A Convex Approximation of Optimal Distributed Controller in Frequency Domain

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***Abstract*— We study the optimal distributed control (ODC) problem and propose a design method based on the approximation of the $\mathcal{H}_2$ performance using the optimal centralized controller. The designed distributed controller obeys a prescribed sparsity pattern in the frequency domain. The proposed method enables a convex approximation of the distributed controller problem, which has a closed-form solution with optimality guarantees. After introducing the notion of rank-preserving weights, we give sufficient conditions for the controller to be proper, and furthermore characterize the set of controllers that can be obtained by the proposed technique via fine-tuning the algorithm parameters. The results are applied to the linear-quadratic regulator problem and to the purely decentralized case with a diagonal sparsity pattern, where certain connections to the simple methods of thresholding and averaging are discovered. Numerical examples are provided to demonstrate the effectiveness of the developed method.**

I. Introduction

Optimal distributed control (ODC) studies complex large-scale systems and has been an area of growing interest. There are various communication and structural constraints in many real-world systems, such as power networks and transportation networks, which demand a distributed controller. Significant efforts have been devoted to the complexity analysis of the distributed controller design problem and to the development of optimization methods that precisely or approximately solve the problem.

ODC is known to admit nonlinear solutions and is NP-hard in the worst case [1], [2]. In principle, ODC can be solved using nonlinear programming techniques [3]. However, these methods are often based on first-order optimality conditions and lack global optimality guarantees. In contrast, if one regards ODC as a polynomial optimization problem, it is possible to use convex relaxations and reformulation techniques [4]–[7]. Convex formulations are desirable since they can be solved to global optimality in polynomial time (up to any accuracy) using local search algorithms [5]. They can also exploit sparsity structures that reduces the complexity in a principled way [8], and they allow for a disciplined modeling framework [9]. It should be noted that the complexity of different convex formulations could vary significantly and there have been several works on the approximation of convex relaxations with linear programs and second-order cone programs [10].

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Recognizing the core difficulty of the structural constraints in a distributed control problem and the fact that many techniques developed for classic centralized control problems are inapplicable in such settings, there is a line of work on the identification of structures that break down the complexity of the ODC problem. Some of these structures are as follows: spatially invariant systems [11], partially nested systems [12], positive systems [13], localized systems [14], and quadratic invariant systems [15]. More recently, a new System Level Framework [16] identifies a large set of problems that have a convex formulation.

Unlike the above-mentioned papers that focus on the structural properties of the control system or convex relaxations of the problem, we propose a new convex approximation technique based on the optimal centralized controller. This is motivated by the fact that with sufficient knowledge one can make the distributed control trajectories arbitrarily close to the centralized control trajectories [17]. Moreover, promising results on approximation techniques for the design of a static distributed controller has been developed in [18]. The paper [19] studies the control sparsification problem (where the control topology is to be co-optimized) by a rank-constrained formulation that matches frequency characteristics with linear matrix inequality (LMI) constraints. The ODC problem is known to be more difficult if the controller structure is imposed as a constraint rather than being flexible in the optimization problem.

In this work, we consider the gap between the optimal centralized and distributed control problems, and focus on systems for which this gap is not very large. This enables us to develop an approximation technique for the distributed control problem based on the centralized control problem. If the gap is large, the proposed method still works, but the obtained controller may not have a satisfactory performance or even be stabilizing. This implicit technical assumption allows us to overcome the NP-hardness of the problem. It is shown that the proposed approximation method reduces to a least-squares problem at all frequencies, which has a closed-form solution. This explicit formula makes it possible to obtain optimality guarantees on the designed controller. We prove that the controller design method automatically produces proper controllers, based on a new notion of rank-preserving weight pattern. It is shown that the proposed method has interesting connections to the basic thresholding and averaging methods. We also apply the results to the linear-quadratic regulator (LQR) problem and fully decentralized controller problems.

The remainder of the paper is organized as follows. After formulating the ODC problem in Section II, we present an approximation scheme in Section III. Two special cases of LQR and diagonal controllers are studied in Section IV. Numerical examples are provided in Section V. Section VI and VII offer final discussions and draw some concluding remarks.

