On the Convexity of Optimal Decentralized Control Problem and Sparsity Path

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Abstract— **This paper is concerned with an important special case of the stochastic optimal decentralized control (SODC) problem, where the objective is to design a static structurally constrained controller for a stable stochastic system. This problem is non-convex and hard to solve in general. We show that if either the measurement noise covariance or the input weighting matrix is not too small, the problem is locally convex. Under such circumstances, the design of a decentralized controller with a bounded norm subject to an arbitrary sparsity pattern is naturally a convex problem. We also study the problem of designing a sparse controller using a regularization technique, where the control structure is not pre-specified but penalized in the objective function. Under some genericity assumptions, we prove that this method is able to design a decentralized controller with any arbitrary sparsity level. Although this paper is focused on stable systems, the results can be generalized to unstable systems as long as an initial stabilizing controller with a desirable structure is known *a priori*.**

I. Introduction

The area of optimal decentralized control (ODC) is of a high importance for real-world systems, where the objective is to control complex systems with many interacting subsystems. As opposed to the centralized control problem, the decentralized controller under design is required to be sparse and/or have a certain sparsity pattern. The structural constraints of the controller impose hard or soft penalties on the interactions (information exchange) between different subsystems in order to reduce the communication and computation complexity of the control system. In the case where the controller is distributed over a geographical area, the term *distributed control* is often used instead of *decentralized control*. It is well known that if no structural constraints are imposed on the control system, the resulting centralized controller can be efficiently found using Riccati equations. However, the ODC problem is intractable in the worst case since it amounts to an NP-hard problem [1], [2]. Several methods have been developed to solve ODC for special control systems, such as spatially distributed systems [3]–[5], localizable systems [6], [7], strongly connected systems [8], optimal static distributed systems [9], decentralized systems over graphs [10], [11], and quadratically-invariant systems [12]. Recently, we have studied the possibility of designing a near-optimal static decentralized controller via a transformation of the optimal centralized controller [13]. Furthermore, the stability analysis of decentralized control systems has attracted much attention [14]–[16].

The difficulty of solving the long-standing ODC problem stems from the fact that polynomial optimization problems are NP-hard in their general form. Due to the NP-hardness of finding a optimal minimum of such problems, several convex relaxations have been introduced to reduce the complexity at the expense of obtaining a near-global (or sub-optimal) solution. These methods include, but are not restricted to, Linear Matrix Inequality (LMI), Second-Order Cone Programming (SOCP), and Semidefinite Programming (SDP) [17], [18]. Recently, we have shown that the underlying structure of a nonlinear optimization problem can be mapped into a *generalized weighted graph*, where the exactness of the convex relaxation of the problem depends on the specifications of this graph [19], [20]. By building on this result, we have shown in [21]–[23] that the SDP relaxations of the finite- and infinite-horizon ODC problems have guaranteed low-rank solutions, from which near-global solutions may be recovered. We have demonstrated the efficacy of this technique on the control of power systems in [24], [25].

In this work, we consider an important special case of the stochastic optimal decentralized control (SODC) problem, where the controller under design is considered static and the open-loop system is stable. First, we prove that the measurement noise and the control effort are both able to indirectly convexify the problem. More precisely, we show that if either the noise covariance or the input weighting matrix is not too small, the design of an optimal decentralized controller subject to any arbitrary sparsity pattern is naturally a convex problem if the controller is sought within a convex stability region around the origin.

We also investigate the SODC problem in the case where the goal is to design a sparse controller whose structure is not pre-specified but softly penalized via a regularization term. This sparsity-promoting technique has been introduced in [26] for static controllers, and studied in [27] as a rank constrained optimization problem and in [28] for dynamic controllers under a quadratic-invariance assumption. In this work, we aim to show that this sparsity-promoting technique is able to design sparse static controllers with any given sparsity level. To this end, we prove that the cardinality of the controller as a function of the regularization coefficient changes by one at each breakpoint, under some genericity assumptions. The results of this work can be readily generalized to the design of a dynamic controller for an

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unstable system, provided that an initial stabilizing controller with a desirable sparsity pattern is given. The details of this generalization are omitted due to space restrictions.

Notations: Lowercase, bold lowercase and uppercase letters are used for scalars, vectors and matrices, respectively (say x, $\mathbf{x}$, X). The symbol $\mathrm{trace}\{W\}$ denotes the trace of a matrix W. The notation $\mathrm{vec}\{W\}$ refers to the vectorization of the matrix W. The symbol $(\cdot)^T$ is used for transpose. The inner product of two matrices M and N is denoted as $\langle M, N\rangle$. The set of real numbers is denoted as $\mathbb{R}$. The notation $\otimes$ is used for the Kronecker product. The notation $W \succ 0$ means that W is positive definite. The symbols $|x|$ and $|\mathcal{X}|$ denote the absolute value of a real number x and the cardinality of a set $\mathcal{X}$, respectively.

II. Preliminaries

Infinite-Horizon Deterministic ODC (DODC): Consider the system

$$x[\tau+1] = Ax[\tau] + Bu[\tau], \qquad \tau = 0, 1, ... \tag{1}$$

with the known matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $x[0] \in \mathbb{R}^n$. The objective is to design a stabilizing static controller $u[\tau] = Kx[\tau]$ to minimize the quadratic objective

$$J(K) = \sum_{\tau=0}^{\infty} \left(x[\tau]^T Q x[\tau] + u[\tau]^T R u[\tau]\right) \tag{2}$$

for given positive semidefinite matrices $Q \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times m}$, subject to the structural constraint $K \in \mathcal{K}$. Note that $\mathcal{K} \in \mathbb{R}^{m\times n}$ is a subspace that captures the sparsity pattern of all permissible controllers, and therefore it imposes forced zeros in certain entries of the unknown controller K.

