Network Topologies Guaranteeing Zero Duality Gap for Optimal Power Flow Problem

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***Abstract*—We have recently shown that the optimal power flow (OPF) problem with a quadratic cost function can be solved in polynomial time for a large class of power networks, including IEEE benchmark systems, due to their physical properties. In this work, our previous zero-duality-gap result is extended to OPF with arbitrary convex cost functions, and then it is proved that adding phase shifters to the network makes it possible to solve OPF efficiently. More precisely, it is first shown that the Lagrangian dual of OPF can be used to find a globally optimal solution of OPF (i.e., the duality gap is zero) if and only if a linear matrix inequality (LMI) optimization has a specific solution. Since both the dual of OPF and this LMI problem might be expensive to solve for a large-scale network, the sparsity structure of the power network is exploited to significantly reduce the computational complexity of solving these problems. Furthermore, it is proved that adding multiple controllable phase shifters (with variable phases) to certain lines of the power network simplifies the verification of the duality gap. Interestingly, if a sufficient number of phase shifters are added to the network, the duality gap becomes zero for OPF, provided load over-satisfaction (over delivery) is allowed. This result implies that every network topology can be modified by the integration of phase shifters to guarantee the solvability of OPF in polynomial time for all possible values of loads, physical limits and convex cost functions.**

I. Introduction

The optimal power flow (OPF) problem is concerned with finding an optimal operating point of a power system, which minimizes the total generation cost subject to certain network and physical constraints [1]. This optimization problem has been extensively studied since half a century ago [2]. Due to the nonlinear interrelation among active power, reactive power and voltage magnitude, OPF is described by nonlinear equations and may have a nonconvex/disconnected feasibility region [3]. Several algorithms have been proposed for solving this highly nonconvex problem, including linear programming, Newton Raphson, quadratic programming, nonlinear programming, Lagrange relaxation, interior point methods, artificial intelligence, artificial neural network, fuzzy logic, genetic algorithm, evolutionary programming and particle swarm optimization [4], [5], [6], [7], [8], [9], [10]. Most of these methods are built on Karush-Kuhn-Tucker (KKT) necessary conditions, which can only find a local solution (as opposed to a globally optimal solution) [11].

In the past decade, much attention has been paid to devising efficient algorithms for solving OPF. The papers [12] and [13] propose nonlinear interior-point algorithms for an equivalent current injection model of the problem. An implementation of the automatic differentiation technique for OPF is studied in [14]. A second-order cone program (SCOP) is proposed in [15] as a convex relaxation for the load flow problem of a radial distribution system. The generalization of this result to meshed networks requires the incorporation of nonconvex arctangent equality constraints into the SOCP problem [16]. Another convex relaxation is suggested in [17] to solve OPF via a semidefinite program (SDP).

We have recently studied the classical OPF problem with a quadratic cost function in [3], [18], [19]. The Lagrangian dual of OPF is obtained in [3] as an SDP optimization, from which a globally optimal solution of OPF can be found (in polynomial time) if the duality gap between OPF and its dual is zero. It is then shown that the duality gap is zero for IEEE benchmark systems with 14, 30, 57, 118 and 300 buses, in addition to several randomly generated power networks. The paper [3] proves that the duality gap is zero for a large class of power networks due to the passivity of transmission lines and transformers. In particular, there exists an infinite number of network topologies (admittance matrices) that make the duality gap zero for all possible values of loads, provided load over-satisfaction is allowed. These results are extended in [18] to the case when there are more sources of non-convexity in OPF, such as variable transformer ratios, variable shunt elements and contingency constraints.

The present paper extends our previous results on the zero duality gap of OPF to a great extent. Given an arbitrary convex cost function, it is shown that the dual of OPF can be expressed as the maximization of a concave function subject to a linear matrix inequality (LMI) (see [20] or [3] for the definition of LMI). In the case when the cost function is quadratic, this dual optimization simply becomes an SDP. A sufficient condition is derived to verify the zero duality gap of OPF based only on the optimal values of the Lagrange multipliers. It is also proved that the duality gap is zero if and only if a different LMI optimization has a specific solution. Since solving this LMI optimization might be expensive for a large-scale power network, this problem is transformed into a much simpler optimization whose computational complexity depends partly on the number of cycles (loops) in the network. In particular, if the network is radial (acyclic), the resulting optimization turns out to be a generalized SOCP. Interestingly, this simplified optimization is inherently identical to the optimization proposed in [15], which implies that the convex relaxation proposed therein for the power flow problem in radial networks works correctly if and only if the duality gap is zero.

It is known that phase shifters with variable phases make OPF harder to solve, in general. However, it is shown in this paper that adding phase shifters to certain lines of the network reduces the computational complexity of OPF. Moreover, if every cycle of the network contains a line with a controllable

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phase shifter, then the duality gap can be verified by solving a simple generalized SOCP problem. It is also shown that if load over-satisfaction is allowed, then the duality gap always becomes zero, in light of the effect of phase shifters on the topology of the network. In other words, if it is allowed to express the power balance equations as inequalities (rather than equalities), then OPF is always solvable in polynomial time for all possible values of loads, physical limits and cost functions. As stated in [10], [1], [3], allowing load over-satisfaction rarely changes the solution of OPF in practice due to the non-decreasing nature of generation cost functions. This work demonstrates that every network topology can be manipulated by phase shifters to make OPF an easy problem.

Notation 1: The following notations will be used throughout the paper.

- i : Imaginary unit.
- $\mathbf{R}$: Set of real numbers.
- $\mathbf{H}^{n\times n}$: Set of $n \times n$ Hermitian matrices.
- Re$\{\cdot\}$ and Im$\{\cdot\}$: Real and imaginary parts of a complex matrix.
- $*$: Conjugate transpose operator.
- T : Transpose operator.
- $(\cdot)^{\text{opt}}$: Superscript "opt" used to denote an optimal point.
- $\succeq$: Matrix inequality sign in the positive semidefinite sense [20].

II. Problem Formulation

Consider a power network with the set of buses $\mathcal{N} := \{1, 2, ..., n\}$, the set of generator buses $\mathcal{G} \subseteq \mathcal{N}$ and the set of flow lines $\mathcal{L} \subseteq \mathcal{N} \times \mathcal{N}$. Let some notations be introduced below:

- $P_{D_k} + Q_{D_k}\mathrm{i}$: Apparent power of the load connected to bus $k \in \mathcal{N}$ (this value is zero whenever bus k is not connected to any load).
- $P_{G_k} + Q_{G_k}\mathrm{i}$: Output apparent power of generator $k \in \mathcal{G}$.
- V_k: Complex voltage at bus $k \in \mathcal{N}$.
- P_{lm}: Active power transferred from bus $l \in \mathcal{N}$ to the rest of the network through the line $(l, m) \in \mathcal{L}$.
- S_{lm}: Apparent power transferred from bus $l \in \mathcal{N}$ to the rest of the network through the line $(l, m) \in \mathcal{L}$.
- $f_k(P_{G_k})$: A convex function representing the generation cost for generator $k \in \mathcal{G}$.

