Pi_i^B(\cdot,\cdot)$ is the optimal single-period revenue if and only if the following first-order condition is satisfied

$$\begin{aligned}
&\frac{\partial \Pi_i(\mathbf{p},\mathbf{r})}{\partial p_i}\bigg|_{p_i=p_i^B(r_i,\mathbf{u}_{-i}(\mathbf{p}_{-i},\mathbf{r}_{-i}))}=0\\
\Leftrightarrow\quad & d_i^B-(b_i+c_i)\cdot p_i^B(r_i,\mathbf{u}_{-i}(\mathbf{p}_{-i},\mathbf{r}_{-i}))\cdot d_i^B\big(1-d_i^B\big)=0\\
\Leftrightarrow\quad & 1=(b_i+c_i)\cdot\big(p_i^B(r_i,\mathbf{u}_{-i}(\mathbf{p}_{-i},\mathbf{r}_{-i}))-\Pi_i^B(r_i,\mathbf{u}_{-i}(\mathbf{p}_{-i},\mathbf{r}_{-i}))\big)\\
\Leftrightarrow\quad & p_i^B(r_i,\mathbf{u}_{-i}(\mathbf{p}_{-i},\mathbf{r}_{-i}))=\Pi_i^B\big(r_i,\mathbf{u}_{-i}(\mathbf{p}_{-i},\mathbf{r}_{-i})\big)+\frac{1}{b_i+c_i},
\end{aligned}\tag{L.44}$$

where we use d_i^B to denote the demand at the best-response price, i.e., $d_i^B:=d_i\big((p_i^B,\mathbf{p}_{-i}),\mathbf{r}\big)$. From Eq. (L.44), we observe that $p_i^B(r_i,\mathbf{u}_{-i}(\mathbf{p}_{-i},\mathbf{r}_{-i}))$ and $\Pi_i^B\big(r_i,\mathbf{u}_{-i}(\mathbf{p}_{-i},\mathbf{r}_{-i})\big)$ only differs by a constant $1/(b_i+c_i)$. Hence, it is equivalent to derive the Lipschitz continuity of $\Pi_i^B\big(r_i,\mathbf{u}_{-i}(\mathbf{p}_{-i},\mathbf{r}_{-i})\big)$.

As the information of $\mathbf{p}_{-i}$ and $\mathbf{r}_{-i}$ is already absorbed in their utility functions, we adopt the shorthand notation $\mathbf{u}_{-i}:=\mathbf{u}_{-i}(\mathbf{p}_{-i},\mathbf{r}_{-i})$ and $u_j:=u_j(p_j,r_j)$ for all $j\in N\setminus\{i\}$. In addition, when it is clear from the context, we may also use the simplified notations $p_i^B:=p_i^B(r_i,\mathbf{u}_{-i})$ and $\Pi_i^B:=\Pi_i^B\big(r_i,\mathbf{u}_{-i}\big)$. Using the definition of the revenue function and the relation in Eq. (L.44), we can express Π_i^B using an implicit equation:

$$\begin{aligned}
&\Pi_i^B=p_i^Bd_i^B\overset{(\Delta_1)}{=}\Big(\Pi_i^B+\frac{1}{b_i+c_i}\Big)\frac{\exp\big(u_i(p_i^B,r_i)\big)}{1+\exp\big(u_i(p_i^B,r_i)\big)+\sum_{j\neq i}\exp\big(u_j\big)}\\
\Leftrightarrow\ &\Pi_i^B\Big(1+\sum_{j\neq i}\exp\big(u_j\big)\Big)=\frac{1}{b_i+c_i}\exp\big(u_i(p_i^B,r_i)\big)\\
\overset{(\Delta_2)}{\Leftrightarrow}\ &(b_i+c_i)\cdot\Pi_i^B\Big(1+\sum_{j\neq i}\exp\big(u_j\big)\Big)=\exp\big(a_i-(b_i+c_i)\cdot\Pi_i^B+c_ir_i-1\big),
\end{aligned}\tag{L.45}$$

where the expressions in (Δ_1) and (Δ_2) are obtained by substituting p_i^B with the first-order condition in the last line of Eq. (L.44).

With the goal of computing the Lipschitz coefficients, we use the implicit function theorem to derive the partial derivatives of $\Pi_i^B\big(r_i,\mathbf{u}_{-i}\big)$ with respect to r_i and $\mathbf{u}_{-i}$. To begin with, we first define a function $\Psi(\Pi_i,r_i,\mathbf{u}_{-i})$ as below

$$\Psi\big(\Pi_i,r_i,\mathbf{u}_{-i}\big)=(b_i+c_i)\cdot\Pi_i\cdot\Big(1+\sum_{j\neq i}\exp(u_j)\Big)-\exp\big(a_i-(b_i+c_i)\Pi_i+c_ir_i-1\big).\tag{L.46}$$

Since $\Psi\big(\Pi_i^B,r_i,\mathbf{u}_{-i}\big)=0$ by Eq. (L.45), we apply the implicit function theorem to derive that

$$\begin{aligned}
\frac{\partial \Pi_i^B(r_i,\mathbf{u}_{-i})}{\partial r_i}&=-\frac{\dfrac{\partial}{\partial r_i}\Psi\big(\Pi_i^B,r_i,\mathbf{u}_{-i}\big)}{\dfrac{\partial}{\partial \Pi_i}\Psi\big(\Pi_i^B,r_i,\mathbf{u}_{-i}\big)}\\
&=\frac{c_i\exp\big(a_i-(b_i+c_i)\Pi_i^B+c_ir_i-1\big)}{(b_i+c_i)\cdot\big(1+\sum_{j\neq i}\exp(u_j)\big)+(b_i+c_i)\cdot\exp\big(a_i-(b_i+c_i)\Pi_i^B+c_ir_i-1\big)}.
\end{aligned}\tag{L.47}$$

Hence, we can upper-bound the above partial derivative as detailed below

$$\begin{aligned}\left|\frac{\partial \Pi_i^B(r_i,\mathbf{u}_{-i})}{\partial r_i}\right| &= \left|\frac{c_i \exp\big(a_i-(b_i+c_i)\Pi_i^B+c_ir_i-1\big)}{(b_i+c_i)\cdot\big(1+\sum_{j\neq i}\exp(u_j)\big)+(b_i+c_i)\cdot\exp\big(a_i-(b_i+c_i)\Pi_i^B+c_ir_i-1\big)}\right|\\ &\overset{(\Delta)}{=} \left|\frac{c_i\exp\big(u_i(p_i^B,r_i)\big)}{(b_i+c_i)\cdot\Big[1+\exp\big(u_i(p_i^B,r_i)\big)+\sum_{j\neq i}\exp(u_j)\Big]}\right| \\ &= \frac{c_i}{b_i+c_i}\cdot d_i^B \le \frac{c_i}{b_i+c_i},\end{aligned} \tag{L.48}$$

where in step (Δ), we use the fact that $\Pi_i^B = p_i^B - 1/(b_i+c_i)$ from Eq. (L.44).