Infinite-Horizon Stochastic ODC (SODC) problem: Consider the stochastic system

$$\begin{aligned} x[\tau+1] &= Ax[\tau] + Bu[\tau] + Ed[\tau], \qquad &\tau = 0, 1, ... \\ y[\tau] &= x[\tau] + Fv[\tau], &\tau = 0, 1, ... \end{aligned} \tag{3}$$

where A, B, E, F are some given matrices. The random variables $d[\tau]$ and $v[\tau]$ represent disturbance and measurement noise, which are assumed to be independent from each other at different times. The objective is to design a stabilizing static controller to minimize the objective function

$$J(K) = \lim_{\tau\to+\infty} \mathcal{E}\left(x[\tau]^T Q x[\tau] + u[\tau]^T R u[\tau]\right) \tag{4}$$

where $\mathcal{E}(\cdot)$ is the expectation operator.

The design of an optimal controller with a pre-defined structure is NP-hard and intractable in the worst case. More precisely, the above problem is non-convex with respect to K even in the centralized case $\mathcal{K} = \mathbb{R}^{m\times n}$. Riccati equations convexify the problem by eliminating K through a reformulation, but this technique cannot be used in the decentralized problem where there are explicit structural constraints on K. Note that the computational complexity of ODC is related to two constraints: (i) $K \in \mathcal{K}$, (ii) closed-loop stability. In this paper, we only focus on issue (i) by assuming that there is a known initial controller K_0 and a bounded convex region $\mathcal{S} \subset \mathbb{R}^{m\times n}$ containing K_0 such that

- All eigenvalues of $A + BK$ are inside the unit circle for every $K \in \mathcal{S}$.
- The solution of the ODC problem belongs to $\mathcal{S}$ (or alternatively, the controller under design is restricted to be in this set).

The above assumption implies that $K = K_0$ is a feasible solution for both DODC and SODC problems. With no loss of generality, we assume that $K_0 = 0$ (this can be achieved by replacing the variable K with $K + K_0$ and redefining the system (1) or (3) based on K_0). Our assumption implies that A is a stable matrix, but we will show that the problem is still challenging and the controller gain $K = 0$ is not an optimal solution under some genericity assumptions.

III. Local Convexity of SODC

In this section, we study the local convexity of the ODC problem. Before stating our main results, we reformulate the stochastic optimal control problem.

Lemma 1. *For every $K \in \mathcal{S}$, the function $J(K)$ associated with the SODC problem can be written as*

$$\begin{aligned} J(K) = &\mathrm{vec}\{\Sigma_d\}^T \tilde{A}(K)^{-1} \mathrm{vec}\{Q + K^T R K\} \\ &+ \mathrm{vec}\{\Sigma_v\}^T \mathrm{vec}\{(BK)^T P(K)(BK) + K^T R K\} \end{aligned} \tag{5}$$

where

$$\tilde{A}(K) = I - (A + BK)^T \otimes (A + BK)^T \tag{6}$$

and $P(K)$ is the solution of the Lyapunov equation

$$(A+BK)^T P(K)(A+BK) - P(K) + Q + K^T R K = 0 \tag{7}$$

Proof. It is easy to verify that

$$J(K) = \langle P(K), \Sigma_d\rangle + \langle (BK)^T P(K)(BK) + K^T R K, \Sigma_v\rangle \tag{8}$$

where

$$P(K) := \sum_{t=0}^{\infty} \left((A+BK)^t\right)^T (Q + K^T R K)(A + BK)^t \tag{9a}$$

$$\Sigma_d := \mathcal{E}\{Ed[\tau]d[\tau]^T E^T\} \tag{9b}$$

$$\Sigma_v := \mathcal{E}\{Fv[\tau]v[\tau]^T F^T\} \tag{9c}$$

Note that (9a) is equivalent to (7). Moreover, Σ_d and Σ_v are covariance matrices for the disturbance and measurement noises. Notice that since $Q + K^T R K \succeq 0$ and $(A + BK)$ is stable, (7) has a unique positive semidefinite solution $P(K)$. Applying the vectorization operation to (7) yields that

$$\begin{aligned} \mathrm{vec}\{P(K)\} = &\left(I - (A + BK)^T \otimes (A + BK)^T\right)^{-1} \\ &\times \mathrm{vec}\{Q + K^T R K\} \end{aligned} \tag{10}$$

This implies that (8) and (5) are equivalent, which completes the proof. $\square$

The reason behind writing $J(K)$ as (5) is twofold:

1. This allows us to study the convexity of the function $J(K)$. Based on this analysis, we aim to obtain sufficient conditions under which the SODC problem is a convex problem, no matter what the structural set $\mathcal{K}$ is.

2. In the case where $J(K)$ is non-convex, we can study how to penalize it to make the resulting objective function convex. This may lead to a near-global solution for the controller $K \in \mathcal{K}$.