Define $\mathbf{V}$, $\mathbf{P}_G$, $\mathbf{Q}_G$, $\mathbf{P}_D$ and $\mathbf{Q}_D$ as the vectors $\{V_k\}_{k\in\mathcal{N}}$, $\{P_{G_k}\}_{k\in\mathcal{G}}$, $\{Q_{G_k}\}_{k\in\mathcal{G}}$, $\{P_{D_k}\}_{k\in\mathcal{N}}$ and $\{Q_{D_k}\}_{k\in\mathcal{N}}$, respectively. The power network has some controllable parameters (e.g. $|V_k|$ and P_{G_k} for every $k \in \mathcal{G}$), which can all be recovered from $\mathbf{V}$, $\mathbf{P}_G$ and $\mathbf{Q}_G$. In order to optimize these controllable parameters, the optimal power flow (OPF) problem can be solved. Given the known vectors $\mathbf{P}_D$ and $\mathbf{Q}_D$, OPF minimizes the total generation cost $\sum_{k\in\mathcal{G}} f_k(P_{G_k})$ over the unknown parameters $\mathbf{V}$, $\mathbf{P}_G$ and $\mathbf{Q}_G$ subject to the power balance equations at all buses as well as the physical constraints

$$P_k^{\min} \le P_{G_k} \le P_k^{\max}, \qquad \forall\, k \in \mathcal{G} \tag{1a}$$
$$Q_k^{\min} \le Q_{G_k} \le Q_k^{\max}, \qquad \forall\, k \in \mathcal{G} \tag{1b}$$
$$V_k^{\min} \le |V_k| \le V_k^{\max}, \qquad \forall\, k \in \mathcal{N} \tag{1c}$$
$$P_{lm} \le P_{lm}^{\max}, \qquad \forall\, (l,m) \in \mathcal{L} \tag{1d}$$

where the limits $P_k^{\min}, P_k^{\max}, Q_k^{\min}, Q_k^{\max}, V_k^{\min}, V_k^{\max}$, $P_{lm}^{\max} = P_{ml}^{\max}$ are given. Instead of the flow limit constraint (1d), one may impose a restriction on the value of $|V_l - V_m|$ or S_{lm}, and the results of this work will be still valid.

To formulate OPF, the first step is to derive an equivalent lumped model of the power network by replacing every transmission line and transformer with their equivalent Π models [3]. Define the following notations:

- Let y_{lm} denote the mutual admittance between buses l and m, and y_{kk} denote the admittance-to-ground at bus k, for every $k \in \mathcal{N}$ and $(l, m) \in \mathcal{L}$.
- Let $\mathbf{Y}$ represent the admittance matrix of the power network, which is defined as an $n \times n$ complex-valued matrix whose $(l, m) \in \mathcal{N} \times \mathcal{N}$ entry is equal to $-y_{lm}$ if $l \neq m$ and $y_{ll} + \sum_{k\in\mathcal{N}(l)} y_{lk}$ otherwise, where $\mathcal{N}(l)$ denotes the set of those buses that are directly connected to bus k (y_{lm} is zero by convention if $l \neq m$ and $(l, m) \notin \mathcal{L}$).
- Define the current vector $\mathbf{I} := \begin{bmatrix} I_1 & I_2 & \cdots & I_n \end{bmatrix}^T$ as $\mathbf{Y}\mathbf{V}$, where I_k is the net current injected to bus $k \in \mathcal{N}$.
- Define $e_1, e_2, ..., e_n$ as the standard basis vectors in $\mathbf{R}^n$.

With no loss of generality, assume that $\mathcal{G} = \mathcal{N}$. The case when a bus $k \in \mathcal{N}$ is not connected to any generator can be accommodated by setting $P_k^{\min}$ and $P_k^{\max}$ to 0. One can write:

$$\begin{aligned}(P_{G_k} - P_{D_k}) + (Q_{G_k} - Q_{D_k})\mathrm{i} &= \mathbf{V}_k \mathbf{I}_k^* \\ &= (e_k^*\mathbf{V})(e_k^*\mathbf{I})^* = \text{trace}\{\mathbf{V}\mathbf{V}^*\mathbf{Y}^* e_k e_k^*\}, \quad k \in \mathcal{N}\end{aligned} \tag{2}$$

The above relation, regarded as the power balance equation, motivates the introduction of a new matrix variable $\mathbf{W} \in \mathbf{H}^{n\times n}$ as a substitute for $\mathbf{V}\mathbf{V}^*$. In order to make this change of variable invertible, $\mathbf{W}$ must be constrained to be positive semidefinite and rank-one. Hence, OPF can be formulated as:

OPF: Minimize the function

$$\sum_{k\in\mathcal{G}} f_k(P_{G_k}) \tag{3}$$

over $\mathbf{W} \in \mathbf{H}^{n\times n}$, $\mathbf{P}_G \in \mathbf{R}^n$ and $\mathbf{Q}_G \in \mathbf{R}^n$, subject to

$$P_k^{\min} \le P_{G_k} \le P_k^{\max} \tag{4a}$$
$$Q_k^{\min} \le Q_{G_k} \le Q_k^{\max} \tag{4b}$$
$$(V_k^{\min})^2 \le W_{kk} \le (V_k^{\max})^2 \tag{4c}$$
$$\text{Re}\left\{(W_{ll} - W_{lm})y_{lm}^*\right\} \le P_{lm}^{\max} \tag{4d}$$
$$\text{trace}\{\mathbf{W}\mathbf{Y}^* e_k e_k^*\} = P_{G_k} - P_{D_k} + (Q_{G_k} - Q_{D_k})\mathrm{i} \tag{4e}$$
$$\mathbf{W} = \mathbf{W}^* \succeq 0 \tag{4f}$$
$$\text{rank}\ \{\mathbf{W}\} = 1 \tag{4g}$$

for every $k \in \mathcal{N}$ and $(l, m) \in \mathcal{L}$.

The details of the above formulation can be found in [3]. Note that if there is a transformer on the line (l, m), an extra term may be required in the left side of (4d), depending on how the transformer is modeled.

Notation 2: Given complex values a_1 and a_2, the inequality $a_1 \ge a_2$ means $\text{Re}\{a_1\} \ge \text{Re}\{a_2\}$ and $\text{Im}\{a_1\} \ge \text{Im}\{a_2\}$.

The transmission lines and transformers of a power network are all passive (dissipative). This implies that

$$\text{Re}\{\mathbf{Y}\} \succeq 0, \qquad y_{lm}^* \ge 0, \quad \forall\, (l,m) \in \mathcal{L} \tag{5}$$

We have shown in [3] that although OPF is NP-hard in the worst case, it can be solved in polynomial-time for a large class of admittance matrices $\mathbf{Y}$ satisfying the above circuit property, provided the cost function f_k is quadratic. Under the assumption (5), the objective of this paper is threefold:

- Extend the results of [3] to every convex function f_k.
- Simplify the computational complexity of solving OPF by exploiting the topology of the power network.
- Introduce a minimum number of power electronic devices into the network so that OPF can always be solved in polynomial time, independent of the values of loads, physical limits and cost functions.

III. Main Results

The Lagrangian dual of OPF has been obtained in [3] for quadratic cost functions, by transforming the formulas (3)–(4) into the field of real numbers. The dual problem can ultimately be re-expressed in terms of complex variables, rather than real variables. The derivation of this complex dual optimization will be outlined in the sequel. Given an index $k \in \mathcal{N}$, define the convex conjugate function $\bar{f}_k(x) : \mathbf{R} \to \mathbf{R}$ as:

$$\bar{f}_k(x) = -\min_{P_{G_k}} \left(f_k(P_{G_k}) - xP_{G_k} \right), \qquad \forall\ x \in \mathbf{R} \tag{6}$$

Let $\underline{\lambda}_k$, $\overline{\lambda}_k$ and λ_k denote the Lagrange multipliers for the power constraints $P_k^{\min} \leq P_{G_k}$, $P_{G_k} \leq P_k^{\max}$ and $\text{Re}\{\text{trace}(\mathbf{W}\mathbf{Y}^* e_k e_k^*)\} = P_{G_k} - P_{D_k}$, respectively. Define $\boldsymbol{\Theta}$ as the vector of all Lagrange mutipliers associated with OPF. In line with the technique used in [3], the Lagrangian dual of OPF can be written as:

$$\max_{\boldsymbol{\Theta}} \left\{ -\sum_{k\in\mathcal{N}} \bar{f}_k(\underline{\lambda}_k - \overline{\lambda}_k + \lambda_k) + h(\boldsymbol{\Theta}) \right\} \tag{7a}$$

$$\text{subject to} \qquad A(\boldsymbol{\Theta}) \succeq 0 \tag{7b}$$

where $h(\boldsymbol{\Theta}) \in \mathbf{R}$ is a linear function and $A(\boldsymbol{\Theta}) \in \mathbf{H}^{n\times n}$ is a linear matrix function. The above convex optimization, referred to as *Dual OPF*, has a concave objective and an SDP constraint. Hence, this optimization can be solved efficiently in polynomial time. However, its optimal objective value is only a lower bound on the optimal objective value of OPF. Whenever OPF and Dual OPF have the same optimal values, it is said that *strong duality holds* or *duality gap is zero for OPF*. In the case when the duality gap is zero, a globally optimal solution of OPF can be found in polynomial time. The duality gap will be studied in the subsequent subsections.