Next, to bound the gradient of $\Pi_i^B(r_i,\mathbf{u}_{-i})$ with respect to $\mathbf{u}_{-i}$, we first calculate its partial derivative for product $j\in N\setminus\{i\}$ by the implicit function theorem

$$\begin{aligned}\frac{\partial \Pi_i^B(r_i,\mathbf{u}_{-i})}{\partial u_j} &= -\frac{\frac{\partial}{\partial u_j}\Psi\big(\Pi_i^B,r_i,\mathbf{u}_{-i}\big)}{\frac{\partial}{\partial \Pi_i}\Psi\big(\Pi_i^B,r_i,\mathbf{u}_{-i}\big)}\\ &= \frac{-(b_i+c_i)\cdot\Pi_i^B\cdot\exp(u_j)}{(b_i+c_i)\cdot\big(1+\sum_{k\neq i}\exp(u_k)\big)+(b_i+c_i)\cdot\exp\big(a_i-(b_i+c_i)\Pi_i^B+c_ir_i-1\big)}\\ &= -\Pi_i^B\cdot d_j\big((p_i^B,\mathbf{p}_{-i}),\mathbf{r}\big).\end{aligned} \tag{L.49}$$

Then, we can bound the gradient as follows

$$\big\|\nabla_{\mathbf{u}_{-i}}\Pi_i^B\big(r_i,\mathbf{u}_{-i}\big)\big\| \le \Pi_i^B\cdot\sqrt{\sum_{j\neq i}\Big(d_j\big((p_i^B,\mathbf{p}_{-i}),\mathbf{r}\big)\Big)^2} \le \Pi_i^B = p_i^B\cdot d_i^B \le \overline{p}. \tag{L.50}$$

where the last inequality results from the fact that $d_i^B = d_i\big((p_i^B,\mathbf{p}_{-i}),\mathbf{r}\big)\le 1$ and $p_i^B\in[\underline{p},\overline{p}]$. As $p_i^B = \Pi_i^B + 1/(b_i+c_i)$ from Eq. (L.44), calculations in Eqs. (L.48) and (L.49) conclude the proof. ☐

L.6 Lemma EC.7

LEMMA EC.7. *In the loss-averse scenario, let $G_i^\diamond(\mathbf{p},\mathbf{r})$ be the scaled true/virtual derivative defined in Eq. (G.1), where $\diamond\in\{+,-\}$. Then, $G_i^\diamond(\mathbf{p},\mathbf{r})$ and its (sub)-gradients are bounded as follows*

$$|G_i^\diamond(\mathbf{p},\mathbf{r})|\le \tilde{M}_G,\ \|\nabla_{\mathbf{r}}G_i^\diamond(\mathbf{p},\mathbf{r})\|\le\tilde{\ell}_{r,i},\ \|\nabla_{\mathbf{p}}G_i^\diamond(\mathbf{p})\|\le\tilde{\ell}_{p,i},\ \forall\diamond\in\{+,-\},\ \forall\mathbf{p},\mathbf{r}\in\mathcal{P}^n,\ \forall i\in N, \tag{L.51}$$

where $G_i^\diamond(\mathbf{p}):=G_i^\diamond(\mathbf{p},\mathbf{p})$. The upper bound $\tilde{M}_G$ and the Lipschitz constants $\tilde{\ell}_{r,i}$, $\tilde{\ell}_{p,i}$ are defined as

$$\tilde{M}_G := \max_{i\in N}\left\{\frac{1}{(b_i+c_i^+)\underline{p}}\right\}+1,\quad \tilde{\ell}_{r,i} := \frac{1}{4}\sqrt{(c_i^-)^2+\max_{j\neq i}\big\{(c_j^-)^2\big\}}, \tag{L.52}$$

$$\tilde{\ell}_{p,i} := \sqrt{\frac{1}{(b_i+c_i^+)^2\underline{p}^4}+\frac{b_i}{2(b_i+c_i^+)\underline{p}^2}+\frac{b_i^2+\max_{j\neq i}\big\{b_j^2\big\}}{16}}. \tag{L.53}$$

Proof of Lemma EC.7. The first two bounds presented in Eq. (L.51) are analogous to their loss-neutral equivalents found in Eq. (L.30). It is straightforward to show that $|G_i^\diamond(\mathbf{p},\mathbf{r})| \le \tilde{M}_G$ and $\|\nabla_{\mathbf{r}} G_i^\diamond(\mathbf{p},\mathbf{r})\| \le \tilde{\ell}_{r,i}$ by similar procedures outlined in Eq. (L.34) and Eq. (L.35), respectively.

Now, we are left to show the last bound in Eq. (L.51). We start with computing the following partial derivative

$$\frac{\partial G_i^\diamond(\mathbf{p})}{\partial p_j} = \begin{cases} -\dfrac{1}{b_i + c_i^\diamond} \cdot \dfrac{1}{p_i^2} - b_i \cdot d_i(\mathbf{p}) \cdot \big(1 - d_i(\mathbf{p})\big) & \text{if } j = i, \\ b_j \cdot d_i(\mathbf{p}) \cdot d_j(\mathbf{p}) & \text{if } j \neq i, \end{cases} \tag{L.54}$$

where we recall that $d_i(\mathbf{p}) := d_i(\mathbf{p},\mathbf{p})$. With the information in Eq. (L.54), we are ready to derive the final bound:

$$\begin{aligned} \|\nabla_{\mathbf{p}} G_i^\diamond(\mathbf{p})\|^2 &\le \frac{1}{(b_i + c_i^\diamond)^2 \underline{p}^4} + \frac{2 b_i d_i(\mathbf{p})\big(1 - d_i(\mathbf{p})\big)}{(b_i + c_i^\diamond)\underline{p}^2} + b_i^2 \Big(d_i(\mathbf{p})\big(1 - d_i(\mathbf{p})\big)\Big)^2 + \sum_{j\neq i} \Big(b_j d_i(\mathbf{p}) d_j(\mathbf{p})\Big)^2 \\ &\overset{(\Delta)}{\le} \frac{1}{(b_i + c_i^\diamond)^2 \underline{p}^4} + \frac{b_i}{2(b_i + c_i^\diamond)\underline{p}^2} + \frac{b_i^2 + \max_{j\neq i}\{b_j^2\}}{16} \\ &\le \frac{1}{(b_i + c_i^+)^2 \underline{p}^4} + \frac{b_i}{2(b_i + c_i^+)\underline{p}^2} + \frac{b_i^2 + \max_{j\neq i}\{b_j^2\}}{16}, \end{aligned} \tag{L.55}$$

where the second term in step (Δ) follows from the fact that $d_i(\mathbf{p})\big(1 - d_i(\mathbf{p})\big) \le 1/4$, and the last term is derived via the same method used in (Δ_2) of Eq. (L.35). In the last line, we replace $c_i^\diamond$ with c_i^+ to ensure the bound works for both $\diamond \in \{+,-\}$, as $c_i^+ \le c_i^-$ in the loss-averse scenario. Taking the square root of both sides in Eq. (L.55) yields the final result, and this completes the proof of Lemma EC.7. □