Definition 1. Consider a function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ with respect to K. Denote the gradient and Hessian of f as $\nabla f(K)$ and $\nabla^2 f(K)$, respectively. Note that $\nabla f : \mathbb{R}^{m\times n} \to \mathbb{R}^{mn\times 1}$ and $\nabla^2 f : \mathbb{R}^{m\times n} \to \mathbb{R}^{mn\times mn}$. The ij^{th} element of $\nabla f(K)$ is the partial derivative of $f(K)$ with respect to k_{ij}, and the $(ij, i'j')^{\text{th}}$ element of $\nabla^2 f(K)$ is the second-order partial derivative of $f(K)$ with respect to k_{ij} and $k_{i'j'}$. For the sake of simplicity, we drop the argument K and simply use the notations ∇f and $\nabla^2 f$ for the gradient and Hessian of $f(K)$. Henceforth, $\nabla f(ij)$ and $\nabla^2 f(ij, i'j')$ denote the ij^{th} and $(ij, i'j')^{\text{th}}$ entries of gradient and Hessian, respectively. Given a function $G : \mathbb{R}^{m\times n} \to \mathbb{R}^{r\times l}$ with respect to K, the notation $D_{ij}G(K)$ shows the partial derivative of $G(K)$ with respect to k_{ij}, and $D^2_{ij,i'j'}G(K)$ denotes the second-order partial derivative of $G(K)$ with respect to k_{ij} and $k_{i'j'}$.

Before proceeding with the main results of this paper, we decompose $J(K)$ into three functions:

$$\begin{aligned} J(K) =& \underbrace{\mathrm{vec}\{\Sigma_d\}^T \tilde{A}(K)^{-1}\mathrm{vec}\{Q\}}_{J_1(K)} \\ &+ \underbrace{\mathrm{vec}\{\Sigma_d\}^T \tilde{A}(K)^{-1}\mathrm{vec}\{K^T R K\}}_{J_2(K)} \\ &+ \underbrace{\mathrm{vec}\{\Sigma_v\}^T \mathrm{vec}\{(BK)^T P(K)(BK) + K^T R K\}}_{J_3(K)} \end{aligned} \tag{11}$$

The SODC can be reformulated as below.

Reformulated SODC: Minimize the function $J_1(K) + J_2(K) + J_3(K)$ subject to the constraint $K \in \mathcal{K} \cap \mathcal{S}$.

We aim to study the convexity of the reformulated SODC problem.

Lemma 2. *The following statements hold:*

- *If $\Sigma_d \succ 0$ and $R \succ 0$, then $\nabla^2 J_2|_{K=0} \succ 0$.*
- *If $\Sigma_v \succ 0$ and $R \succ 0$, then $\nabla^2 J_3|_{K=0} \succ 0$.*

Proof. To study the Hessian of $J_2(K)$ at $K = 0$, consider a matrix $\tilde{P}(K)$ such that

$$\mathrm{vec}\{\tilde{P}(K)\}^T = \mathrm{vec}\{\Sigma_d\}^T \tilde{A}(K)^{-1} \tag{12}$$

We can write

$$\tilde{A}(K)^T \mathrm{vec}\{\tilde{P}(K)\} = \mathrm{vec}\{\Sigma_d\} \tag{13}$$

and

$$(I - (A+BK)\otimes(A+BK))\mathrm{vec}\{\tilde{P}(K)\} = \mathrm{vec}\{\Sigma_d\} \tag{14}$$

Setting $K = 0$ yields that

$$(I - A\otimes A)\mathrm{vec}\{\tilde{P}(0)\} = \mathrm{vec}\{\Sigma_d\} \tag{15}$$

which is equivalent to the Lyapunov equation

$$A\tilde{P}(0)A^T - \tilde{P}(0) + \Sigma_d = 0 \tag{16}$$

For simplicity, we denote $\tilde{P}(0)$ by $\tilde{P}$. Since $\Sigma_d \succ 0$ and A is stable, we have $\tilde{P} \succ 0$. Define M_{ij} as a 0-1 matrix such that $M_{ij}(k,l) = 1$ if $(k,l) = (i,j)$ and $M_{ij}(k,l) = 0$ otherwise. It can be verified that

$$\nabla^2 J_2(ij, i'j')|_{K=0} = \mathrm{vec}\{\tilde{P}\}\mathrm{vec}\{M_{ij}^T R M_{i'j'} + M_{i'j'}^T R M_{ij}\} \tag{17}$$

On the other hand,

$$M_{ij}^T R M_{i'j'} = R(i,i')M_{jj'} \tag{18a}$$

$$M_{i'j'}^T R M_{ij} = R(i',i)M_{j'j} \tag{18b}$$

Therefore,

$$\begin{aligned} \nabla^2 J_2(ij, i'j')|_{K=0} &= \langle \tilde{P}, R(i,i')M_{jj'} + R(i',i)M_{j'j}\rangle \\ &= 2\tilde{P}(j,j')R(i,i') \end{aligned} \tag{19}$$

There is a permutation matrix V such that $\nabla^2 J_2|_{K=0} = V^{-1}(\tilde{P}\otimes R)V$. Since each eigenvalue of $\tilde{P}\otimes R$ is the multiplication of two eigenvalues of the positive definite matrices $\tilde{P}$ and R, the matrix $\nabla^2 J_2|_{K=0}$ is positive definite.