A. Various SDP Relaxations and Zero Duality Gap

Define $\mathcal{G} := (\mathcal{N}, \mathcal{L})$ as the graph corresponding to the power network. With no loss of generality, assume that $\mathcal{G}$ is a connected graph (otherwise, it can be partitioned into a set of disconnected sub-networks). This graph may have several cycles, which all together establish a cycle space of dimension $|\mathcal{L}| - |\mathcal{N}| + 1$. Let $\{\mathcal{C}_1, \mathcal{C}_2, ..., \mathcal{C}_{|\mathcal{L}|-|\mathcal{N}|+1}\}$ be an arbitrary basis for this cycle space, meaning that $\mathcal{C}_1, ..., \mathcal{C}_{|\mathcal{L}|-|\mathcal{N}|+1}$ are all cycles of $\mathcal{G}$ from which every other cycle of $\mathcal{G}$ can be constructed.

Definition 1: Define the subgraph set $\mathcal{S}$ as

$$\mathcal{S} := \mathcal{L} \cup \{\mathcal{C}_1, \mathcal{C}_2, ..., \mathcal{C}_{|\mathcal{L}|-|\mathcal{N}|+1}\} \tag{8}$$

(note that since each edge of $\mathcal{G}$ can be regarded as a two-vertex subgraph, $\mathcal{L}$ is indeed a set of $|\mathcal{L}|$ subgraphs).

Definition 2: Given a Hermitian matrix $\mathbf{W} \in \mathbf{H}^{n\times n}$ and an arbitrary subgraph $\mathcal{G}_s \in \mathcal{S}$, define $\mathbf{W}(\mathcal{G}_s)$ as a matrix obtained from $\mathbf{W}$ by removing those columns and rows of $\mathbf{W}$ whose indices do not appear in the vertex set of $\mathcal{G}_s$.

To clarify Definition 2, let $\mathcal{G}_s$ be a single edge $(l, m) \in \mathcal{L}$. Since the vertex set of this subgraph has only two elements l and m, one can write:

$$\mathbf{W}(\mathcal{G}_s) = \begin{bmatrix} W_{ll} & W_{lm} \\ W_{ml} & W_{mm} \end{bmatrix} \tag{9}$$

where W_{lm} denotes the (l, m) entry of $\mathbf{W}$. Three convex relaxations of OPF will be introduced below.

Relaxed OPF 1 (ROPF 1): This optimization is obtained from the OPF problem formulated in (3)–(4) by removing its rank constraint (4g).

Relaxed OPF 2 (ROPF 2): This optimization is obtained from ROPF 1 by replacing its constraint $\mathbf{W} \succeq 0$ with the set of constraints

$$\mathbf{W}(\mathcal{G}_s) \succeq 0, \qquad \forall\ \mathcal{G}_s \in \mathcal{S} \tag{10}$$

Relaxed OPF 3 (ROPF 3): This optimization is obtained from ROPF 2 by replacing its constraint (10) with

$$W_{11}, W_{22}, ..., W_{nn} \geq 0, \tag{11a}$$

$$W_{ll}W_{mm} \geq |W_{lm}|^2, \qquad \forall\ (l, m) \in \mathcal{L} \tag{11b}$$

Note that ROPF 1 and ROPF 2 have convex objectives with SDP constraints, whereas ROPF 3 has a convex objective with SOCP constraints. The relationship between the above optimizations and OPF will be studied in the sequel.

Theorem 1: The following statements hold:

- The duality gap is zero for OPF if and only if ROPF 1 has a solution $(\mathbf{W}^{opt}, \mathbf{P}_G^{\text{opt}}, \mathbf{Q}_G^{\text{opt}})$ with the property $\text{rank}\{\mathbf{W}^{opt}\} = 1$, in which case $\mathbf{W}^{opt} = \mathbf{V}^{\text{opt}}(\mathbf{V}^{\text{opt}})^T$.
- The duality gap is zero if $\text{rank}\{A(\boldsymbol{\Theta}^{\text{opt}})\} = n - 1$, in which case $A(\boldsymbol{\Theta}^{\text{opt}})\mathbf{V}^{\text{opt}} = 0$.

Proof: This theorem is a natural extension of the results of [3], which were developed for quadratic cost functions and real-valued Dual OPF (rather than complex Dual OPF). The techniques used in [3] can be used to prove this theorem. ■

Although ROPF 1 can be solved in polynomial time, it has a matrix variable $\mathbf{W}$ with n^2 unknown entries. Since the number of scalar variables of ROPF 1 is on the order of $O(n^2)$, this optimization may not be solved efficiently for a large value of n. The same argument is valid for Dual OPF. Due to this drawback, the goal is to reduce the computational complexity of these optimizations.

Lemma 1: Given a Hermitian matrix $\mathbf{W} \in \mathbf{H}^{n\times n}$, the following two statements are equivalent:

i) There exists a matrix $\mathbf{W}^{(1)} \in \mathbf{H}^{n\times n}$ such that

$$W_{lm}^{(1)} = W_{lm},\ \forall\ (l, m) \in \mathcal{L} \cup \{(1, 1), ..., (n, n)\} \tag{12a}$$

$$\mathbf{W}^{(1)} \succeq 0, \tag{12b}$$

$$\text{rank}\{\mathbf{W}^{(1)}\} = 1 \tag{12c}$$

ii) There exists a matrix $\mathbf{W}^{(2)} \in \mathbf{H}^{n\times n}$ such that

$$W^{(2)}_{lm} = W_{lm},\ \forall\ (l,m) \in \mathcal{L} \cup \{(1,1),...,(n,n)\} \quad (13a)$$
$$\mathbf{W}^{(2)}(\mathcal{G}_s) \succeq 0, \qquad \forall\ \mathcal{G}_s \in \mathcal{S} \quad (13b)$$
$$\text{rank}\{\mathbf{W}^{(2)}(\mathcal{G}_s)\} = 1, \qquad \forall\ \mathcal{G}_s \in \mathcal{S} \quad (13c)$$

Proof: In order to prove that Condition (i) implies Condition (ii), consider a matrix $\mathbf{W}^{(1)}$ satisfying the relations given in (12). Since $\mathbf{W}^{(1)}$ is both rank-one and positive semidefinite, its principal minors $\mathbf{W}^{(1)}(\mathcal{G}_s)$, $\mathcal{G}_s \in \mathcal{S}$, are also rank-one and positive semidefinite. Hence, Condition (ii) holds if $\mathbf{W}^{(2)}$ is taken as $\mathbf{W}^{(1)}$.