II. Problem Formulation

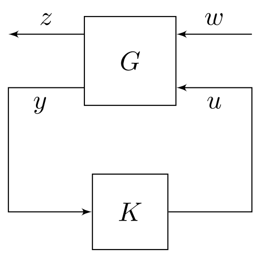

Fig. 1. Synthesis arrangement

Based on Figure 1, consider a plant G described as

$$G(s)=\left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right]=\left[\begin{array}{cc} G_{11} & G_{12} \\ G_{21} & G_{22} \end{array}\right].$$

where the following standard notation is used for transfer functions:

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]=C(sI-A)^{-1}B+D.$$

The corresponding state-space representation is

$$\begin{aligned} \dot{x} &= Ax+B_1w+B_2u \\ z &= C_1x \qquad\quad +D_{12}u \\ y &= C_2x+D_{21}w, \end{aligned}$$

where x, y, z, u, and w denote the state of the system, output, measurement, control action, and disturbance, respectively. The control action associated with the controller K is described in the frequency domain as $\hat{u}(s)=K(s)\hat{y}(s)$, where $\hat{y}$ (resp. $\hat{u}$) is the Laplace transform of the signal y (resp. u). Moreover, T_{zw} denotes the transfer function from z to w, which can be expressed as

$$\begin{aligned} T_{zw} &= \left[\begin{array}{c|c} A+B_2KC_2 & B_1+B_2KD_{21} \\ \hline C_1+D_{12}KC_2 & D_{12}KD_{21} \end{array}\right] \\ &= G_{11}+G_{12}K(I-G_{22}K)^{-1}G_{21}. \end{aligned}$$

The $\mathcal{H}_2$ norm of the system G is defined as

$$\begin{aligned} ||G||^2_{\mathcal{H}^2} &= \int_{-\infty}^{\infty}\mathrm{Tr}[G^*(jw)G(jw)]dw \\ &= \int_{-\infty}^{\infty}||G(jw)||_F^2 dw, \end{aligned}$$

where "$*$", Tr$(\cdot)$ and $\|\cdot\|_F$ denote the conjugate transpose, trace and Frobenius norm operators.

Notations: We use the operator $\mathrm{vec}(\cdot)$ to vectorize a matrix by stacking its columns on top of each other from left to right. V^T denotes the transpose of V. $V\otimes U$ is the Kronecker product of V and U. The set $\mathcal{RH}_\infty^{k\times m}$ includes all rational transfer matrices of dimension $k\times m$ that do not have poles in the closed right-half plane. $RP^{k\times m}$ is a larger transfer matrix set that only requires the matrix elements to be proper. The subscripts c and d refer to "centralized" and "distributed", respectively. The unknown distributed controller K_d is assigned a sparsity pattern Σ. This is a 0-1 matrix that specifies the nonzero elements of K_d, and we say $K_d\in\Sigma$ if $K_d(i,j)=0$ whenever $\Sigma(i,j)=0$. Given a transfer matrix whose sparsity pattern Σ has r nonzero terms, we define a corresponding vector $\mathrm{vec}_\Sigma(K)\in RP^r$ associated with the nonzero elements of K. With a slight abuse of notation, for a matrix $A\in\mathbb{C}^{(km)\times(km)}$, we use the notation $A\Sigma\in\mathbb{C}^{(km)\times r}$ to select those columns of A corresponding to the nonzero elements of Σ, and $\Sigma^TA\in\mathbb{C}^{r\times(km)}$ to select the associated rows. Here, Σ can be regarded as an $(mk)\times r$ zero-one matrix with the property $\mathrm{vec}_\Sigma(K)=\mathrm{vec}(K)\Sigma$.

Throughout the paper, we make the standard assumption that G_{22} is strictly proper to ensure that stabilizing G_{22} is the same as stabilizing G [20]. The $\mathcal{H}_2$ optimal distributed control problem can be stated as

$$\begin{aligned} \min \quad & ||T_{zw,d}||_{\mathcal{H}_2} & (1)\\ \text{s.t.} \quad & K_d\in\Sigma & (2)\\ & K_d \text{ stabilize the system.} & (3) \end{aligned}$$