L.7 Lemma EC.8

LEMMA EC.8. *In the loss-averse scenario, let* $\tilde{\mathbf{p}}^\star$ *be the unique SNE that satisfies*

$$\tilde{p}_i^\star = \frac{1}{(b_i + \tilde{c}_i)\cdot(1 - d_i(\tilde{\mathbf{p}}^\star, \tilde{\mathbf{p}}^\star))}, \quad \forall i \in N, \tag{L.56}$$

where $\tilde{c}_i \in [c_i^+, c_i^-]$*. Define the function* $\tilde{\mathcal{G}}(\mathbf{p})$ *as*

$$\tilde{\mathcal{G}}(\mathbf{p}) := \sum_{i\in N} \mathtt{sign}\,(\tilde{p}_i^\star - p_i)\left[\frac{1}{(b_i + \tilde{c}_i)p_i} + d_i(\mathbf{p},\mathbf{p}) - 1\right], \tag{L.57}$$

where the function $\mathtt{sign}(\cdot)$ *is defined in Eq.* (D.4)*. Then, it holds that*

$$\tilde{\mathcal{G}}(\mathbf{p}) \ge \frac{1}{\overline{p}\,\|\tilde{\mathbf{p}}^\star\|_\infty} \cdot \sum_{i\in N} \frac{|\tilde{p}_i^\star - p_i|}{b_i + \tilde{c}_i}, \quad \forall \mathbf{p} \in \mathcal{P}^n. \tag{L.58}$$

Proof of Lemma EC.8. This lemma is the loss-averse version of Lemma EC.3. Its proof follows a similar scheme as that of Lemma EC.3. □

Appendix M Proofs for Local Convergence of OPGA

M.1 Proof of Lemma EC.1

LEMMA EC.1 **(Restated)**. *In the loss-neutral scenario, define function* $\mathcal{H}(\mathbf{p})$ *as follows:*

$$\mathcal{H}(\mathbf{p}) := \sum_{i\in N}(p_i^\star - p_i)\cdot \frac{\partial \log\big(\Pi(\mathbf{p},\mathbf{r})\big)}{\partial p_i}\bigg|_{\mathbf{r}=\mathbf{p}} = \sum_{i\in N}\left[\frac{1}{p_i} + (b_i+c_i)\left(d_i(\mathbf{p},\mathbf{p})-1\right)\right](p_i^\star - p_i), \tag{M.1}$$

where $\mathbf{p}^\star$ *is the unique SNE. Then, there exist* $\gamma>0$ *and a open set* $U_\gamma \ni \mathbf{p}^\star$ *such that*

$$\mathcal{H}(\mathbf{p}) \geq \gamma\cdot\|\mathbf{p}-\mathbf{p}^\star\|^2, \quad \forall \mathbf{p}\in U_\gamma. \tag{M.2}$$

Proof of Lemma EC.1. We consider the second-order Taylor expansion of $\mathcal{H}(\mathbf{p})$ at $\mathbf{p}^\star$. For all $\mathbf{p}\in\mathcal{P}^n$, there exists $\widehat{\mathbf{p}}$ on the line segment between $\mathbf{p}$ and $\mathbf{p}^\star$ such that

$$\begin{aligned}\mathcal{H}(\mathbf{p}) &= \mathcal{H}(\mathbf{p}^\star) + \nabla\mathcal{H}(\mathbf{p}^\star)\cdot\big(\mathbf{p}-\mathbf{p}^\star\big) + \frac{1}{2}\big(\mathbf{p}-\mathbf{p}^\star\big)^\top\nabla^2\mathcal{H}(\widehat{\mathbf{p}})\cdot\big(\mathbf{p}-\mathbf{p}^\star\big)\\ &= \nabla\mathcal{H}(\mathbf{p}^\star)\cdot\big(\mathbf{p}-\mathbf{p}^\star\big) + \frac{1}{2}\big(\mathbf{p}-\mathbf{p}^\star\big)^\top\nabla^2\mathcal{H}(\widehat{\mathbf{p}})\cdot\big(\mathbf{p}-\mathbf{p}^\star\big),\end{aligned} \tag{M.3}$$

where the second equality arises from $\mathcal{H}(\mathbf{p}^\star)=0$. We first compute the the gradient $\nabla\mathcal{H}(\mathbf{p}) = (\partial\mathcal{H}(\mathbf{p})/\partial p_1,\ldots,\partial\mathcal{H}(\mathbf{p})/\partial p_n)$, where we adopt the shorthand notation $d_i(\mathbf{p}) := d_i(\mathbf{p},\mathbf{p})$ and use the partial derivative of $d_i(\mathbf{p})$ in Eq. (L.39):

$$\begin{aligned}\frac{\partial\mathcal{H}(\mathbf{p})}{\partial p_i} &= -\left[\frac{1}{p_i} + (b_i+c_i)\big(d_i(\mathbf{p})-1\big)\right] + (p_i^\star - p_i)\left[-\frac{1}{p_i^2} - (b_i+c_i)b_i\cdot d_i(\mathbf{p})\big(1-d_i(\mathbf{p})\big)\right]\\ &\quad + \sum_{j\neq i}(p_j^\star - p_j)\cdot(b_j+c_j)b_i\cdot d_j(\mathbf{p})d_i(\mathbf{p})\\ &= -\left[\frac{1}{p_i} + (b_i+c_i)\big(d_i(\mathbf{p})-1\big)\right] + b_i\cdot d_i(\mathbf{p})\sum_{k\in N}(p_k^\star - p_k)\cdot(b_k+c_k)d_k(\mathbf{p})\\ &\quad - (p_i^\star - p_i)\left[\frac{1}{p_i^2} + (b_i+c_i)b_i\cdot d_i(\mathbf{p})\right].\end{aligned} \tag{M.4}$$

When this partial derivative evaluates at $\mathbf{p}^\star$, the first term becomes $1/p_i^\star + (b_i+c_i)\big(d_i(\mathbf{p}^\star)-1\big)=0$, since $\mathbf{p}^\star$ satisfies the first-order condition in Eq. (C.16). Hence, it follows that $\nabla\mathcal{H}(\mathbf{p}^\star)=0$, and Eq. (M.3) simplifies to $\mathcal{H}(\mathbf{p}) = \frac{1}{2}\big(\mathbf{p}-\mathbf{p}^\star\big)^\top\nabla^2\mathcal{H}(\widehat{\mathbf{p}})\cdot\big(\mathbf{p}-\mathbf{p}^\star\big)$.