Now, assume that Condition (ii) is satisfied for a matrix $\mathbf{W}^{(2)}$. The goal is to find a matrix $\mathbf{W}^{(1)}$ for which Condition (i) holds. To this end, notice that

$$\sum_{(l,m)\in\vec{\mathcal{C}}_j} \measuredangle W^{(2)}_{lm} = 0, \qquad j = 1,...,|\mathcal{L}| - |\mathcal{N}| + 1 \quad (14)$$

where $\vec{\mathcal{C}}_j$ denotes a directed cycle obtained from $\mathcal{C}_j$ by giving an appropriate orientation to each edge of the cycle and $\measuredangle$ represents the phase of a complex number. Regard the graph $\mathcal{G}$ as a weighted directed graph, where the weights $\measuredangle W^{(2)}_{lm}$ and $\measuredangle W^{(2)}_{ml}$ are assigned to each edge $(l,m) \in \mathcal{L}$ in the forward and backward directions, respectively. Equation (14) can be interpreted as the directed sum of the edge weights around every cycle $\vec{\mathcal{C}}_j$ is zero. Since the set $\{\mathcal{C}_1,...,\mathcal{C}_{|\mathcal{L}|-|\mathcal{N}|+1}\}$ constitutes a basis for the cycle space of the graph $\mathcal{G}$, it can be concluded that the relation (14) holds even if the cycle $\vec{\mathcal{C}}_j$ is replaced by an arbitrary directed cycle of the graph $\mathcal{G}$. Therefore, it is straightforward to show that the n vertices of the graph $\mathcal{G}$ can be labeled by some angles $\theta_1,...,\theta_n$ such that

$$\measuredangle W^{(2)}_{lm} = \theta_l - \theta_m, \qquad \forall\ (l,m) \in \mathcal{L} \quad (15)$$

Define $\mathbf{W}^{(1)}$ to be a matrix with the $(l,m) \in \mathcal{N}\times\mathcal{N}$ entry

$$W^{(1)}_{lm} = \sqrt{W^{(2)}_{ll}}\sqrt{W^{(2)}_{mm}}\ \measuredangle(\theta_l - \theta_m) \quad (16)$$

It is easy to observe that (12b) and (12c) are satisfied for this choice of $\mathbf{W}^{(1)}$. On the other hand, (12a) obviously holds for an index $(l,m) \in \{(1,1),...,(n,n)\}$. It remains to show the validity of (12a) for an edge $(l,m) \in \mathcal{L}$. One can write (13c) for the subgraph $\mathcal{G}_s = (l,m)$ to obtain

$$W^{(2)}_{ll} W^{(2)}_{mm} = \left|W^{(2)}_{lm}\right|^2$$

Thus, it follows from (13a), (15) and (16) that

$$\begin{aligned} W_{lm} = W^{(2)}_{lm} &= \sqrt{W^{(2)}_{ll} W^{(2)}_{mm}}\ \measuredangle W^{(2)}_{lm} \\ &= \sqrt{W^{(2)}_{ll} W^{(2)}_{mm}}\ \measuredangle(\theta_l - \theta_m) = W^{(1)}_{lm} \end{aligned} \quad (17)$$

This completes the proof. ■

Lemma 1 will be exploited next to propose an efficient way for checking the duality gap of OPF, as an alternative to solving ROPF 1,

Theorem 2: The duality gap is zero for OPF if and only if ROPF 2 has a solution $(\mathbf{W}^{\text{opt}}, \mathbf{P}^{\text{opt}}_G, \mathbf{Q}^{\text{opt}}_G)$ with the property that $\text{rank}\{\mathbf{W}^{\text{opt}}(\mathcal{G}_s)\} = 1$ for every $\mathcal{G}_s \in \mathcal{S}$.

Sketch of proof: Given arbitrary matrices $\mathbf{W}$, $\mathbf{P}_G$ and $\mathbf{Q}_G$, consider two matrices $\mathbf{W}^{(1)}$ and $\mathbf{W}^{(2)}$ satisfying Conditions (i) and (ii) in Lemma 1, respectively. Notice that an entry W_{lm} of the matrix $\mathbf{W}$ does not appear in the constraints (4a)–(4e) of OPF unless $l = m$ or $(l,m) \in \mathcal{L}$, which means that some entries of $\mathbf{W}$ may not be important. Using this fact and Lemma 1, one can argue that the triple $(\mathbf{W}, \mathbf{P}_G, \mathbf{Q}_G)$ is a feasible point of OPF if and only if $(\mathbf{W}^{(1)}, \mathbf{P}_G, \mathbf{Q}_G)$ is a feasible point of ROPF 1, or equivalently if and only if $(\mathbf{W}^{(2)}, \mathbf{P}_G, \mathbf{Q}_G)$ is a feasible point of ROPF 2. The proof follows immediately from this property and Theorem 1. ■

Recall that ROPF 1 has a matrix constraint $\mathbf{W} \succeq 0$, which makes it hard to handle for a large value of n. To avoid this issue, Theorem 2 states that this matrix constraint can be broken down into the small-sized matrix constraints given in (10), and yet the resulting optimization can be deployed to measure the duality gap for OPF. The efficacy of Theorem 2 will be later elucidated in Example 1.

B. Acyclic Networks

Throughout this subsection, assume that the power network is radial so that the graph $\mathcal{G}$ has no cycle. Note that this network does not necessarily represent a distribution network with a single feeder (generator), and indeed it can have an arbitrary number of generators.

Corollary 1: The duality gap is zero for OPF if and only if ROPF 3 has an optimal solution at which every inequality in (11b) becomes an equality.

Proof: Since the graph $\mathcal{G}$ is assumed to be acyclic, constraint (10) in ROPF 2 can be expressed as

$$\begin{bmatrix} W_{ll} & W_{lm} \\ W_{ml} & W_{mm} \end{bmatrix} \succeq 0, \qquad \forall\ (l,m) \in \mathcal{L} \quad (18)$$

The proof follows from Theorem 2 as well as the equivalence between (11) and (18). ■

As pointed out earlier, the SDP constraint $\mathbf{W} \succeq 0$ in ROPF 1 makes it hard to solve the problem numerically for a large value of n. Nonetheless, Corollary 1 states that this constraint can be replaced by the SCOP constraint (11), and yet the zero duality gap of OPF can be verified from this optimization.

Remark 1: The paper [15] proposes an SOCP relaxation for the power flow problem in the radial case, by formulating the problem in terms of the variables $|V_k|^2$, $|V_l||V_m|\cos(\measuredangle V_l - \measuredangle V_m)$ and $|V_l||V_m|\sin(\measuredangle V_l - \measuredangle V_m)$ for every $k \in \mathcal{N}$ and $(l,m) \in \mathcal{L}$. It can be shown that this relaxation is tantamount to ROPF 3. As a result, the SOCP relaxation provided in [15] for solving the power flow problem in radial networks works correctly if and only if the duality gap is zero for the corresponding power flow problem.

The power balance equations

$$\text{trace}\{\mathbf{W}\mathbf{Y}^* e_k e_k^*\} = P_{G_k} - P_{D_k} + (Q_{G_k} - Q_{D_k})\text{i} \quad (19)$$

$\forall\ k \in \mathcal{N}$, appear in OPF and ROPF 1–3. It is said that *load over-satisfaction is allowed* if these equality constraints in the aforementioned optimizations are permitted to be replaced by

$$\text{trace}\{\mathbf{W}\mathbf{Y}^* e_k e_k^*\} \leq P_{G_k} - P_{D_k} + (Q_{G_k} - Q_{D_k})\text{i} \quad (20)$$

The main idea behind this notion is that the amount of active/reactive power delivered to each node is allowed to be more than the requested amount of power. In other words, this notion allows over-delivery of power, in which case the excess power should be thrown away (wasted/stored). However, it is generally true that even when load over-satisfaction is permitted, a practical power network is maintained in a normal condition so that no node of the network receives extra power for free. This is due to two properties: (i) transmission lines are lossy, and (ii) cost fucntions are monotonically increasing. Note that the notion of over-satisfaction has already been used by other researchers [10], [1]. This notion has also been studied in our recent work [3] via the name *modified OPF (MOPF)*. In the rest of the paper, MOPF and RMOPF 1–3 will refer to the optimizations OPF and ROPF 1–3 in the load over-satisfaction case. It can be easily shown that all of the results derived so far hold true when load over-satisfaction is allowed.

The goal of this part is to show that the duality gap is zero for acyclic networks, when load over-satisfaction is allowed. To this end, the following lemma is needed, which holds for both acyclic and cyclic networks.

Lemma 2: RMOPF 3 has an optimal solution at which every inequality in (11b) becomes an equality.