In the centralized case, the sparsity constraint (2) is removed, and K_c, if it exists, can be found efficiently using the Riccati equation or through the Youla parameterization. There are also powerful LMI characterizations of the optimal state feedback solution. In this work, we assume that K_c exists. In the distributed case, Problem (1) with an arbitrary sparsity constraint is difficult to solve. In light of the significant difference between the complexities of the centralized and distributed cases, we propose to find a distributed controller that minimizes an approximate error between the centralized transfer function and the distributed transfer function. Roughly speaking, if we can find an approximate error

$$E_{dc}\approx T_{zw,d}-T_{zw,c},$$

whose corresponding optimization problem

$$\begin{aligned} \min \quad & ||E_{dc}||_{\mathcal{H}_2} & (4a)\\ \text{s.t.} \quad & K_d\in\Sigma & (4b)\\ & K_d \text{ stabilize the system.} & (4c) \end{aligned}$$

is easy to solve, then we can recover a suboptimal distributed controller.

III. Approximation

An approximation scheme will be proposed in this section. Note that

$$\begin{aligned} T_{zw,d} &= G_{11}+G_{12}K_d(I-G_{22}K_d)^{-1}G_{21} \\ T_{zw,c} &= G_{11}+G_{12}K_c(I-G_{22}K_c)^{-1}G_{21}. \end{aligned}$$

The difference between the centralized and distributed transfer functions can be calculated as

$$\begin{aligned}&T_{zw,d}-T_{zw,c}\\&=G_{12}\left[K_d(I-G_{22}K_d)^{-1}-K_c(I-G_{22}K_c)^{-1}\right]G_{21},\end{aligned}$$

where the expression within the brackets can be simplified to

$$\begin{aligned}&\left[K_d-K_c(I-G_{22}K_c)^{-1}(I-G_{22}K_d)\right](I-G_{22}K_d)^{-1}\\&=\left[K_d-K_c+K_c(I-G_{22}K_c)^{-1}G_{22}(K_d-K_c)\right]\\&\times(I-G_{22}K_d)^{-1}\\&=\left[I+K_c(I-G_{22}K_c)^{-1}G_{22}\right](K_d-K_c)(I-G_{22}K_d)^{-1}\\&=(I-K_cG_{22})^{-1}(K_d-K_c)(I-G_{22}K_d)^{-1}\\&\approx(I-K_cG_{22})^{-1}(K_d-K_c)(I-G_{22}K_c)^{-1}.\end{aligned}$$

The only approximation we make is the replacement of the unknown term $(I-G_{22}K_d)^{-1}$ with the known term $(I-G_{22}K_c)^{-1}$. This approximation is completely negligible at high frequencies since G_{22} is strictly proper. Therefore, we propose the following error:

$$E_{dc}=G_{12}(I-K_cG_{22})^{-1}(K_d-K_c)(I-G_{22}K_c)^{-1}G_{21}.$$

Note that this is linear in K_d with two weighting matrices on both sides of K_d-K_c. Using the *Youla* parameterization of the centralized controller, these matrices can be computed efficiently. Consider a coprime factorization of G_{22} as

$$G_{22}=NM^{-1}=\bar{M}^{-1}\bar{N},$$

where

$$\begin{bmatrix}\bar{X} & -\bar{Y}\\ -\bar{N} & \bar{M}\end{bmatrix}\begin{bmatrix}M & Y\\ N & X\end{bmatrix}=I,$$

and $K_c=(Y-MQ)(X-NQ)^{-1}=(\bar{X}-Q\bar{N})^{-1}(\bar{Y}-Q\bar{M})$. The following lemma is borrowed from [20].

Lemma 1: The equations

$$\begin{aligned}(I-K_cG_{22})^{-1}&=M(\bar{X}-Q\bar{N}),\\(I-G_{22}K_c)^{-1}&=(X-NQ)\bar{M},\end{aligned}$$

hold and, therefore, both inverses belong to $\mathcal{RH}_\infty$.