Similarly, it can be shown that the eigenvalues of $\nabla^2 J_3|_{K=0}$ are equal to those of $\Sigma_v \otimes B^T P(0)B + \Sigma_v \otimes R$. Since $P(0), \Sigma_v, R \succ 0$, we conclude that $\nabla^2 J_3|_{K=0} \succ 0$. □

Remark 1. The reason behind splitting the objective function $J(K)$ into three terms is that the local convexity of $J_2(K)$ and $J_3(K)$ is guaranteed by the positive definiteness of the parameters R, Σ_d and Σ_v. In other words, if $J(K)$ is locally non-convex at $K = 0$, it is due to $J_1(K)$. Notice that the multiplication of R by a factor α greater than one increases the eigenvalues of both $\nabla^2 J_2|_{K=0}$ and $\nabla^2 J_3|_{K=0}$, while keeping $J_1(K)$ unchanged. Similarly, if the covariance of the measurement noise is elevated by a factor of α, the eigenvalues of $\nabla^2 J_3|_{K=0}$ are multiplied by α. This implies that the noise covariance and the matrix R could make the objective function $J(K)$ convex. However, since $J_1(K)$ is not necessarily locally convex, elevating the disturbance covariance would increase the non-convexity of the problem.

Notation 1. Since $J(K)$ depends on the fixed parameters R, Σ_d and Σ_v, the notation $J(K|R, \Sigma_d, \Sigma_v)$ would be used instead of $J(K)$.

Theorem 1. *Assume that R, Σ_d and Σ_v are positive definite. There exist two positive numbers α' and β' such that the following statements hold for every $\alpha \geq \alpha'$ and $\beta \geq \beta'$:*

i) *The function $J(K|\alpha R, \Sigma_d, \beta\Sigma_v)$ is strictly convex at a neighborhood of $K = 0$.*
ii) *Generically, $K = 0$ is not a local minimum of $J(K|\alpha R, \Sigma_d, \beta\Sigma_v)$.*

Proof. According to Lemma 2 and Remark 1, if the elements of the pair (α, β) are sufficiently large, the local convexity of $J_2(K|\alpha R, \Sigma_d, \beta\Sigma_v) + J_3(K|\alpha R, \Sigma_d, \beta\Sigma_v)$ dominates the possible non-convexity of $J_1(K|\alpha R, \Sigma_d, \beta\Sigma_v)$. As a result, the eigenvalues of the Hessian of $J(K|\alpha R, \Sigma_d, \beta\Sigma_v))$ become strictly positive at $K = 0$. Moreover, since $A + BK$ is stable for every $K \in \mathcal{S}$, the function $J(K|\alpha R, \Sigma_d, \beta\Sigma_v))$

is infinitely differentiable over this region. Therefore, this function is convex around the origin.

To prove Part (ii), the partial derivatives of $J(K)$ are calculated in [29]. It is straightforward to observe that the gradient of $J(K)$ is nonzero at $K = 0$, for a generic choice of $(A, B, Q, R, \Sigma_d, \Sigma_v)$. $\square$

To find α' and β' satisfying the conditions of Theorem 1, it suffices to seek a pair (α', β') for which the Hessian of $J(K|\alpha' R, \Sigma_d, \beta' \Sigma_v)$ is positive definite. Due to (11), this Hessian is a linear function of α' as well as a linear function of β' (it is bilinear in terms of both parameters). Hence, we can set α' or β' equal to 1 and solve an LMI problem to find the other parameter (note that there are infinitely many choices for (α', β')).

Example 1: Assume that $m = 5$, $n = 10$, and Q and R are identity matrices. We generate random matrices A and B as follows:

- We write $A = V^{-1} D V$, where each entry of V is chosen randomly from a normal Gaussian distribution and D is a diagonal matrix. The eigenvalues of A are generated as $r_i e^{\theta_i \sqrt{-1}}$, where r_i's and θ_i's are uniformly chosen from $[0, 0.99]$ and $[0, 2\pi]$, respectively.
- Each entry of B is chosen randomly from a normal Gaussian distribution.

Assume that $\Sigma_d = \Sigma_v = 0.5I$. The minimum eigenvalue of the Hessian of $J(K|\alpha R, \Sigma_d, \Sigma_v)$ at $K = 0$ is plotted for $\alpha \in [0, 20]$ in Figure 1a. It can be seen that the function is increasing and becomes positive around $\alpha = 8$. Similarly, the minimum eigenvalue of the Hessian of $J(K|R, \Sigma_d, \beta\Sigma_v)$ at $K = 0$ is drawn for $\beta \in [0, 10]$ in Figure 1b. As before, the function is increasing and becomes positive around $\beta = 0.8$. We have generated many random systems according to the above probability distributions and obtained similar numbers.

One of the main results of this work is stated below.

Theorem 2. *Consider the bounded convex region $\mathcal{S}$. There exist two positive numbers $\alpha_{\mathcal{S}}$ and $\beta_{\mathcal{S}}$ such that the reformulated SODC problem is convex for every arbitrary control pattern $\mathcal{K}$ if $R \succeq \alpha_{\mathcal{S}} I$ and $\Sigma_v \succeq \beta_{\mathcal{S}} I$. Moreover, if the tuple $(A, B, Q, R, \Sigma_d, \Sigma_v)$ is generic, then $K = 0$ is not a solution of SODC.*

Proof. The proof immediately follows from Theorem 1. The details are omitted for brevity. $\square$

Theorems 1 and 2 state that the measurement covariance and the matrix R can both convexify the SODC problem. Moreover, the minimum value of R or Σ_v needed to ensure this convexity can be found via an LMI problem.