Below, we aim to show that there exists $\gamma>0$ such that $\nabla^2\mathcal{H}(\mathbf{p})\succ 2\gamma I_n$ when $\mathbf{p}$ belongs to some neighborhood U_γ of $\mathbf{p}^\star$, where I_n is the $n\times n$ identity matrix. Then, for any $\mathbf{p}\in U_\gamma$, it follows that

$$\mathcal{H}(\mathbf{p}) = \frac{1}{2}\big(\mathbf{p}-\mathbf{p}^\star\big)^\top\nabla^2\mathcal{H}(\widehat{\mathbf{p}})\cdot\big(\mathbf{p}-\mathbf{p}^\star\big) \geq \gamma\,\|\mathbf{p}-\mathbf{p}^\star\|^2. \tag{M.5}$$

We first compute the Hessian matrix $\nabla^2 \mathcal{H}(\mathbf{p})$ evaluated at $\mathbf{p}^\star$. The second-order partial derivatives can be calculated as follows

$$\begin{aligned}
\frac{\partial^2 \mathcal{H}(\mathbf{p}^\star)}{\partial p_i^2} &= \frac{1}{(p_i^\star)^2} + (b_i + c_i) b_i \cdot d_i(\mathbf{p}^\star)\big(1 - d_i(\mathbf{p}^\star)\big) - (b_i + c_i) b_i \cdot \big(d_i(\mathbf{p}^\star)\big)^2 + \frac{1}{(p_i^\star)^2} + (b_i + c_i) b_i \cdot d_i(\mathbf{p}^\star) \\
&= \frac{2}{(p_i^\star)^2} + 2(b_i + c_i) b_i \cdot d_i(\mathbf{p}^\star)\big(1 - d_i(\mathbf{p}^\star)\big) \\
&\overset{(\Delta)}{=} 2(b_i + c_i)^2 \big(1 - d_i(\mathbf{p}^\star)\big)^2 + 2(b_i + c_i) b_i \cdot d_i(\mathbf{p}^\star)\big(1 - d_i(\mathbf{p}^\star)\big) \\
&= 2(b_i + c_i) \cdot \big(1 - d_i(\mathbf{p}^\star)\big)\big(b_i + c_i - c_i d_i(\mathbf{p}^\star)\big) \\
&= (b_i + c_i)^2 \cdot 2\big(1 - d_i(\mathbf{p}^\star)\big)\left(1 - \frac{c_i}{b_i + c_i} d_i(\mathbf{p}^\star)\right),
\end{aligned} \tag{M.6}$$

were step (Δ) utilizes the first-order condition in Eq. (C.16) and substitutes $1/p_i^\star$ with $(b_i + c_i)\big(1 - d_i(\mathbf{p}^\star)\big)$. Similarly, we can compute the second-order cross derivatives as

$$\begin{aligned}
\frac{\partial^2 \mathcal{H}(\mathbf{p}^\star)}{\partial p_i \partial p_j} &= -(b_i + c_i) b_j \cdot d_i(\mathbf{p}^\star) d_j(\mathbf{p}^\star) - b_i (b_j + c_j) \cdot d_i(\mathbf{p}^\star) d_j(\mathbf{p}^\star) \\
&= -\big[(b_j + c_j) b_i + (b_i + c_i) b_j\big] \cdot d_i(\mathbf{p}^\star) d_j(\mathbf{p}^\star) \\
&= -(b_i + c_i)(b_j + c_j) \cdot \left(\frac{b_i}{b_i + c_i} + \frac{b_j}{b_j + c_j}\right) d_i(\mathbf{p}^\star) d_j(\mathbf{p}^\star).
\end{aligned} \tag{M.7}$$

Based on Eqs. (M.6) and Eq. (M.7), we observe that the Hessian matrix can be decomposed as $\nabla^2 \mathcal{H}(\mathbf{p}^\star) = AQA$, where A is a diagonal matrix defined as

$$A := \begin{bmatrix} b_1 + c_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & b_n + c_n \end{bmatrix},$$

and matrix Q is defined such that its (i, j)-th entry is equal to

$$Q_{ij} = \begin{cases} 2\big(1 - d_i(\mathbf{p}^\star)\big)\left(1 - \dfrac{c_i}{b_i + c_i} d_i(\mathbf{p}^\star)\right) & \text{if } i = j, \\ -\left(\dfrac{b_i}{b_i + c_i} + \dfrac{b_j}{b_j + c_j}\right) d_i(\mathbf{p}^\star) d_j(\mathbf{p}^\star) & \text{if } i \neq j. \end{cases} \tag{M.8}$$

Since A is clearly an invertible matrix, demonstrating $\nabla^2 \mathcal{H}(\mathbf{p}^\star)$ is positive definite is equivalent to proving Q is positive definite. Then, it suffices to show for any vector $\mathbf{x}$, $\mathbf{x}^\top Q \mathbf{x} > 0$. Without loss of generality, we assume $\|\mathbf{x}\|=1$, and it follows that

$$\begin{aligned}
\mathbf{x}^\top Q\mathbf{x} &= 2\sum_{i\in N}\left[x_i^2\big(1-d_i(\mathbf{p}^\star)\big)\left(1-\frac{c_i}{b_i+c_i}d_i(\mathbf{p}^\star)\right)-\frac{1}{2}\sum_{j\neq i}x_ix_j\left(\frac{b_i}{b_i+c_i}+\frac{b_j}{b_j+c_j}\right)d_i(\mathbf{p}^\star)d_j(\mathbf{p}^\star)\right]\\
&\geq 2\sum_{i\in N}\left[x_i^2\big(1-d_i(\mathbf{p}^\star)\big)\left(1-\frac{c_i}{b_i+c_i}d_i(\mathbf{p}^\star)\right)-\frac{1}{2}\sum_{j\neq i}|x_i|\,|x_j|\left(\frac{b_i}{b_i+c_i}+\frac{b_j}{b_j+c_j}\right)d_i(\mathbf{p}^\star)d_j(\mathbf{p}^\star)\right]\\
&> 2\sum_{i\in N}\left[x_i^2\big(1-d_i(\mathbf{p}^\star)\big)^2-\sum_{j\neq i}|x_i|\,|x_j|\,d_i(\mathbf{p}^\star)d_j(\mathbf{p}^\star)\right]\\
&= 2\sum_{i\in N}\left\{x_i^2\left[\big(1-d_i(\mathbf{p}^\star)\big)^2+\big(d_i(\mathbf{p}^\star)\big)^2\right]-\sum_{j\in N}|x_i|\,|x_j|\,d_i(\mathbf{p}^\star)d_j(\mathbf{p}^\star)\right\}\\
&\overset{(\Delta_1)}{=} 2\left\{\sum_{i\in N}x_i^2\left[\big(1-d_i(\mathbf{p}^\star)\big)^2+\big(d_i(\mathbf{p}^\star)\big)^2\right]-\left(\sum_{i\in N}|x_i|\,d_i(\mathbf{p}^\star)\right)^2\right\}\\
&\overset{(\Delta_2)}{>} 2\left\{\sum_{i\in N}x_i^2\left[\left(\sum_{j\neq i}d_j(\mathbf{p}^\star)\right)^2+\big(d_i(\mathbf{p}^\star)\big)^2\right]-\sum_{i\in N}\big(d_i(\mathbf{p}^\star)\big)^2\right\}\\
&> 2\left\{\sum_{i\in N}x_i^2\cdot\sum_{i\in N}\big(d_i(\mathbf{p}^\star)\big)^2-\sum_{i\in N}\big(d_i(\mathbf{p}^\star)\big)^2\right\}=0,
\end{aligned}$$