Using Lemma 1, the approximate error can be simplified as

$$E_{dc}=G_{12}M(\bar{X}-Q\bar{N})(K_d-K_c)(X-NQ)\bar{M}G_{21}.$$

We design a sub-optimal distributed controller by minimizing this approximate performance difference. Writing $U=G_{12}M(\bar{X}-Q\bar{N})$, and $V^T=(X-NQ)\bar{M}G_{21}$, the approximation problem (4) is now fully specified by $E_{dc}=U(jw)(K_d(jw)-K_c(jw))V^T(jw)$ as follows:

$$\begin{aligned}\min\ &\int_{-\infty}^{\infty}||U(jw)(K_d(jw)-K_c(jw))V(jw)^T||_F^2dw \quad (5)\\ \text{s.t.}\ \ &K_d\in\Sigma\\ &K_d \text{ stabilize the system.}\end{aligned}$$

This problem is in the frequency domain. If the stabilizability constraint is removed, the above optimization problem (5) reduces to a separate least-squares problem for every $w\in\mathbb{R}$:

$$\begin{aligned}\min\ \ &||U(jw)(K_d(jw)-K_c(jw))V(jw)^T||_F^2 \quad \text{(LS-DC)}\\ \text{s.t.}\ \ &K_d(jw)\in\Sigma.\end{aligned}$$

This is an unconstrained convex quadratic problem with a closed-form solution. Note that the weighting matrices $U(jw)$ and $V(jw)$ do not need to be chosen as before and can be designed to obtain a better suboptimal distributed controller.

Theorem 1: Assuming that G_{21} is invertible on the imaginary axis, the least-squares problem (LS-DC) produces a solution K_d^{opt} as a function of K_c with the optimality guarantee

$$||T_{zw,d}||_{\mathcal{H}^2}-||T_{zw,c}||_{\mathcal{H}^2}\le C_{dc}||E_{dc}^{opt}||_{\mathcal{H}_2} \quad (6)$$

where the constant

$$C_{dc}=||G_{21}^{-1}(I-G_{22}K_c)(I-G_{22}K_d^{opt})^{-1}G_{21}||_\infty.$$

depends only on the centralized controller after writing K_d^{opt} in terms of K_c.

Proof: It follows from the triangle inequality that

$$||T_{zw,d}||_{\mathcal{H}^2}-||T_{zw,c}||_{\mathcal{H}^2}\le||T_{zw,d}-T_{zw,c}||_{\mathcal{H}^2},$$

Therefore,

$$\begin{aligned}&||T_{zw,d}-T_{zw,c}||_{\mathcal{H}^2}^2=\int_{-\infty}^{\infty}||T_{zw,d}(jw)-T_{zw,c}(jw)||_F^2dw\\&=\int_{-\infty}^{\infty}||E_{dc}G_{21}^{-1}(I-G_{22}K_c)(I-G_{22}K_d^{opt})^{-1}G_{21}||_F^2dw\\&\le\int_{-\infty}^{\infty}||G_{21}^{-1}(I-G_{22}K_c)(I-G_{22}K_d^{opt})^{-1}G_{21}||_2^2||E_{dc}^{opt}||_F^2dw\\&\le||G_{21}^{-1}(I-G_{22}K_c)(I-G_{22}K_d^{opt})^{-1}G_{21}||_\infty^2\int_{-\infty}^{\infty}||E_{dc}^{opt}||_F^2dw\\&=C_{dc}^2||E_{dc}^{opt}||_{\mathcal{H}^2}^2.\end{aligned}$$

■

We should be cautious when we remove the constraint that K_d is a stabilizing controller — the controller obtained from (LS-DC) may not be proper. This is illustrated below.

Example 1: Consider the weighting matrix $(V\otimes U)(s)=\begin{bmatrix}1 & \frac{1}{s+1}\end{bmatrix}$, which is in $\mathcal{RH}_\infty$. Assume that the objective is to design a distributed controller $K_d=\begin{bmatrix}0\\ *\end{bmatrix}$ from a centralized controller $K_c=\begin{bmatrix}1\\0\end{bmatrix}$. The optimization problem

$$\min_{K_d(s)\in\Sigma}\left|\left|\begin{bmatrix}1 & \frac{1}{s+1}\end{bmatrix}\left(K_d(s)-\begin{bmatrix}1\\0\end{bmatrix}\right)\right|\right|$$

has a unique solution $K_d(s)=\begin{bmatrix}0\\ s+1\end{bmatrix}$, which is not proper and cannot be implemented. We can verify that $(V\otimes U)\Sigma=[\frac{1}{s+1}]$ loses rank at infinity. It turns out if we can avoid such rank losses at infinity, then (LS-DC) returns a proper controller. This will be formalized below.