Example 2: Suppose A and B in (3) are equal to the randomly generated matrices

$$A = \begin{bmatrix} 0.0848 & 1.7378 \\ -0.4183 & 0.9798 \end{bmatrix}, \; B = \begin{bmatrix} 0.1323 & -0.1551 \\ 0.0529 & -0.2594 \end{bmatrix} \quad (20)$$

Furthermore, suppose that $Q = I$ and $\Sigma_d = 0.05 \times I$. The goal is to design a fully decentralized controller in a pre-defined feasible region with the structure

$$K = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \quad (21)$$

where k_1 and k_2 are the to-be-designed elements of K. Stability and local convexity regions corresponding to different settings of R and Σ_v are shown in Figure 2. The blue area demonstrates the stability region of the controller (if (k_1, k_2) belongs to this region, the closed-loop system is stable). The red area that is included in the stability region corresponds to the set of points for which the reformulated SODC problem is locally convex. The black ellipse shows the pre-selected feasible region $\mathcal{S}$ for k_1 and k_2. First, assume that $R = 0.01 \times I$ and $\Sigma_v = 0.001 \times I$. The stability and convexity regions for k_1 and k_2 are depicted in Figure 2a. As can be seen, the problem is not always locally convex in the feasible region. Now, suppose that R is increased to $2 \times I$ while Σ_v is kept unchanged. As can be seen in Figure 2b, the convexity region expands and covers most of the feasible region for k_1 and k_2. If in addition to R, we increase Σ_v to $0.1 \times I$, the convexity region in Figure 2c covers the entire feasible region and the reformulated SODC problem becomes convex in $\mathcal{S}$, as expected from Theorem 2.

IV. Sparsity Promoting via L_1 Regularization

In the preceding section, we studied the problem of designing a controller with a given sparsity pattern $\mathcal{K}$. In this section, the objective is to design a controller whose sparsity pattern is not fixed but is instead softly enforced via a regularization term.

Recently, a sparsity promotion method using the L_1 norm has been studied in [26]. We adopt the same strategy in this work. Consider the regularized SODC problem

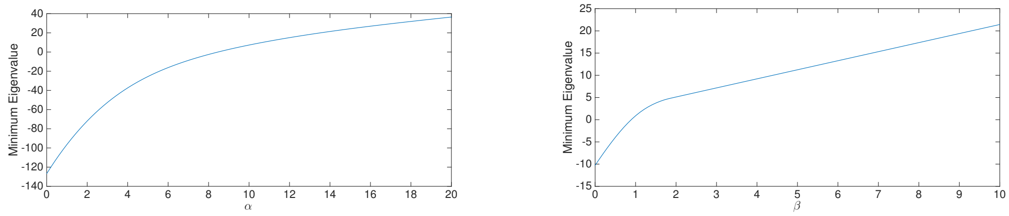
$$\min \quad J(K) + \lambda ||K||_1 \quad (22a)$$

$$\text{s.t.} \quad K \in \mathcal{S} \quad (22b)$$

where λ is a nonnegative number and $\|\cdot\|_1$ denotes the sum of the absolute values of all entries of K. The results of this paper can be easily generalized to the case where $||K||_1$ is replaced by a weighted sum of some or all absolute values of the entries of K. We assume that $J(K)$ is convex over $\mathcal{S}$ to ensure the convexity of (22), which is guaranteed by a relatively large matrix R or Σ_v. The objective is to study the sparsity level of the optimal control K as a function of λ.

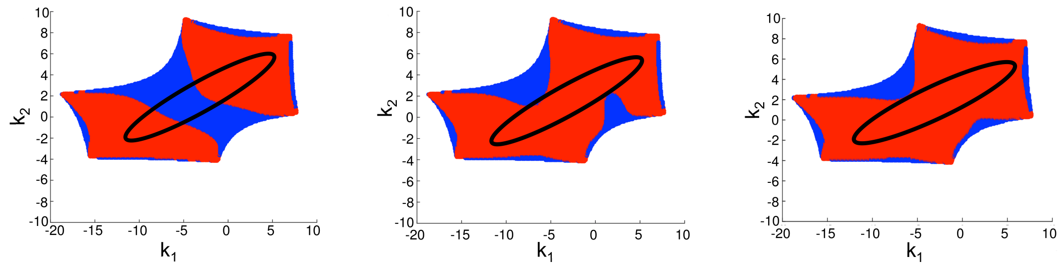
The main result of this section is to show that, for a generic system, the cardinality of the control gain K (obtained via the regularized SODC problem) as a function of λ is a piecewise constant function, where the cardinality changes by one at each breakpoint. In general, there are 2^{mn} different sparsity patterns for the controller K. Therefore, finding the best structure(s) among this exponential number of structures via enumeration is neither practical nor efficient. Due to this difficulty, we use the L_1 regularization and prove the following results under generic conditions:

1. The optimal controller corresponding to $\lambda = 0$ is dense, whereas the one for large λ's is highly sparse ($K = 0$).



(a) Convexity guaranteed by α (b) Convexity guaranteed by β

Fig. 1: The minimum eigenvalue of the Hessian of $J(K)$ as a function of R and Σ_v in Example 1.