where step (Δ_1) follows from $\sum_{i\in N}\sum_{j\in N}|x_i|\,|x_j|\,d_i(\mathbf{p}^\star)d_j(\mathbf{p}^\star)=\left(\sum_{i\in N}|x_i|\,d_i(\mathbf{p}^\star)\right)^2$. In step (Δ_2), we first use the fact that $1-d_i(\mathbf{p}^\star)=d_0(\mathbf{p}^\star)+\sum_{j\neq i}d_j(\mathbf{p}^\star)>\sum_{j\neq i}d_j(\mathbf{p}^\star)$, where $d_0(\mathbf{p}^\star)$ is the no-purchase probability. Then, we use the Cauchy-Schwarz inequality, i.e., $\left(\sum_{i\in N}|x_i|\,d_i(\mathbf{p}^\star)\right)^2\leq\|\mathbf{x}\|^2\sum_{i\in N}\big(d_i(\mathbf{p})\big)^2$.

As a result, we conclude that $\nabla^2\mathcal{H}(\mathbf{p}^\star)$ is positive definite. By the continuity of $\nabla^2\mathcal{H}(\mathbf{p})$, there exists some constant $\gamma>0$ and a open set $U_\gamma\ni\mathbf{p}^\star$ such that $\nabla^2\mathcal{H}(\mathbf{p}^\star)\succ 2\gamma I_n$ for all $\mathbf{p}\in U_\gamma$. Together with Eq. (M.5), this completes the proof of Lemma EC.1. □

M.2 Proof of Proposition EC.1

Proposition EC.1. **(Restated)** *In the loss-neutral scenario, let the step-sizes $\{\eta^t\}_{t\geq 0}$ be a non-increasing sequence such that $\lim_{t\to\infty}\eta^t=0$ and $\sum_{t=0}^{\infty}\eta^t=\infty$ hold. Then, there exists some neighborhood $\mathcal{B}$ of $\mathbf{p}^\star$ such that when the price path $\{\mathbf{p}^t\}_{t\geq 0}$ enters $\mathcal{B}$ with a sufficiently small step-size, the price path will stay in $\mathcal{B}$ during subsequent periods.*

Furthermore, suppose the step-sizes satisfy $\eta^t=\frac{C_\eta}{t+1}$ for all $t\geq 1$, where C_η is some general constant. Then, the local convergence rate of $\{(\mathbf{p}^t,\mathbf{r}^t)\}_{t\geq 0}$ after the path stays in $\mathcal{B}$ satisfies that

$$\left\|\mathbf{p}^\star-\mathbf{p}^t\right\|^2\leq\mathcal{O}\left(\frac{1}{t}\right),\quad \left\|\mathbf{p}^\star-\mathbf{r}^t\right\|^2\leq\mathcal{O}\left(\frac{1}{t}\right). \tag{M.9}$$

Proof of Proposition EC.1. Let $\{\mathbf{p}^t\}_{t\geq 0}$ be the price path generated by Algorithm 1 with step-sizes $\{\eta^t\}_{t\geq 0}$ such that $\lim_{t\to\infty}\eta^t=0$ and $\sum_{t=0}^{\infty}\eta^t=\infty$. In the following, we use Lemma EC.1

demonstrate that when the price path $\{\mathbf{p}^t\}_{t\geq 0}$ enters the ℓ^2-neighborhood $\mathcal{B}_{\epsilon_0} := \big\{\mathbf{p}\in\mathcal{P}^n \mid \|\mathbf{p}-\mathbf{p}^\star\| < \epsilon_0\big\}$ for some sufficiently small $\epsilon_0 > 0$ with small enough step-sizes, the price path will stay in $\mathcal{B}_{\epsilon_0}$ during subsequent periods. In particular, we prove it by induction, where we show that when $\mathbf{p}^t \in \mathcal{B}_{\epsilon_0}$ for some sufficiently large t, then it also holds that $\mathbf{p}^{t+1}\in\mathcal{B}_{\epsilon_0}$. The value of ϵ_0 will be specified later in the proof.

By the update rule of Algorithm 1, it follows that

$$\begin{aligned}
\left|p_i^\star - p_i^{t+1}\right|^2 &= \left|p_i^\star - \texttt{Proj}_{\mathcal{P}}\left(p_i^t+\eta^t D_i^t\right)\right|^2 \\
&\leq \left|p_i^\star - \left(p_i^t+\eta^t D_i^t\right)\right|^2 \\
&\overset{(\Delta_1)}{=} \left|\left(p_i^\star - p_i^t\right) - \eta^t(b_i+c_i)G_i(\mathbf{p}^t,\mathbf{r}^t)\right|^2 \\
&= \left|p_i^\star - p_i^t\right|^2 - 2\left(p_i^\star - p_i^t\right)\cdot\eta^t(b_i+c_i)G_i(\mathbf{p}^t,\mathbf{r}^t) + \left[\eta^t(b_i+c_i)G_i(\mathbf{p}^t,\mathbf{r}^t)\right]^2 \\
&= \left|p_i^\star - p_i^t\right|^2 - 2\left(p_i^\star - p_i^t\right)\cdot\eta^t(b_i+c_i)G_i(\mathbf{p}^t,\mathbf{p}^t) + \left[\eta^t(b_i+c_i)G_i(\mathbf{p}^t,\mathbf{r}^t)\right]^2 \\
&\quad + 2\left(p_i^\star - p_i^t\right)\cdot\eta^t(b_i+c_i)\left[G_i(\mathbf{p}^t,\mathbf{p}^t) - G_i(\mathbf{p}^t,\mathbf{r}^t)\right] \\
&\overset{(\Delta_2)}{\leq} \left|p_i^\star - p_i^t\right|^2 - 2\left(p_i^\star - p_i^t\right)\cdot\eta^t(b_i+c_i)G_i(\mathbf{p}^t,\mathbf{p}^t) + \left(\eta^t(b_i+c_i)M_G\right)^2 \\
&\quad + 2\left|p_i^\star - p_i^t\right|\cdot\eta^t(b_i+c_i)\cdot\ell_{r,i}\left\|\mathbf{p}^t-\mathbf{r}^t\right\|,
\end{aligned} \tag{M.10}$$

where step (Δ_1) uses the definition of the scaled derivative $G_i(\mathbf{p},\mathbf{r})$ in Eq. (D.2) and the equivalence that $D_i^t = (b_i+c_i)G_i(\mathbf{p}^t,\mathbf{r}^t)$ from Eq. (D.3). In step (Δ_2), we use $|G_i(\mathbf{p},\mathbf{r})|\leq M_G$ and the mean value theorem with the fact that $\left\|\nabla_{\mathbf{r}}G_i(\mathbf{p},\mathbf{r})\right\|\leq \ell_{r,i}$ (see Lemma EC.4).