Definition 1: Given the sparsity pattern Σ and weight matrix $U, V \in \mathcal{RH}_\infty$, the weight-pattern triplet (U, V, Σ) is said to be *rank-preserving* if there is some $h \geq 0$ such that $B(s) = s^h \cdot V(s) \otimes U(s)$ has a nonzero limit at infinity and $\text{rank}(B(s)\Sigma) = \text{rank}(B(\infty)\Sigma)$ when s is large enough.

Theorem 2: Assuming that (U, V, Σ) is rank-preserving, the solution to (LS-DC) is a proper controller.

Proof: The least-squares problem (LS-DC) has a closed-form solution based on the Moore-Penrose pseudo-inverse (shown as †):

$$\begin{aligned}
&\text{vec}_\Sigma(K_d^{opt}(s)) = \underset{K_d \in \Sigma}{\arg\min} \, ||B(s)\text{vec}(K_d - K_c(s))||_2^2 \\
&= \underset{K_d \in \Sigma}{\arg\min} \, ||B(s)\Sigma\text{vec}_\Sigma(K_d) - B(s)\text{vec}(K_c(s))||_2^2 \\
&= (B(s)\Sigma)^\dagger B(s)\text{vec}(K_c(s))
\end{aligned}$$

By assumption the rank does not change when s becomes large enough, the continuity of Moore-Penrose inverse [21] implies that $(B(s)\Sigma)^\dagger \to (B(\infty)\Sigma)^\dagger$, and

$$\begin{aligned}
\text{vec}_\Sigma(K_d^{opt}(s)) &= (B(s)\Sigma)^\dagger B(s)\text{vec}(K_c(s)) \\
&\to (B(\infty)\Sigma)^\dagger B(\infty)\text{vec}(K_c(\infty)).
\end{aligned}$$

Since this is finite by assumption, K_d^{opt} is proper. ■

Remark 1: The condition in Theorem 2 is sufficient to guarantee the recovery of a proper controller. However, this is not the only way to enforce properness: one can solve a modified optimization problem by adding a robust term with some unbounded norm $|| \cdot ||$:

$$\min_{K_d(jw) \in \Sigma} ||U(jw)(K_d(jw) - K_c(jw))V^T(jw)||_F^2 + \lambda ||K_d||.$$

With the added term, the optimization problem always has a bounded solution and properness can be attained for free.

Recognizing the special pattern of the optimization problem (LS-DC), it is not necessary to select the weighting matrices U and V based on the definitions made before, as long as the condition in Theorem 2 is satisfied. The following corollary characterizes the set of all proper controllers that can be obtained from the proposed least-squares problem.

Corollary 1: The set of proper controller that can be obtained from the optimization problem (LS-DC) is equal to

$$\begin{aligned}
\{K_d(s) \mid \text{vec}_\Sigma(K_d(s)) &= (B(s)\Sigma)^\dagger B(s)\text{vec}(K_c(s)), \\
B(s) &= V(s) \otimes U(s), (U, V, \Sigma) \text{ is rank-preserving}\}.
\end{aligned}$$

Proof: The proof follows from that of Theorem 2. ■

This corollary implies that each element of the designed distributed controller is a weighted average of the elements of the centralized controller.

Corollary 2: In the special case where $B(s) = V(s) \otimes U(s)$ is diagonal, the optimization problem (LS-DC) obtains the thresholded centralized controller, namely $\text{vec}_\Sigma(K_d) = \text{vec}_\Sigma(K_c)$.

Proof: If $B(s)$ is diagonal, there is no coupling between the entries in K_d that are forced to be zero and those that are free in the corresponding optimization problem:

$$\begin{aligned}
K_d^{opt}(s) &= \underset{K_d \in \Sigma}{\arg\min} \, ||B(s)\text{vec}(K_d - K_c(s))||_2^2 \\
&= \underset{K_d \in \Sigma}{\arg\min} \, ||B(s)\Sigma\text{vec}_\Sigma(K_d - K_c(s))||_2^2,
\end{aligned}$$

The proof follows from the above relation. ■

IV. Special Cases

Corollary 1 discovers a strong feature: averaging the elements of the centralized controller by weights in the form of transfer functions to obtain a distributed controller. On the other hand, Corollary 2 reveals that the proposed method is related to the simple thresholding method in a special case. Examples in this section make these two observations more concrete and illustrate the importance of the choice of (U, V, Σ).