(a) $R = 0.01 \times I$, $\Sigma_v = 0.001 \times I$ (b) $R = 2 \times I$, $\Sigma_v = 0.001 \times I$ (c) $R = 2 \times I$, $\Sigma_v = 0.1 \times I$

Fig. 2: Stability and local convexity regions of the decentralized control problem in Example 2.

2. As λ decreases from $+\infty$ to 0, the sparsity of the optimal controller changes gradually and there is no big jump in the cardinality of the controller.
2. Given any arbitrary integer from 0 to mn, there is a range of values for λ such that the cardinality of the corresponding optimal controllers is the same as that number.
3. As λ changes from $+\infty$ to 0, at least $mn + 1$ sparsity patterns are obtained. This induces an ordering on mn sparsity patterns out of 2^{mn} patterns. This ordering shows the trade-off between the optimal performance and the sparsity level.

Before presenting the main results of this section, it is desirable to examine the cardinality of the optimal controller in a numerical example.

Example 3: Consider a stochastic continuous-time system with the parameters $A \in \mathbb{R}^{5\times5}$ and $B \in \mathbb{R}^{5\times5}$ whose entries are chosen randomly from Gaussian distributions with a zero mean and variances 25 and 1, respectively. The matrix A is then normalized to make it stable. The cardinality of the optimal controller and the trajectory of its entries are obtained using the local-search method described in [26]. As can be observed in Figure 3, the cardinality changes by one at each breakpoint point, and it reduces from 25 to 12. If λ increases beyond 30, the cardinality gradually reduces to 0.

To streamline our presentation, we make two assumptions:

1. The convex function $J(K)$ is strictly convex over $\mathcal{S}$. Notice that this is indeed the case for a relatively large matrix R or Σ_v.
3. The solution path of the optimal controller resides in the interior of $\mathcal{S}$. Note that our proof can be generalized to the case where the path hits the boundary of $\mathcal{S}$.

Definition 2. Let $K^*(\lambda)$ denote the unique optimal solution of (22) for a given λ. Define the active set of $K^*(\lambda)$ as the set of nonzero entries of $K^*(\lambda)$ for a given λ, which is denoted as $I_{ac}(\lambda)$. It is said that λ^* is a breakpoint if $I_{ac}(\lambda)$ changes at $\lambda = \lambda^*$. Denote the set of all breakpoints as Λ_b.

Lemma 3. *The minimizer $K^*(\lambda)$ is continuous in λ.*

Proof. Note that $J(K)$ is coercive since $\mathcal{S}$ is bounded. The proof follows immediately from the result in [30]. □

Corollary 1. *Λ_b is a countable set.*

Proof. It is straightforward to verify that the continuity of $K^*(\lambda)$ (due to Lemma 3) implies that Λ_b is a countable set. □

Definition 3. It is said that the polynomial function $p(\cdot)$ with a finite degree is **generic** with respect to another polynomial function $\tilde{p}(\cdot)$ with the same degree if it is generated by adding infinitesimal random perturbations to the coefficients of $\tilde{p}(\cdot)$. For simplicity, we simply call $p(\cdot)$ generic without mentioning $\tilde{p}(\cdot)$ if $\tilde{p}(\cdot)$ is implied by the context.

In theory, the function $I_{ac}(\lambda)$ could have a breakpoint at which $|I_{ac}(\lambda)|$ remains unchanged (i.e., the sparsity pattern changes at a breakpoint but the sparsity level remains the same). In what follows, we prove that this case cannot occur if $J(K)$ is a generic polynomial with a finite degree.

Theorem 3. *If the function $J(K)$ is a generic polynomial $p(K)$ with respect to a strictly convex arbitrary polynomial $\tilde{p}(K)$, then with probability 1 the function $|I_{ac}(\lambda)|$ would change by 1 at each breakpoint $\lambda_b \in \Lambda_b$.*

Proof. By contradiction, suppose that $|I_{ac}(\lambda)|$ does not change by 1 at some breakpoint λ_b. Then, there are three possibilities: (i) $|I_{ac}(\lambda)|$ increases by at least 2, (ii) $|I_{ac}(\lambda)|$ decreases by at least 2, (iii) the cardinality remains the same but at least one new element enters $I_{ac}(\lambda)$ and exactly the same number of elements leave $I_{ac}(\lambda)$. With no loss of generality, we investigate only scenario (i) in this proof. First, we show that as λ approaches λ_b from both sides, it must satisfy a particular set of equations. Then, we prove that these equations are satisfied with probability zero for the generic polynomial $p(K)$.