Let $\mathcal{H}(\mathbf{p})$ be the function defined as

$$\mathcal{H}(\mathbf{p}) := \sum_{i\in N}(p_i^\star - p_i)\cdot\frac{\partial\log\left(\Pi(\mathbf{p},\mathbf{r})\right)}{\partial p_i}\bigg|_{\mathbf{r}=\mathbf{p}} = \sum_{i\in N}(b_i+c_i)\cdot G_i(\mathbf{p},\mathbf{p})\cdot(p_i^\star - p_i). \tag{M.11}$$

Then, by summing Eq. (M.10) over all products $i\in N$, we have that

$$\begin{aligned}
\left\|\mathbf{p}^\star - \mathbf{p}^{t+1}\right\|^2 &\leq \left\|\mathbf{p}^\star - \mathbf{p}^t\right\|^2 - 2\eta^t\sum_{i\in N}\left(p_i^\star - p_i^t\right)\cdot(b_i+c_i)G_i(\mathbf{p}^t,\mathbf{p}^t) \\
&\quad + (\eta^t M_G)^2\sum_{i\in N}(b_i+c_i)^2 + 2\eta^t\left\|\mathbf{p}^t-\mathbf{r}^t\right\|\sum_{i\in N}\ell_{r,i}(b_i+c_i)\left|p_i^\star - p_i^t\right| \\
&= \left\|\mathbf{p}^\star - \mathbf{p}^t\right\|^2 - 2\eta^t\mathcal{H}(\mathbf{p}^t) + (\eta^t M_G)^2\sum_{i\in N}(b_i+c_i)^2 \\
&\quad + 2\eta^t\left\|\mathbf{p}^t-\mathbf{r}^t\right\|\sum_{i\in N}\ell_{r,i}(b_i+c_i)\left|p_i^\star - p_i^t\right| \\
&\leq \left\|\mathbf{p}^\star - \mathbf{p}^t\right\|^2 - \eta^t\Big(2\mathcal{H}(\mathbf{p}^t) - \eta^t\omega_1 - \omega_2\left\|\mathbf{p}^t-\mathbf{r}^t\right\|\Big),
\end{aligned} \tag{M.12}$$

where we denote $\omega_1 := M_G^2\cdot\sum_{i\in N}(b_i+c_i)^2$ and $\omega_2 = 2|\overline{p}-\underline{p}|\cdot\sum_{i\in N}\ell_{r,i}(b_i+c_i)$.

By Lemma EC.1, there exist $\gamma > 0$ and a open set $U_\gamma \ni \mathbf{p}^\star$ such that $\mathcal{H}(\mathbf{p}) \geq \gamma \cdot \|\mathbf{p} - \mathbf{p}^\star\|^2$, $\forall \mathbf{p} \in U_\gamma$. Consider $\epsilon_0 > 0$ such that the ℓ^2-neighborhood $\mathcal{B}_{\epsilon_0} = \left\{\mathbf{p} \in \mathcal{P}^n \mid \|\mathbf{p} - \mathbf{p}^\star\| < \epsilon_0\right\} \subset U_\gamma$. Furthermore, let T_γ be some period such that for all $t \in T_\gamma$, it holds that

$$\eta^t \left(\eta^t \omega_1 + \sqrt{n}\omega_2(\overline{p} - \underline{p})\right) \leq \frac{\epsilon_0^2}{4} \quad \text{and} \quad \eta^t \omega_1 + \omega_2 \left\|\mathbf{p}^t - \mathbf{r}^t\right\| \leq \frac{\gamma \epsilon_0^2}{2}. \tag{M.13}$$

The existence of such a T_γ follows from the fact that $\lim_{t\to\infty} \eta^t = 0$ and $\lim_{t\to\infty} \|\mathbf{p}^t - \mathbf{r}^t\| = 0$ (see Lemma EC.2). Below, we discuss two cases depending on the location of $\mathbf{p}^t$ in $\mathcal{B}_{\epsilon_0}$.

Case 1. $\mathbf{p}^t \in \mathcal{B}_{\epsilon_0/2} \subset \mathcal{B}_{\epsilon_0}$, i.e., $\|\mathbf{p}^\star - \mathbf{p}^t\| < \epsilon_0/2$.

Since $\mathcal{H}(\mathbf{p}) \geq 0$, $\forall \mathbf{p} \in U_\gamma$ by Lemma EC.1, it follows from Eq. (M.12) and Eq. (M.13) that

$$\begin{aligned} \left\|\mathbf{p}^\star - \mathbf{p}^{t+1}\right\|^2 &\leq \left\|\mathbf{p}^\star - \mathbf{p}^t\right\|^2 + \eta^t \Big(\eta^t \omega_1 + \omega_2 \left\|\mathbf{p}^t - \mathbf{r}^t\right\|\Big) \\ &\overset{(\Delta)}{\leq} \frac{\epsilon_0^2}{4} + \eta^t \left(\eta^t \omega_1 + \sqrt{n}\omega_2(\overline{p} - \underline{p})\right) \\ &\leq \frac{\epsilon_0^2}{4} + \frac{\epsilon_0^2}{4} < \epsilon_0^2, \end{aligned} \tag{M.14}$$

where inequality (Δ) is due to $\|\mathbf{p}^t - \mathbf{r}^t\| \leq \sqrt{n}(\overline{p} - \underline{p})$. Eq. (M.14) implies that $\mathbf{p}^{t+1} \in \mathcal{B}_{\epsilon_0}$.

Case 2. $\mathbf{p}^t \in \mathcal{B}_{\epsilon_0} \backslash \mathcal{B}_{\epsilon_0/2}$, i.e., $\|\mathbf{p}^\star - \mathbf{p}^t\| \in [\epsilon_0/2, \epsilon_0)$.