Proof: Consider an arbitrary solution $(\mathbf{W}^{\text{opt}}, \mathbf{P}_G^{\text{opt}}, \mathbf{Q}_G^{\text{opt}})$ of RMOPF 3. Define $\widehat{\mathbf{W}}^{\text{opt}}$ as a matrix whose $(l,m) \in \mathcal{N} \times \mathcal{N}$ entry, denoted by $\widehat{W}_{lm}^{\text{opt}}$, is equal to W_{lm}^{opt} if $(l,m) \notin \mathcal{L}$ and

$$\widehat{W}_{lm}^{\text{opt}} = \sqrt{W_{ll}^{\text{opt}} W_{mm}^{\text{opt}} - \left(\text{Im}\{W_{lm}^{\text{opt}}\}\right)^2} + \text{Im}\{W_{lm}^{\text{opt}}\}\text{i}$$

otherwise. It is evident that inequality (11b) becomes an equality for $\mathbf{W} = \widehat{\mathbf{W}}^{\text{opt}}$. Furthermore, given an index $(l,m) \in \mathcal{L}$, since (11b) is satisfied for $\mathbf{W} = \mathbf{W}^{\text{opt}}$, one can write

$$\begin{aligned}
&\left(\widehat{W}_{ll}^{\text{opt}} - \widehat{W}_{lm}^{\text{opt}}\right) y_{lm}^* - \left(W_{ll}^{\text{opt}} - W_{lm}^{\text{opt}}\right) y_{lm}^* = \\
&= \left(\text{Re}\{W_{lm}^{\text{opt}}\} - \sqrt{W_{ll}^{\text{opt}} W_{mm}^{\text{opt}} - \text{Im}\{W_{lm}^{\text{opt}}\}^2}\right) y_{lm}^* \leq 0
\end{aligned} \tag{21}$$

Note that the above inequality is inferred from the non-negativity of the real and imaginary parts of y_{lm}^*. Besides,

$$\begin{aligned}
&\text{trace}\{\widehat{\mathbf{W}}^{\text{opt}} \mathbf{Y}^* e_k e_k^*\} - \text{trace}\{\mathbf{W}^{\text{opt}} \mathbf{Y}^* e_k e_k^*\} = \\
&= \sum_{l \in \mathcal{N}(k)} \left(\text{Re}\{W_{kl}^{\text{opt}}\} - \sqrt{W_{kk}^{\text{opt}} W_{ll}^{\text{opt}} - \text{Im}\{W_{kl}^{\text{opt}}\}^2}\right) y_{kl}^* \leq 0
\end{aligned}$$

for every $k \in \mathcal{N}$. This inequality, together with (21), yields that $(\widehat{\mathbf{W}}^{\text{opt}}, \mathbf{P}_G^{\text{opt}}, \mathbf{Q}_G^{\text{opt}})$ is another solution of RMOPF 3 at which every inequality in (11b) becomes an equality. ■

Theorem 3: The duality gap is zero for OPF if load over-satisfaction is allowed.

Proof: The proof follows immediately from Corollary 1 and Lemma 2. ■

As shown in Theorem 3, the physical properties of the power network make the duality gap zero for OPF if load over-satisfaction is allowed. The following steps are needed to find a globally optimal solution of OPF in this case:

- *Step 1:* Find an arbitrary solution $(\mathbf{W}^{\text{opt}}, \mathbf{P}_G^{\text{opt}}, \mathbf{Q}_G^{\text{opt}})$ of RMOPF 3.
- *Step 2:* Construct the matrix $\widehat{\mathbf{W}}^{\text{opt}}$ from $\mathbf{W}^{\text{opt}}$
- *Step 3:* By considering $\mathbf{W}$ and $\mathbf{W}^{(2)}$ in Lemma 1 as $\mathbf{W}^{\text{opt}}$, find a matrix $\mathbf{W}^{(1)}$ satisfying Condition (i) of this lemma.
- *Step 4:* $(\mathbf{V}^{\text{opt}}, \mathbf{P}_G^{\text{opt}}, \mathbf{Q}_G^{\text{opt}})$ is a globally optimal solution of OPF, where $\mathbf{W}^{(1)} = \mathbf{V}^{\text{opt}}(\mathbf{V}^{\text{opt}})^T$.

C. General Networks

In this part, the results of the preceding subsection will be generalized to the case when the graph $\mathcal{G}$ has at least one cycle (loop). Define *controllable phase shifter* as an ideal (lossless) phase-shifting transformer with the ratio $e^{\gamma \text{i}}$, where the phase shift γ is a variable of the OPF problem. If there are some controllable phase shifters in the network, the term *OPF* refers to the optimal power flow problem with the variables $\mathbf{V}, \mathbf{P}_G, \mathbf{Q}_G$ and the phase shifts of these transformers. Note that adding a controllable phase shifter to a transmission line may need reformulating OPF to incorporate the unknown phase shift of the transformer. The objective is to investigate the role of controllable phase shifters in diminishing the duality gap of OPF.

A bridge of the graph $\mathcal{G}$ is an edge of this graph whose removal makes $\mathcal{G}$ disconnected. The next theorem studies the importance of a phase shifter installed on a bridge line.

Lemma 3: Assume that $(l,m) \in \mathcal{L}$ is a bridge of the graph $\mathcal{G}$ and that the line (l,m) of the power network has a controllable phase shifter. The OPF problem has a globally optimal solution at which the optimal phase shift of the phase shifter is 0.

Sketch of Proof: Define $\mathcal{N}_1$ and $\mathcal{N}_2$ as the sets of vertices of the two disconnected subgraphs obtained by removing the edge (l,m) from the graph $\mathcal{G}$. Assume that $l \in \mathcal{N}_1$ and that the phase shifter of the line (l,m) is located on the side of bus l. Let $(\mathbf{V}^{\text{opt}}, \mathbf{P}_G^{\text{opt}}, \mathbf{Q}_G^{\text{opt}}, \gamma^{\text{opt}})$ denote an optimal solution of OPF, where γ represents the phase of the phase-shifting transformer on the line (l,m). Define $\tilde{\mathbf{V}}^{\text{opt}} \in \mathbf{R}^n$ with the entries

$$\tilde{V}_j^{\text{opt}} = \begin{cases} V_j^{\text{opt}} e^{\gamma^{\text{opt}} \text{i}} & \text{if } j \in \mathcal{N}_1 \\ V_j^{\text{opt}} & \text{if } j \in \mathcal{N}_2 \end{cases}$$

Note that $\tilde{V}_j^{\text{opt}}$ is uniquely defined above, because of the relations $\mathcal{N}_1 \cup \mathcal{N}_2 = \mathcal{N}$ and $\mathcal{N}_1 \cap \mathcal{N}_2 = \phi$. It is straightforward to observe that $(\tilde{\mathbf{V}}^{\text{opt}}, \mathbf{P}_G^{\text{opt}}, \mathbf{Q}_G^{\text{opt}}, 0)$ is another solution of OPF. This completes the proof. ■

Lemma 3 states that a controllable phase shifter on a bridge line of the power network has no effect on the optimal value of OPF. In particular, since every line of a radial network is a bridge, as far as the OPF problem is concerned, phase shifters are not useful for this type of network. In contrast, adding a phase shifter to a non-bridge line of a cyclic network may improve the performance of the network. From the optimization perspective, this addition may require to modify the formulation of OPF by introducing new variables.

Given a natural number t, assume that t phase shifters are added to the lines (l_j, m_j) for $j = 1, ..., t$. Let each phase shifter j be located on the side of bus l_j, with the variable phase shift γ_j. One can write:

$$S_{l_jm_j} = |V_{l_j}|^2 \left(y^*_{l_jm_j} + \frac{1}{2} b_{l_jm_j} \mathrm{i} \right) - V_{l_j} V^*_{m_j} y^*_{l_jm_j} e^{\gamma_j \mathrm{i}} \quad (22)$$

where $b_{l_jm_j}$ denotes the capacitance of the line (l_j, m_j). In the previous formulation of OPF, $W_{l_jm_j} = W^*_{m_jl_j}$ represented the parameter $V_{l_j}V^*_{m_j}$. In order to account for the inclusion of the phase shifters, the equation (22) suggests two modifications:

- Use the new notations $\overline{W}_{l_jm_j} = \overline{W}^*_{l_jm_j}$ for $V_{l_j}V^*_{m_j}$.
- Use the previous notations $W_{l_jm_j} = W^*_{m_jl_j}$ for $V_{l_j}V^*_{m_j}e^{\gamma_j \mathrm{i}}$.