A. LQR Problem

The LQR problem with a state feedback corresponds to the setting:

$$\left[\begin{array}{c|cc}
A & I & B \\
\hline
\begin{pmatrix} 0 \\ Q^{1/2} \end{pmatrix} & 0 & \begin{pmatrix} R^{1/2} \\ 0 \end{pmatrix} \\
I & 0 & 0
\end{array}\right].$$

where Q and R are positive definite matrices. Such arrangement naturally admits a limiting behavior related to the thresholding method.

Corollary 3: Consider the LQR problem with a diagonal weighting matrix R and an arbitrary sparsity pattern Σ. The optimization problem (LS-DC) recovers a controller K_d with the property $\lim_{s \to \infty} K_d(s)(i,j) = K_c(s)(i,j), \forall (i,j) \in \Sigma$.

Proof: One can write:

$$\begin{aligned}
G_{12} &= \begin{pmatrix} 0 \\ Q^{1/2} \end{pmatrix} (sI - A)^{-1}B + \begin{pmatrix} R^{1/2} \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} R^{1/2} \\ Q^{1/2}(sI - A)^{-1}B \end{pmatrix} \\
G_{22} &= (sI - A)^{-1}B \\
G_{21} &= (sI - A)^{-1}.
\end{aligned}$$

Also, the multiplier matrices on both sides of $K_d - K_c$ are

$$\begin{aligned}
U &= G_{12}(I - K_c G_{22})^{-1} \\
&= \begin{pmatrix} R^{1/2} \\ Q^{1/2}(sI - A)^{-1}B \end{pmatrix} (I - K_c(sI - A)^{-1}B)^{-1} \\
&= \begin{pmatrix} R^{1/2} + R^{1/2}K_c(sI - A - BK_c)^{-1}B \\ Q^{1/2}(sI - A - BK_c)^{-1}B \end{pmatrix} \\
V^T &= (I - G_{22}K_c)^{-1}G_{21} \\
&= (I - (sI - A)^{-1}BK_c)^{-1}(sI - A)^{-1} \\
&= (sI - A - BK_c)^{-1}.
\end{aligned}$$

Note that at infinity the big matrix of the least-squares problem is equal to

$$sV(s) \otimes U(s) = \left(I - \frac{(A - BK_c)^T}{s} \right)^{-1} \otimes \begin{pmatrix} R^{1/2} + R^{1/2} K_c (sI - A - BK_c)^{-1} B \\ Q^{1/2} (sI - A - BK_c)^{-1} B \end{pmatrix} \to I \otimes \begin{pmatrix} R^{1/2} \\ O \end{pmatrix} \quad (s \to \infty).$$

While taking the limit, the rank of $sV(s) \otimes U(s)\Sigma$ does not change because I and R both have full rank and a small perturbation cannot change the rank. Theorem 2 states that one can always obtain a proper controller using the proposed approximation technique. Moreover, when R is a positive definite diagonal matrix, it follows from Corollary 2 and Theorem 2 that as $s \to \infty$ the designed controller approaches the thresholded centralized controller in the frequency domain. ■

B. Diagonal Sparsity Pattern

We write $U^*U = H = (h_{ij})_{ij=1}^m$ and $V^*V = L = (l_{ij})_{ij=1}^m$, in the case where m is the number of measurements and is the same as the number of scalar controller outputs. The solution of the optimization problem (LS-DC) with a diagonal sparsity pattern has a simple formula, as given in Corollary 4.

Corollary 4: The optimal solution to

$$\min_{K_d = \text{ diagonal}} \quad \mathrm{Tr}(K_d - K_c)^* H (K_d - K_c) L^T \tag{7}$$

is given by the system of linear equations

$$\sum_u h_{iu} l_{iu} K_d(u,u) = \sum_{u,v} h_{iu} l_{iv} K_c(u,v), i = 1, \ldots, m. \tag{8}$$

Proof: (7) is a convex optimization problem that has an explicit solution given by the first-order optimality conditions. Taking the derivative with respect to the real and complex parts of K_d, we obtain the following for all $i = 1, \ldots, m$:

$$\mathrm{Tr}(E_{ii} H (K_d - K_c) L^T + (K_d - K_c)^* H E_{ii} L^T) = 0,$$
$$\mathrm{Tr}(-E_{ii} H (K_d - K_c) L^T + (K_d - K_c)^* H E_{ii} L^T) = 0.$$