By Lemma EC.1, we have that $\mathcal{H}(\mathbf{p}^t) \geq \gamma \left\|\mathbf{p}^\star - \mathbf{p}^t\right\|^2 \geq \gamma \epsilon_0^2/4$. Thus, again by Eq. (M.12) and Eq. (M.13), we have that

$$\begin{aligned} \left\|\mathbf{p}^\star - \mathbf{p}^{t+1}\right\|^2 &\leq \left\|\mathbf{p}^\star - \mathbf{p}^t\right\|^2 - \eta^t \Big(2\mathcal{H}(\mathbf{p}^t) - \eta^t \omega_1 - \omega_2 \left\|\mathbf{p}^t - \mathbf{r}^t\right\|\Big) \\ &\leq \left\|\mathbf{p}^\star - \mathbf{p}^t\right\|^2 - \eta^t \Big(\frac{\gamma \epsilon_0^2}{2} - \eta^t \omega_1 - \omega_2 \left\|\mathbf{p}^t - \mathbf{r}^t\right\|\Big) \\ &\leq \left\|\mathbf{p}^\star - \mathbf{p}^t\right\|^2 \leq \epsilon_0^2, \end{aligned} \tag{M.15}$$

which implies $\mathbf{p}^{t+1} \in \mathcal{B}_{\epsilon_0}$. Therefore, we conclude by induction that the price path will stay in the ℓ^2-neighborhood $\mathcal{B}_{\epsilon_0}$.

Next, we proceed to show the local convergence rate in Eq. (M.9). Using the fact that $\mathcal{H}(\mathbf{p}) \geq \gamma \cdot \|\mathbf{p} - \mathbf{p}^\star\|^2$ for all $\mathbf{p} \in U_\gamma$ from Lemma EC.1, we can further derive from Eq. (M.12) that

$$\begin{aligned} \left\|\mathbf{p}^\star - \mathbf{p}^{t+1}\right\|^2 &\leq \left\|\mathbf{p}^\star - \mathbf{p}^t\right\|^2 - 2\eta^t \gamma \left\|\mathbf{p}^\star - \mathbf{p}^t\right\|^2 + \omega_1 (\eta^t)^2 + 2\eta^t \left\|\mathbf{p}^t - \mathbf{r}^t\right\| \sum_{i \in N} \ell_{r,i}(b_i + c_i) \left|p_i^\star - p_i^t\right| \\ &\overset{(\Delta_1)}{\leq} \left\|\mathbf{p}^\star - \mathbf{p}^t\right\|^2 - 2\eta^t \gamma \left\|\mathbf{p}^\star - \mathbf{p}^t\right\|^2 + \omega_1 (\eta^t)^2 + 2\eta^t \left\|\mathbf{p}^t - \mathbf{r}^t\right\| \cdot \hat{k} \left\|\mathbf{p}^\star - \mathbf{p}^t\right\| \\ &\overset{(\Delta_2)}{\leq} \left\|\mathbf{p}^\star - \mathbf{p}^t\right\|^2 - 2\eta^t \gamma \left\|\mathbf{p}^\star - \mathbf{p}^t\right\|^2 + \omega_1 (\eta^t)^2 + \eta^t \hat{k} \left[\frac{\hat{k}}{\gamma} \left\|\mathbf{p}^t - \mathbf{r}^t\right\|^2 + \frac{\gamma}{\hat{k}} \left\|\mathbf{p}^\star - \mathbf{p}^t\right\|^2\right] \\ &\overset{(\Delta_3)}{\leq} \left\|\mathbf{p}^\star - \mathbf{p}^t\right\|^2 - \eta^t \gamma \left\|\mathbf{p}^\star - \mathbf{p}^t\right\|^2 + \omega_1 (\eta^t)^2 + \eta^t k \left\|\mathbf{p}^t - \mathbf{r}^t\right\|^2, \end{aligned} \tag{M.16}$$

where (Δ_1) utilizes the fact that $\sum_{i \in N} \ell_{r,i}(b_i + c_i) \left|p_i^\star - p_i^t\right| \leq \max_{i \in N} \{\ell_{r,i}(b_i + c_i)\} \left\|\mathbf{p}^\star - \mathbf{p}^t\right\|_1 \leq \sqrt{n} \max_{i \in N} \{\ell_{r,i}(b_i + c_i)\} \left\|\mathbf{p}^\star - \mathbf{p}^t\right\|$, and we define $\hat{k} := \sqrt{n} \max_{i \in N} \{\ell_{r,i}(b_i + c_i)\}$. Step (Δ_2) follows from the inequality of arithmetic and geometric means, i.e., $2xy \le Ax^2 + (1/A)y^2$ for any constant $A > 0$. The value of constant k in (Δ_4) is given by $k := \hat{k}^2/\gamma = n\big(\max_{i\in N}\{\ell_{r,i}(b_i + c_i)\}\big)^2/\gamma$.

Our goal is to upper-bound the right-hand side of Eq. (M.16). By a similar technique used in Case 1 of Lemma EC.2, we can demonstrate that $\|\mathbf{p}^t - \mathbf{r}^t\|^2 = \mathcal{O}(1/t^2)$ for reasonably large t. Together with the step-sizes of $\eta_t = C_\eta/(t+1)$, we can further bound Eq. (M.16) as

$$\begin{aligned} \left\|\mathbf{p}^\star - \mathbf{p}^{t+1}\right\|^2 &\le \left(1 - \frac{\gamma C_\eta}{t+1}\right)\left\|\mathbf{p}^\star - \mathbf{p}^t\right\|^2 + \frac{\omega_1 C_\eta^2}{(t+1)^2} + \frac{kC_\eta}{t+1}\cdot \mathcal{O}\left(\frac{1}{t^2}\right) \\ &\le \left(1 - \frac{\gamma C_\eta}{t+1}\right)\left\|\mathbf{p}^\star - \mathbf{p}^t\right\|^2 + \frac{\omega_3}{t(t+1)}, \end{aligned} \tag{M.17}$$

where ω_3 in the last line is some constant that satisfies $\frac{\omega_3}{t(t+1)} \ge \frac{\omega_1 C_\eta^2}{(t+1)^2} + \frac{kC_\eta}{t+1}\cdot \mathcal{O}\left(\frac{1}{t^2}\right)$.