This implies that the following modifications must be made to the OPF problem formulated in (3) and (4):

- Introduce a new matrix variable $\overline{\mathbf{W}} \in \mathbf{H}^{n\times n}$.
- Replace the constraints $\mathbf{W} \succeq 0$ and $\mathrm{rank}\{\mathbf{W}\} = 1$ with $\overline{\mathbf{W}} \succeq 0$ and $\mathrm{rank}\{\overline{\mathbf{W}}\} = 1$.
- Add the new constraints

$$\mathbf{W}(\mathcal{G}_s) \succeq 0, \quad \mathrm{rank}\{\mathbf{W}(\mathcal{G}_s)\} = 1, \qquad \forall\, \mathcal{G}_s \in \mathcal{L}$$

- Impose the constraint that the corresponding entries of $\mathbf{W}$ and $\overline{\mathbf{W}}$ are equal to each other, with the exception of the entries $(l_1, m_1), ..., (l_t, m_t)$ and $(m_1, l_1), ..., (m_t, l_t)$.

One can drop the rank constraints in the above formulation of OPF to obtain a convex relaxation. It can be verified that the duality gap is zero for OPF with the controllable phase shifters if and only if the rank constraints are automatically satisfied at an optimal solution of this convex relaxation. The next theorem proves that if the phase shifters are added to the network in a certain way, the formulation of OPF becomes even simpler than the case with no phase shifters and indeed the new variables $\overline{W}_{l_jm_j}$'s need not be introduced.

Theorem 4: Consider a subset of the cycle basis $\{\mathcal{C}_1, ..., \mathcal{C}_{|\mathcal{L}|-|\mathcal{N}|+1}\}$, say $\mathcal{C}_1, ..., \mathcal{C}_t$ for a given number $t \leq |\mathcal{L}| - |\mathcal{N}| + 1$. For every $j \in \{1, 2, ..., t\}$, assume that a controllable phase shifter is added to a line (l_j, m_j) of the cycle $\mathcal{C}_j$ such that

i) The graph $\mathcal{G}$ remains connected after removing the edges $(l_1, m_1), ..., (l_t, m_t)$.
ii) The set $\{l_j, m_j\}$ is not a subset of the vertex set of any of the reminding cycles $\mathcal{C}_{t+1}, , ..., \mathcal{C}_{|\mathcal{L}|-|\mathcal{N}|+1}$.

Consider the optimization obtained from ROPF 2 by replacing its constraint (10) with the reduced set of constraints

$$\mathbf{W}(\mathcal{G}_s) \succeq 0, \qquad \forall\, \mathcal{G}_s \in \mathcal{S} \backslash \{\mathcal{C}_1, ..., \mathcal{C}_t\} \qquad (23)$$

The duality gap is zero for OPF if and only if every matrix $\mathbf{W}(\mathcal{G}_s)$ in the above inequality becomes rank-one at an optimal solution.

Proof of sufficiency: Let $(\mathbf{W}^{\mathrm{opt}}, \mathbf{P}_G, \mathbf{Q}_G)$ denote an optimal solution of the optimization in Theorem 4 for which every matrix $\mathbf{W}(\mathcal{G}_s)$ becomes rank-one. In line with the argument made in the proof of Lemma 1, it can be shown that

$$\sum_{(l,m)\in\vec{\mathcal{C}}_j} \measuredangle W^{\mathrm{opt}}_{lm} = 0, \quad \forall\, j \in \{t+1, ..., |\mathcal{L}| - |\mathcal{N}| + 1\} \quad (24)$$

By Assumption (i) of the theorem, if the edges (l_1, m_1), ..., (l_t, m_t) are removed from $\mathcal{G}$, then $\{\mathcal{C}_{t+1}, ..., \mathcal{C}_{|\mathcal{L}|-|\mathcal{N}|+1}\}$ forms a cycle basis for the resulting subgraph. In light of (24), this implies that there exist angles $\theta^{\mathrm{opt}}_1, ..., \theta^{\mathrm{opt}}_n$ such that $\theta_l - \theta_m = \measuredangle W^{\mathrm{opt}}_{lm}$ for every edge (l, m) of this subgraph. Now, define the following phase shifts and voltage parameters:

$$\gamma^{\mathrm{opt}}_j = \measuredangle W^{\mathrm{opt}}_{l_jm_j} - \theta^{\mathrm{opt}}_{l_j} + \theta^{\mathrm{opt}}_{m_j}, \quad \forall\, j \in \{1, ..., t\}$$
$$V^{\mathrm{opt}}_k = \sqrt{W^{\mathrm{opt}}_{kk}}\, \measuredangle \theta^{\mathrm{opt}}_k, \qquad \forall\, k \in \mathcal{N}$$

It can be verified that the above parameters correspond to a global solution of OPF and that the duality gap is zero.

Proof of necessity: Assume that the duality gap is zero for OPF in presence of the controllable phase shifters. Denote a global solution of OPF with $(\mathbf{W}^{\mathrm{opt}}, \mathbf{P}^{\mathrm{opt}}_G, \mathbf{Q}^{\mathrm{opt}}_G, \overline{\mathbf{W}}^{\mathrm{opt}})$. Due to Assumption (ii) of the theorem and the definition of $\overline{\mathbf{W}}$, the submatrix $\mathbf{W}^{\mathrm{opt}}(\mathcal{C}_r)$ is a principal minor of $\overline{\mathbf{W}}^{\mathrm{opt}}(\mathcal{C}_r)$ for every $r \in \{t+1, ..., |\mathcal{L}| - |\mathcal{N}| + 1\}$. Hence, $\mathbf{W}^{\mathrm{opt}}(\mathcal{C}_r)$ must be both positive semidefinite and rank-one. As a result, $(\mathbf{W}^{\mathrm{opt}}, \mathbf{P}^{\mathrm{opt}}_G, \mathbf{Q}^{\mathrm{opt}}_G)$ is a solution of the optimization given in Theorem 4 with the property that $\mathrm{rank}\{\mathbf{W}^{\mathrm{opt}}(\mathcal{G}_s)\} = 1$. ■

An implication of Theorem 4 is that adding phase shifters to the network in a certain way simplifies the formulation of OPF, instead of increasing the number of variables and/or constraints. This theorem shows that the phase shifters added to the cycles $\mathcal{C}_1, ..., \mathcal{C}_t$ give rise to excluding the t constraints $\mathbf{W}(\mathcal{C}_s) \succeq 0$, $s = 1, ..., t$, from ROPF 3. While the values of the phase shifters do not appear in ROPF 3, the procedure delineated in the proof of sufficiency can be used to find the optimal phases in the case when the duality gap is zero.

A spanning tree of the connected graph $\mathcal{G}$ is an acyclic subgraph of $\mathcal{G}$ with $|\mathcal{N}|$ vertices and $|\mathcal{N}| - 1$ edges. Note that $\mathcal{G}$ might have an exponential number of spanning trees.

Corollary 2: Given a spanning tree $\mathcal{T}$ of the graph $\mathcal{G}$, assume that a controllable phase shifter is added to every line of the network not belonging to this tree. The duality gap is zero for OPF if and only if OPRF 3 has an optimal solution at which every inequality in (11b) becomes an equality.