Notice that $E_{ii} = e_i e_i^T$, $H^T = \bar{H}, L^T = \bar{L}$,

$$e_i^T H (K_d - K_c) L^T e_i = 0,$$

which can be simplified to (8). ■

V. Numerical Examples

A. Cruise Control

To illustrate the performance of the developed method, we consider the planar vertical-takeoff and landing (PVTOL) model borrowed from [22]. The dynamics of aircraft $i \in \{1, 2, 3\}$ is described by the equation

$$\ddot{X}^i(t) = v^i(t)$$
$$\ddot{\theta}^i(t) = \frac{1}{\mu} (\sin(\theta^i(t)) + v^i(t) \cos(\theta^i(t))),$$

where $\mu > 0$ is a coefficient that describes the amount of coupling between the rolling moment and the lateral acceleration. X^i and θ^i are the horizontal position and angle of the aircraft. We consider the linear feedback rule

$$v^i(t) = \alpha \dot{X}^i(t) + \beta \theta^i(t) + \gamma \dot{\theta}^i(t) + u(t),$$

where the gains are selected in such a way that the horizontal velocity and the angle are stabilized at zero with $u = 0$. As discussed in [22], $\mu = 0.1$, $\alpha = 90.62$, $\beta = -42.15$, $\gamma = -13.22$ are an appropriate choice of parameters. Therefore, we select the parameters in the simulations close to these numbers. Let d denote the desired distance between two adjacent aircraft. The state of aircraft i is

$$x^i(t) = [X^i(t) - X^{i+1}(t) - d, \dot{X}^i(t), \theta^i(t), \dot{\theta}^i(t)]^T \tag{9}$$

where the first term is the relative distance that the aircraft is able to access. Each aircraft can be modeled by the following matrices:

$$A_i = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \alpha & \beta & \gamma \\ 0 & 0 & 0 & 1 \\ 0 & \alpha/\mu & (\beta+1)/\mu & \gamma/\mu \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1/\mu \end{bmatrix}.$$

The A and B matrix of the entire system can be formed by stacking the models of the 3 aircraft together, and adding -1 in the upper block diagonal position that corresponds to the $\dot{X}^{i+1}$ part of $\dot{x}^i$. We need to fix the position of one plane since the shifting is uncontrollable. To do so, we delete one state entry that describes the position of the last plane. There are 11 states and 3 control inputs in total for this system with 3 aircraft. The C matrix is identity, and the sparsity pattern of the controller denoted by $\Sigma \in \mathbb{R}^{3 \times 11}$ is equal to

$$\Sigma = \left[\begin{array}{cccc|cccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

To define an $\mathcal{H}_2$ problem, we apply noise and disturbance to the system and obtain the representation

$$\left[\begin{array}{c|cc} A & I & B \\ \hline \begin{pmatrix} 0 \\ I \end{pmatrix} & 0 & \begin{pmatrix} I \\ 0 \end{pmatrix} \\ I & I & 0 \end{array} \right]. \tag{10}$$

Note that $D_{21} = I$ has full row rank, and that D_{12} has full column rank. The optimal centralized controller K_c is obtained by solving this $\mathcal{H}_2$ synthesis problem via MATLAB's `h2syn` function. The controller is dynamic and of order 11. Distributed controllers are designed under different settings, and the results are summarized in Table I (the notation $T(K)$ denotes the transfer function associated with a controller K). It can be observed that the global optimality guarantee (with respect to the best centralized controller) is at least 98%.

TABLE I
CRUISE CONTROL APPROXIMATION

α	β	γ	μ	$\frac{\|\|T(K_c)\|\|}{\|\|T(K_d)\|\|}$
90.62	-42.15	-13.22	0.10	0.9887
96.77	-45.34	-14.20	0.10	0.9904
103.11	-46.12	-15.20	0.10	0.9919