Then, we inductively show that $\|\mathbf{p}^\star - \mathbf{p}^t\|^2 = \mathcal{O}(1/t)$ when $\mathbf{p}^t \in \mathcal{B}_{\epsilon_0}$. According to the first part of Proposition EC.1, there exists $T_{\epsilon_0} > 0$ such that $t \in \mathcal{B}_\epsilon$ for every $t \ge T_{\epsilon_0}$. Suppose there exists a constant d_p such that for a fixed period $t \ge T_{\epsilon_0}$, it holds that

$$\left\|\mathbf{p}^\star - \mathbf{p}^t\right\|^2 \le \frac{d_p}{t}. \tag{M.18}$$

To establish the induction, it is sufficient to show that

$$\left\|\mathbf{p}^\star - \mathbf{p}^{t+1}\right\|^2 \le \left(1 - \frac{\gamma C_\eta}{t+1}\right)\frac{d_p}{t} + \frac{\omega_3}{t(t+1)} \le \frac{d_p}{t+1}, \tag{M.19}$$

where the first inequality follows from Eq. (M.17). Then, Eq. (M.19) is further equivalent to

$$(\gamma C_\eta - 1)d_p \ge \omega_3. \tag{M.20}$$

Hence, as long as $\gamma C_\eta - 1 > 0$, there exists $d_p > 0$ such that the induction in Eq. (M.19) holds. For example, when $C_\eta = 2/\gamma$, we can take $d_p = \max\left\{nT_{\epsilon_0}(\overline{p} - \underline{p})^2, \omega_3\right\}$, where the first term in the maximization bracket ensures the base case of the induction. This completes the proof of the local convergence rate for the price path. Finally, the convergence rate of the reference price path can be deduced from the following triangular inequality

$$\left\|\mathbf{p}^\star - \mathbf{r}^t\right\|^2 = \left\|\mathbf{p}^\star - \mathbf{p}^t + \mathbf{p}^t - \mathbf{r}^t\right\|^2 \le 2\left\|\mathbf{p}^\star - \mathbf{p}^t\right\|^2 + 2\left\|\mathbf{p}^t - \mathbf{r}^t\right\|^2 = \mathcal{O}\left(\frac{1}{t}\right), \tag{M.21}$$

which completes the proof of Proposition EC.1. □

Appendix N Summary of Constants

In the following Table EC.1 and Table EC.2, we summarize the definitions of all constants used in the paper, along with references to their initial occurrences.

Table EC.1 Summary of Constants for the Loss-neutral Scenario

Notation	Definition	Location
λ	$1/(\overline{p}\,\|\mathbf{p}^\star\|_\infty)$	Eq. (D.14)
M_G	$\max_{i\in N}\left\{\frac{1}{(b_i+c_i)\underline{p}}\right\}+1$	Eq. (L.31)
M_κ	$(\overline{p}-\underline{p})\sum_{i\in N}\frac{1}{b_i+c_i}$	Eq. (E.8)
$\ell_{r,i}$	$\frac{1}{4}\sqrt{c_i^2+\max_{j\neq i}\left\{c_j^2\right\}}$	Eq. (L.31)
$\ell_{d,i}$	$\frac{1}{4}\sqrt{b_i^2+\max_{j\neq i}\left\{b_j^2\right\}}$	Eq. (L.37)
$\ell_{p,i}$	$\frac{1}{4}\sqrt{16+\overline{p}^2\left[(b_i+c_i)^2+\max_{j\neq i}\left\{(b_j+c_j)^2\right\}\right]}$	Eq. (L.37)
T_1	$\left\lceil\frac{2\sqrt{3+\alpha^2}-2}{2-\sqrt{3+\alpha^2}}\right\rceil$	Eq. (L.1)
C_{rp}	$\max\left\{\frac{2M_G\sqrt{(1+\alpha^2)\sum_{i\in N}(b_i+c_i)^2}}{1-\alpha^2},\frac{\sqrt{n}(\overline{p}-\underline{p})(T_1+1)}{C_\eta\log(T_1+1)}\right\}$	Eq. (L.1)
$\widehat{C}_{rp}$	$\max\left\{C_{rp},M_G\right\}$	Eq. (E.2)
C_κ	$n\lambda+\sum_{i\in N}\ell_{r,i}+2\sqrt{\sum_{i\in N}(b_i+c_i)^2}\cdot\sum_{i\in N}\ell_{d,i}$	Eq. (D.16)
$\widehat{C}_\kappa$	$4C_\eta\widehat{C}_{rp}\left(2C_\kappa C_\eta+n+\frac{M_\kappa}{2\sqrt{n}(\overline{p}-\underline{p})}\right)$	Eq. (E.15)
C_p	$\max_{i\in N}\left\{(b_i+c_i)^2\right\}\cdot(\widehat{C}_\kappa)^2$	Eq. (E.16)
C_r	$2\big(C_p+(C_{rp}C_\eta)^2\big)$	Eq. (E.17)
h_i	$\frac{1}{4}(b_i+c_i)\big(2+(b_i+c_i)\overline{p}\big)$	Eq. (F.5)
$C_{R,i}$	$h_iC_r\cdot\max\left\{\left(\frac{c_i}{b_i+c_i}\right)^2,\ 2\overline{p}^2\max_{j\neq i}\left\{c_j^2\right\}\right\}$ $+h_iC_p\cdot\max\left\{4\overline{p}^2\max_{j\neq i}\left\{(b_j+c_j)^2\right\},1\right\}$	Eq. (F.11)

Table EC.2 Summary of Constants for the Loss-averse Scenario

Notation	Definition	Location
$\tilde{M}_G$	$\max_{i\in N}\left\{\frac{1}{(b_i+c_i^+)\underline{p}}\right\}+1$	Eq. (L.52)
$\tilde{\ell}_{r,i}$	$\frac{1}{4}\sqrt{(c_i^-)^2+\max_{j\neq i}\left\{(c_j^-)^2\right\}}$	Eq. (L.52)
$\tilde{\ell}_{p,i}$	$\sqrt{\frac{1}{(b_i+c_i^+)^2\underline{p}^4}+\frac{b_i}{2(b_i+c_i^+)\underline{p}^2}+\frac{b_i^2+\max_{j\neq i}\left\{b_j^2\right\}}{16}}$	Eq. (L.53)
$\tilde{C}_\kappa$	$\sum_{i\in N}\frac{3\max_{k\in N}\left\{b_k+c_k^-\right\}}{2(b_i+c_i^+)}$	Eq. (G.28)
$T_{1/2}$	$\left\lceil\frac{2+2\alpha^2}{1-\alpha^2}\right\rceil$	Eq. (H.12)
$\tilde{C}_{rp}$	$\max\left\{\frac{2\tilde{M}_G\sqrt{(1+\alpha^2)\sum_{i\in N}(b_i+c_i^-)^2}}{1-\alpha^2},\frac{\sqrt{n}(\overline{p}-\underline{p})\sqrt{T_{1/2}+1}}{C_\eta}\right\}$	Eq. (H.12)
$\tilde{C}_{1/2}$	$\max_{i\in N}\left\{(b_i+c_i^-)\tilde{\ell}_{r,i}\tilde{C}_{rp},\ (b_i+c_i^-)\tilde{\ell}_{G,i}\tilde{M}_G\sqrt{\sum_{i\in N}\left(b_i+c_i^-\right)^2}\right\}$	Eq. (H.12)
$\tilde{T}$	$\left\lceil\frac{n\tilde{M}_G\overline{p}^2\cdot\max_{i\in N}\left\{b_i+c_i^-\right\}}{2C_\eta\epsilon}+\sqrt{\max\left\{T_{1/2}+1,\left\lceil\left(\frac{2C_\eta\tilde{C}_{1/2}}{\epsilon}\right)^2\right\rceil\right\}}\right\rceil^2$	Eq. (H.11)

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