Proof: Let $(l_1, m_1), ..., (l_t, m_t)$ denote those edges of the graph $\mathcal{G}$ that do not belong to $\mathcal{T}$, where $t = |\mathcal{L}| - |\mathcal{N}| + 1$. Adding each edge (l_j, m_j), $j = 1, ..., t$, to the tree $\mathcal{T}$ creates a cycle. With a slight abuse of notation, let $\mathcal{C}_j$ denote this cycle. Theorem 4 can be applied to the power network with the phase shifters installed on its lines $(l_1, m_1), ..., (l_t, m_t)$. The proof is completed by noting that constraint (23) is equivalent to (11), i.e., the optimization introduced in Theorem 4 and ROPF 3 are identical for this set of phase shifters. ■

In the case of OPF with no controllable phase shifters, one needs to solve Dual OPF, ROPF 1 or ROPF 2, which have SDP constraints. Nonetheless, Corollary 2 states that if enough number of phase shifters are added to the network, then it suffices to solve ROPF 3 with simple SOCP constraints. Hence, phase shifters can significantly reduce the computational complexity of OPF if they are formulated properly. Note that ROPF 3 is independent of the choice of the non-unique spanning tree $\mathcal{T}$. This implies that as long as a sufficient number of phase shifters are added to the network, it may not be important where to place them.

Theorem 5: Given an arbitrary spanning tree $\mathcal{T}$ of the graph $\mathcal{G}$, assume that a controllable phase shifter is added to every line of the network that does not belong to this tree. The duality gap is zero for OPF if load over-satisfaction is allowed.

Proof: The proof can be deduced from Lemma 2 and Corollary 2. The details are omitted for brevity. ■

It can be inferred from Corollary 2 and Theorem 5 that adding a sufficient number of phase shifters makes it possible to not only recast OPF as a generalized SOCP optimization, but also diminish the duality gap to zero, provided load over-satisfaction is allowed. Note that this zero duality gap is due to the (weighted) topology of the network and holds for all possible values of loads, physical limits and cost functions.

Two remarks can be made here. First, the results of this paper are all valid in the general case that the objective function of OPF is replaced by any arbitrary function, as long as it is convex with respect to P_{G_k}, Q_{G_k} and $|V_k|^2$, $\forall\, k \in \mathcal{N}$. Second, since the graph $\mathcal{G}$ is mostly planar in practice, the number of phase shifters needed to guarantee a zero duality gap is $O(n)$.

D. Prohibition of Load Over-satisfaction

Theorems 3 and 5 state that OPF can be solved efficiently in the load over-satisfaction case. Assume that load over-satisfaction is allowed. Since the cost function $f_k(P_{G_k})$, $k \in \mathcal{G}$, is monotonically increasing in practice, over delivery of active power rarely happens because an increase in the supply raises the cost. However, over delivery of reactive power may occur more frequently, which can be avoided by penalizing the reactive power over-satisfaction in the objective function of OPF.

The goal is to derive sufficient conditions to guarantee that over delivery of active/reactive power will not occur at a given node(s). To this end, assume that a controllable phase shifter is added to every line of the network not belonging to a spanning tree $\mathcal{T}$. Consider a certain node $r \in \mathcal{N}$, and suppose that $V_r^{\max}$ and $P_{rl}^{\max}$, $l \in \mathcal{N}(r)$, are all infinity. Define two cases as follows:

- *Case (i):* Load over-satisfaction is allowed at every node $k \in \mathcal{N}$, with the exception that reactive power over-delivery is not allowed at node r.
- *Case (ii):* Load over-satisfaction is allowed at every node $k \in \mathcal{N}$, with the exception that apparent power over-delivery is not allowed at node r.

Theorem 6: Under the assumptions made above, the following statements hold:

i) The duality gap is zero for Case (i) if

$$\frac{\text{Im}\{y_{rr}^*\}}{\text{Re}\{y_{rr}^*\}} \geq \frac{\text{Im}\{y_{rl}^*\}}{\text{Re}\{y_{rl}^*\}}, \qquad \forall\, l \in \mathcal{N}(r) \tag{25}$$

ii) The duality gap is zero for Case (ii) if

$$\frac{\text{Im}\{y_{rr}^*\}}{\text{Re}\{y_{rr}^*\}} = \frac{\text{Im}\{y_{rl}^*\}}{\text{Re}\{y_{rl}^*\}}, \qquad \forall\, l \in \mathcal{N}(r) \tag{26}$$

Proof: The proof will be provided here only for Part (ii), as the other part can be proved analogously. Given a positive scalar ε, consider the optimization obtained from ROPF 3 via two modifications:

- Replace the power balance constraint (19) with (20) for every $k \in \mathcal{N}\backslash\{r\}$;
- Add the negative term $-\varepsilon W_{rr}$ to the objective function.

This optimization will be referred to as ε-ROPF. Similar to Corollary 2, the duality gap is zero for Case (ii) if ε-ROPF has a solution for $\varepsilon = 0$ at which every inequality in (11b) becomes an equality. To show the existence of such a solution, it suffices to prove the following two statements for an arbitrary $\varepsilon > 0$:

- *Claim 1:* ε-ROPF has a solution at which every inequality in (11b) becomes an equality.
- *Claim 2:* The value W_{rr} at an optimal solution of ε-ROPF is upper bounded by some limit that does not depend on ε.

If the above claims hold true, then the proof of Part (ii) follows by letting ε approach zero. To prove Claim 1, consider an arbitrary solution $(\mathbf{W}^{\text{opt}}, \mathbf{P}_G^{\text{opt}}, \mathbf{Q}_G^{\text{opt}})$ of ε-ROPF. Let $\widehat{\mathbf{W}}^{\text{opt}}$ denote the matrix that was constructed from $\mathbf{W}^{\text{opt}}$ in the proof of Lemma 2. Define μ as

$$\mu = \sum_{l \in \mathcal{N}(r)} \left(\sqrt{W_{rr}^{\text{opt}} W_{ll}^{\text{opt}} - \text{Im}\{W_{rl}^{\text{opt}}\}^2} - \text{Re}\{W_{rl}^{\text{opt}}\} \right) \frac{y_{rl}^*}{y_{rr}^*}$$

In light of (26), μ is a nonnegative real number. It can be verified that $(\widehat{\mathbf{W}}^{\text{opt}} + \mu e_r e_r^*, \mathbf{P}_G^{\text{opt}}, \mathbf{Q}_G^{\text{opt}})$ is a feasible point of ε-ROPF at which the objective function is equal to $\sum_{k \in \mathcal{N}} f_k(P_{G_k}^{\text{opt}}) - \mu\varepsilon$. Since it is assumed that the objective function can never become less than $\sum_{k \in \mathcal{N}} f_k(P_{G_k}^{\text{opt}})$, the nonnegative term μ must be zero. This implies that $(\widehat{\mathbf{W}}^{\text{opt}}, \mathbf{P}_G^{\text{opt}}, \mathbf{Q}_G^{\text{opt}})$ is a solution of ε-ROPF with the property that it makes every inequality in (11b) an equality. This completes the proof of Claim 1. Now, it remains to prove Claim 2. With no loss of generality, assume that $r \neq 1$. Due to the Laplacian structure of $\mathbf{Y}$ and the positive semi-definiteness of Re$\{\mathbf{Y}\}$, there exists a positive number ζ such that Re$\{\mathbf{Y}\} + \zeta e_1 e_1^* \succ 0$. As stated right after Theorem 3, $\widehat{\mathbf{W}}^{\text{opt}}$ can be assumed to have rank-one. Hence, one can write:

$$\begin{aligned}
&\sum_{k \in \mathcal{N}} (P_{G_k} - P_{D_k}) \geq \text{trace}\left\{\widehat{\mathbf{W}}^{\text{opt}}\ \text{Re}\{\mathbf{Y}\}\right\} \\
&\geq \text{trace}\left\{\widehat{\mathbf{W}}^{\text{opt}}\ \text{Re}\{\mathbf{Y} + \zeta e_1 e_1^*\}\right\} - \zeta \widehat{W}_{rr}^{\text{opt}} \\
&\geq \text{trace}\{\widehat{\mathbf{W}}^{\text{opt}}\} \times \lambda_{\min}\{\text{Re}\{\mathbf{Y}\} + \zeta e_1 e_1^*\} - \zeta \times (V_1^{\max})^2
\end{aligned}$$

where $\lambda_{\min}\{\cdot\}$ denotes the smallest eigenvalue of a Hermitian matrix. It can be deduced from the above inequality that trace$\{\widehat{\mathbf{W}}^{\text{opt}}\}$ is upper bounded by a limit independent of ε and so is $\widehat{W}_{rr}^{\text{opt}}$. This completes the proof. ■

If all lines connected to node r are of the same type and with negligible capacitance, then (26) holds. Furthermore, (25) holds if at least one line connected to node r is sufficiently long with large capacitance. It is noteworthy that Theorem 6 can be straightforwardly generalized to multiple nodes.