TABLE II
MASS-SPRING APPROXIMATION

p	$\frac{\|\|T(K_c)\|\|}{\|\|T(K_d)\|\|}$
1	0.9481
2	0.9932
3	0.9937
4	0.9942

B. Mass-Spring System

Consider a mass-spring system with unit masses and unit spring constants as in [3]. The system can be described as

$$\left[\begin{array}{c|cc} \begin{pmatrix} 0 & I \\ G & 0 \end{pmatrix} & I & \begin{pmatrix} 0 \\ I \end{pmatrix} \\ \hline \begin{pmatrix} 0 \\ I \end{pmatrix} & 0 & \begin{pmatrix} I \\ 0 \end{pmatrix} \\ I & I & 0 \end{array}\right]. \tag{11}$$

where G is an $n \times n$ symmetric Toeplitz matrix whose first row is given by $[-2, 1, 0, \ldots, 0]$. The sparsity pattern is considered as $\Sigma = [S_p, I]$, where S_p is a banded matrix with 1's on the p upper and lower sub-diagonals; this implies that each mass can only access its own velocity and the displacement of its p neighbors to the left and to the right. We consider the cases with $n = 10$ and $p = 1, 2, 3, 4$. As before, we first obtain an optimal centralized controller and then use the proposed technique to design a sub-optimal distributed controller. The results are summarized in Table II. The global optimality guarantee of each designed controller is at least 94%.

VI. DISCUSSION

The above simulations demonstrate that the developed method is able to obtain structured controllers whose H_2 performances are very close to those of their centralized counterparts. It should be noted that the efficient calculation of the frequency response is based on a relaxation of the closed-loop stability constraint. The underlying procedure is that the designed controller should be applied to the system for checking stability. As future work, it is useful to obtain conditions that guarantee the closed-loop stability before finding the controller.

The distributed controller design was performed in the frequency domain. This was motivated by the fact that dynamic controllers are far more powerful than static controllers in a distributed setting. However, frequency domain samples cannot be implemented directly, and it is important to find a method for translating the designed controller back to the time domain systematically. This is left as future work.

Finally, it should be mentioned that the controller design procedure can be strengthened by making the approximation more accurate. Indeed, there is a natural tradeoff between making the approximation easier to solve and making it more precise. Note that since

$$I - G_{22}K_c = I - G_{22}K_d + G_{22}(K_d - K_c), \tag{12}$$

one can write:

$$\begin{aligned} &(I - G_{22}K_d)^{-1} \\ &= (I - G_{22}K_c)^{-1} + (I - G_{22}K_c)^{-1}G_{22}(K_d - K_c)(I - G_{22}K_d)^{-1} \\ &\approx (I - G_{22}K_c)^{-1} + (I - G_{22}K_c)^{-1}G_{22}(K_d - K_c)(I - G_{22}K_c)^{-1}. \end{aligned}$$

Hence,

$$\begin{aligned} &T_{zw,d} - T_{zw,c} \\ &\approx G_{12}(I - K_cG_{22})^{-1}(K_d - K_c)(I - G_{22}K_c)^{-1} \\ &\quad + G_{12}(I - K_cG_{22})^{-1}(K_d - K_c)(I - G_{22}K_c)^{-1}G_{22} \\ &\qquad \times (K_d - K_c)(I - G_{22}K_c)^{-1}G_{21} \\ &= G_{12}M(\bar{X} - Q\bar{N})(K_d - K_c)(Y - MQ)\bar{M} \\ &\quad + G_{12}M(\bar{X} - Q\bar{N})(K_d - K_c)(Y - MQ)\bar{N} \\ &\qquad \times (K_d - K_c)(Y - MQ)\bar{M}G_{21}. \end{aligned}$$

where all relevant coefficients are linear in terms of the Youla parameters. We can further approximate this difference to third order using this expansion. Since G_{22} is assumed to be strictly proper, it follows that $\bar{N}$ and N are strictly proper. This leads to a better approximation at high frequencies when higher-order terms are used.

VII. CONCLUSION

In this paper, we developed a convex approximation technique in the frequency domain for the design of a near-globally optimal distributed controller based on the optimal centralized controller. This is achieved by first formulating the difference between the centralized and distributed $\mathcal{H}_2$ performances and then approximating it with a convex function. We investigated the properness of the designed controller, the optimality guarantees of the controller, and the space of all controllers that can be obtained using the proposed method via adjusting the algorithm parameters. When applied to the LQR problem and to those problems with a diagonal sparsity pattern, we discovered interesting connections to the basic thresholding and averaging methods. Future research is needed to obtain a state-space representation of the design controller (via a closed-form formula) and guarantee the closed-loop stability.

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