Assume that λ_b^+ is the smallest breakpoint that is greater than λ_b. Similarly, denote λ_b^- as the largest breakpoint that is less than λ_b. Note that it may happen that either λ_b^+ or λ_b^- does not exist if λ_b is the smallest/largest breakpoint in Λ_b, in which case the argument to be presented next needs a slight modification. Because Λ_b is a discrete set in light of Corollary 1, one can write $\lambda_b^- < \lambda_b < \lambda_b^+$. Consider a number r such that $|I_{ac}(\lambda_b+\epsilon^+)| = r$ for every $0 < \epsilon^+ < \lambda_b^+ - \lambda_b$. For simplicity, we index the elements of matrix K with a single number. Denote the nonzero elements of $K^*(\lambda_b+\epsilon^+)$ as $\{k_1^*(\lambda_b+\epsilon^+), k_2^*(\lambda_b+\epsilon^+), ..., k_r^*(\lambda_b+\epsilon^+)\}$. For every $i \notin I_{ac}(\lambda_b+\epsilon^+)$, we have $k_i^*(\lambda_b+\epsilon^+) = 0$. The first-order optimality conditions for $K^*(\lambda_b+\epsilon^+)$ result that

$$\left|\frac{\partial p(K)}{\partial k_i}\right|_{K=K^*(\lambda_b+\epsilon^+)} = \lambda_b + \epsilon^+ \qquad \forall i \in \{1,2,...,r\} \tag{23a}$$

$$\left|\frac{\partial p(K)}{\partial k_j}\right|_{K=K^*(\lambda_b+\epsilon^+)} = s_j(\lambda_b + \epsilon^+) \quad \forall j \notin \{1,2,...,r\} \tag{23b}$$

for some numbers $s_j \in [-1,1]$.

By assumption, the relation $|I_{ac}(\lambda_b-\epsilon^-)| = r+2$ holds for every $0 < \epsilon^- < \lambda_b - \lambda_b^-$. This means that as λ decreases to pass through λ_b, two elements of K will be added to the set of the nonzero elements. Denote these new elements as k_{r+1}^* and k_{r+2}^*. Then, the optimality conditions at $\lambda_b - \epsilon^-$ can be written as

$$\left|\frac{\partial p(K)}{\partial k_i}\right|_{K=K^*(\lambda_b-\epsilon^-)} = \lambda_b - \epsilon^- \qquad \forall i \in \{1,2,...,r+2\} \tag{24a}$$

$$\left|\frac{\partial p(K)}{\partial k_j}\right|_{K=K^*(\lambda_b-\epsilon^-)} = s_j(\lambda_b - \epsilon^-) \;\; \forall j \notin \{1,2,...,r+2\} \tag{24b}$$

for some numbers $s_j \in [-1,1]$. Without loss of generality in our analysis, we drop the absolute values in (23) and (24).

Consider the limiting behavior of $k_{r+1}^*(\lambda)$ and $k_{r+2}^*(\lambda)$. Note that $k_{r+1}^*(\lambda_b+\epsilon^+) = k_{r+2}^*(\lambda_b+\epsilon^+) = 0$, for every $0 < \epsilon^+ < \lambda_b^+ - \lambda_b$. Due to the continuity of $K^*(\lambda)$, one can write

$$k_{r+1}^*(\lambda_b) = \lim_{\epsilon^+\to 0^+} k_{r+1}^*(\lambda_b+\epsilon^+) = 0 \tag{25a}$$

$$k_{r+2}^*(\lambda_b) = \lim_{\epsilon^+\to 0^+} k_{r+2}^*(\lambda_b+\epsilon^+) = 0 \tag{25b}$$

Moreover, since , $p(K)$ is continuously differentiable, it follows from (24) that

$$\frac{\partial p(K)}{\partial k_{r+1}}\Big|_{K=K^*(\lambda_b)} = \lim_{\epsilon^-\to 0^+} \frac{\partial p(K)}{\partial k_{r+1}}\Big|_{K=K^*(\lambda_b-\epsilon^-)} = \lambda_b \tag{26a}$$

$$\frac{\partial p(K)}{\partial k_{r+2}}\Big|_{K=K^*(\lambda_b)} = \lim_{\epsilon^-\to 0^+} \frac{\partial p(K)}{\partial k_{r+2}}\Big|_{K=K^*(\lambda_b-\epsilon^-)} = \lambda_b \tag{26b}$$

Therefore,

$$\frac{\partial p(K)}{\partial k_i}\Big|_{K=K^*(\lambda_b)} = \lambda_b \qquad \forall i \in \{1,2,...,r+2\} \tag{27a}$$

$$\frac{\partial p(K)}{\partial k_j}\Big|_{K=K^*(\lambda_b)} = s_j\lambda_b \quad \forall j \notin \{1,2,...,r+2\} \tag{27b}$$

where $s_j \in [-1,1]$, $k_l^*(\lambda_b) = 0$ for $l \notin \{1,2,...,r\}$ and $k_t^*(\lambda_b) > 0$ for $t \in \{1,2,...,r\}$. Let $\bar{\mathbf{a}}$ denote the vector of all coefficients of $p(K)$. It is desirable to identify the set of all vectors $\bar{\mathbf{a}}$ for which there exist λ_b and $\{k_1^*, ..., k_r^*\}$ such that (27a) holds (recall that $k_{r+1}^* = k_{r+2}^* = 0$). Therefore, one can consider (27a) as a set of polynomial equations in terms of the variables $\lambda_b, k_1^*, ..., k_r^*$. The number of these equations is $r+2$, while the number of variables is $r+1$. This set of equations has a common zero $(\lambda_b, k_1^*, ..., k_r^*)$ if its resultant, denoted as $f(\mathbf{a})$, vanishes at $\mathbf{a} = \bar{\mathbf{a}}$. Note that the resultant is a polynomial function of the coefficients of $p(K)$. One can easily verify that $f(\mathbf{a})$ is not identical to zero. Therefore, the coefficients of $p(K)$ that are generated by a generic perturbation of the coefficient vector for $\tilde{p}(K)$ do not satisfy the equation $f(\bar{\mathbf{a}}) = 0$. This means that (27a) does not hold for a generic polynomial, which is a contradiction. $\square$

Theorem 3 proves that if the objective function of SODC were a generic polynomial with a finite degree, then the cardinality of $I_{ac}(\lambda)$ would change by one at each breakpoint. However, in general $J(K)$ is neither generic (due to the inherent structure of the objective) nor polynomial with a finite degree. In order to make use of this result for the true objective of SODC, we propose two methods to find a generic polynomial with a finite degree that is arbitrarily close to $J(K)$.