IV. Examples

Example 1: To illustrate the efficacy of Theorem 2, consider the network depicted in Figure 1. This network consists of three acyclic (radial) subnetworks $1-10$, $11-20$ and $21-30$, which are interconnected via the cycle (transmission network)

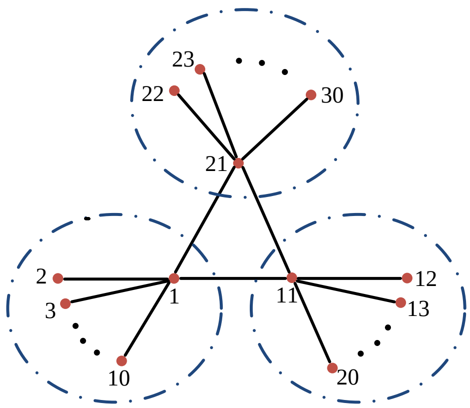

Fig. 1. Power network used to illustrate Theorem 2.

$\{1, 11, 21\}$. Zero duality gap can be verified from both ROPF 1 and ROPF 2. Nonetheless, ROPF 2 exploits the sparsity of the network and therefore is much easier to solve. More precisely,

- ROPF 1 has a 30×30 matrix constraint $\mathbf{W} \succeq 0$ in which $\frac{30 \times 31}{2} = 465$ scalar complex variables are involved.
- The graph $\mathcal{G}$ has 30 edges (i.e. $|\mathcal{L}| = 30$) and a single cycle $\{1, 11, 21\}$. Thus, ROPF 2 has 31 matrix constraints

$$\begin{bmatrix} W_{ll} & W_{lm} \\ W_{ml} & W_{mm} \end{bmatrix} \succeq 0, \quad \forall\ (l,m) \in \mathcal{L},\ l < m \tag{27}$$

and

$$\begin{bmatrix} W_{1,1} & W_{1,11} & W_{1,21} \\ W_{11,1} & W_{11,11} & W_{11,21} \\ W_{21,1} & W_{21,11} & W_{21,21} \end{bmatrix} \succeq 0 \tag{28}$$

(note that $W_{l,m}$ stands for W_{lm}). In light of (28), constraint (27) need not be written for the edges $(1, 11)$, $(1, 21)$ and $(11, 21)$. Hence, ROPF 2 is nearly an SOCP optimization with 28 non-redundant 2×2 and 3×3 matrix constraints in which only 60 scalar variables are involved. Therefore, several entries of $\mathbf{W}$ never appear in the constraints of ROPF 2 and can be simply ignored.

Example 2: Let the results of this paper be illustrated on two IEEE systems [21]. Similar to [3], a small amount of resistance (10^{-5} per unit) is added to a few purely-inductive lines of these networks. The simulations performed here are run on a computer with a Pentium IV 3.0 GHz and 3.62 GB of memory. The toolbox “YALMIP”, together with the solvers “SEDUMI” and “SDPT3”, is used to solve different LMI problems, where the numerical tolerance is chosen as 10^{-15}.

Consider the IEEE 30-bus system with $f_k(P_{G_k}) = P_{G_k}$, $\forall k \in \mathcal{G}$, which has 30 buses and 41 lines. Dual OPF can be solved in 1.2 seconds for this power network to detect the zero duality gap and attain the optimal generation cost 191.09. Alternatively, one can solve ROPF 1 in 9.3 seconds to verify the zero duality gap. Assume now that every line of the network has a controllable phase shifter. Due to the results developed here, at most $41 - 30 + 1$ of the 41 phase shifters are important and the remaining ones can be simply ignored. The duality gap is zero for OPF with 12 controllable phase shifters. Indeed, ROPF 3 can be solved in only 0.4 second, leading to the optimal value 190.66. Note that (i) most of the phase shifters have negligible effect, and (ii) even if load over-satisfaction is allowed, it will never occur. As another experiment, suppose that $f_k(P_{G_k})$ is the quadratic cost function specified in [21]. The solutions of ROPF 1 and ROPF 3 turn out to be 576.90 and 573.59, respectively. To substantiate that most of the 12 controllable phase shifters are not important, notice that

- OPF with a single variable phase shifter on the line $(25, 27)$ has the optimal cost 573.92, corresponding to the phase $7.82°$.
- OPF with two variable phase shifters on the lines $(25, 27)$ and $(8, 28)$ gives the optimal cost 573.67, corresponding to the optimal phases $5.70°$ and $-0.30°$.

The authors have repeated the above experiment for several random cost functions and observed that OPF with 1-2 phase shifters is a good approximation of ROPF 3. One of such random cases will be reported below. Assume that the cost function $f_k(P_{G_k})$ is equal to $c_{k,2}P_{G_k}^2 + c_{k,1}P_{G_k}$, where

$$\begin{aligned} &c_{1,1} = 0.41, && c_{1,2} = 0.92, && c_{2,1} = 0.89, && c_{2,2} = 0.73, \\ &c_{13,1} = 0, && c_{13,2} = 0.91, && c_{22,1} = 0.05, && c_{22,2} = 0.17, \\ &c_{23,1} = 0.81, && c_{23,2} = 0.93, && c_{27,1} = 0.35, && c_{27,2} = 0.40 \end{aligned}$$

The solutions of ROPF 1 and ROPF 3 are obtained as 3671.17 and 3283.34, respectively. This implies that the cost is decreased significantly by adding 12 phase shifters. However, only two of the phase shifters are important. Indeed, OPF with two variable phase shifters on the lines $(25, 27)$ and $(10, 21)$ leads to the optimal cost 3283.62, associated with the optimal phases $7.03°$ and $1.08°$. This example demonstrates that adding a few controllable phase shifters not only guarantees a zero duality gap, but also reduces the generation cost.

Consider now the IEEE 118-bus system with $f_k(P_{G_k}) = P_{G_k}$, $\forall k \in \mathcal{G}$. A global solution of OPF is found by solving Dual OPF in 11.2 seconds, leading to the optimal cost 4251.9. ROPF 1 can also detect the zero duality gap, but its running time is more than 1 minute. In contrast, ROPF 3 is solved in 0.9 second to attain the optimal value 4251.9. Note that (i) OPF and ROPF 3 have the same optimal value, and (ii) the duality gap is zero for OPF without phase shifters. Hence, the total generation $\sum_{k \in \mathcal{G}} P_{G_k}$ is never reduced by adding phase shifters to the lines of the network. However, phase shifters may have an important role for other types of cost functions.

V. Conclusions

We have recently shown that the optimal power flow (OPF) problem with a quadratic cost function can be solved in polynomial time for a large class of power networks, due to their physical structures. The present work fist generalizes this result to arbitrary convex functions and then studies how the incorporation of phase shifters into the network guarantees solvability of OPF in polynomial time. A global solution of OPF can be found from the dual of OPF if the duality gap is zero, or alternatively if a linear matrix inequality (LMI) optimization derived here has a specific solution. It is shown that the computational complexity of verifying the duality gap can be reduced significantly by exploiting the sparsity of the power network’s topology. In particular, if the network has no cycle, the LMI problem can be equivalently converted to a generalized second-order cone program. It is also proved that the integration of controllable phase shifters with variable phases into the cycles of the network makes the verification of the duality gap easier. More importantly, if every cycle (loop) of the network has a line with a controllable phase shifter, then OPF with variable phase shifters is guaranteed to be solvable in polynomial time, provided load over-satisfaction is allowed.

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