A. Functional Perturbation

The objective of this part is to show that there exists a strictly convex and generic polynomial $p(K)$ that is arbitrarily close to $J(K)$. Using Theorem 3, this result implies that the cardinality of the optimal controller changes by one at breakpoints for $p(K)$ which is generated by an infinitesimal generic perturbation of $J(K)$.

Lemma 4. *For a sufficiently small $\epsilon > 0$, there exists a generic polynomial $p(K)$ with a finite degree such that $|J(K) - p(K)| \leq \epsilon$ for every $K \in \mathcal{S}$.*

Proof. Since $\mathcal{S}$ is bounded and $J(K)$ is infinitely differentiable on $\mathcal{S}$, there exists an $\epsilon > 0$ with the property that for every $0 < \epsilon' < \epsilon$, there is a strictly convex polynomial function $\tilde{p}(K)$ with finite degree $d(\epsilon')$ such that

$$|J(K) - \tilde{p}(K)| \leq \epsilon', \qquad \forall K \in \mathcal{S} \tag{28}$$

Now, consider a generic and random infinitesimal perturbation of the coefficients of $\tilde{p}(K)$. Because this perturbation can be arbitrarily small, one can verify that there exists a generic and strictly convex polynomial $p(K)$ with respect to $\tilde{p}(K)$ such that

$$|J(K) - p(K)| \le \epsilon, \qquad \forall K \in \mathcal{S} \tag{29}$$

This completes the proof. $\square$

Theorem 4. *If $J(K)$ is replaced by its arbitrarily close generic polynomial approximate $p(K)$, then the cardinality of the optimal controller of the perturbed regularized SODC problem changes by 1 at each breakpoint.*

Proof. This is an immediate consequence of Theorem 3 and Lemma 4. $\square$

Theorem 4 states that the cardinality of the optimal controller changes by one at each breakpoint under generic conditions.

B. Coefficient Perturbation

Unlike the functional perturbation of the objective function of SODC, we focus on the genericity of the stochastic system in this section.

Definition 4. A generic perturbation of a matrix M is a new matrix $M + \Psi$, where Ψ is an infinitesimal random perturbation.

Theorem 5. *Assume that A has full column rank, and that Q and Σ_d are positive definite. With a generic infinitesimal perturbation of B, the cardinally of the optimal controller of the regularized SODC problem changes by 1 at each breakpoint over a nonempty region around $K = 0$.*

Proof. To prove the theorem, it suffices to show that the polynomial approximation of $J(K)$ is generic under a generic perturbation of B. Using (12) and derivative formulas introduced in [29], one can verify that

$$\begin{aligned}\langle \Sigma_d, D_{ij}P(K)|_{K=0}\rangle =& \text{vec}\{\tilde{P}(0)\}^T \text{vec}\{M_{ij}^T B^T P(0) A \\ &+ A^T P(0) B M_{ij}\}\end{aligned} \tag{30}$$

We aim to show that all partial derivatives of $P(K)$ are linearly independent at $K = 0$, and hence they remain independent in a nonzero region around $K = 0$. To prove this statement, define the $m \times n$ matrix H as

$$H(i,j) = \frac{\partial J(K)}{\partial k_{ij}}\Big|_{K=0} = \langle \Sigma_d, D_{ij}P(K)|_{K=0}\rangle \tag{31}$$

for $i \in \{1, 2, ..., m\}$ and $j \in \{1, 2, ..., n\}$. One can verify that

$$H = 2B^T P(0) A \tilde{P}(0) \tag{32}$$

Since Q and Σ_d are positive definite and A has full rank, $P(0)A\tilde{P}(0)$ is full rank. Therefore, any infinitesimal perturbation of partial derivatives can be mapped to a perturbed B. Since B is generic (with a generic perturbation), the partial derivatives of $J(K)$ at $K = 0$ are generic. Consider the set of equations (27a) after replacing $p(K)$ with $J(K)$. Since $J(K)$ is generic, this system of equations does not have a solution. By following the proof of Theorem 3, this means that the cardinally of the optimal controller of the regularized SODC problem changes by 1 at each breakpoint. $\square$

Remark 2. In this section, the stability condition is captured by a bounded set S. However, note that this constraint can be removed and instead implicitly implied if λ is not too small. In fact, if the regularization coefficient λ is restricted to be greater than a positive number λ_{min}, then

$$\lambda_{\text{min}}||K^*(\lambda)||_1 \le J(K^*(\lambda)) + \lambda||K^*(\lambda)||_1 \le J(0) \tag{33}$$

or equivalently $||K^*(\lambda)||_1 \le J(0).(\lambda_{\text{min}})^{-1}$. This restricts the optimal controller to a neighborhood of the origin, and acts as a surrogate for the explicit constraint $K \in \mathcal{S}$ for an appropriate choice of $\lambda_{\min}$.

C. Penalized SODC

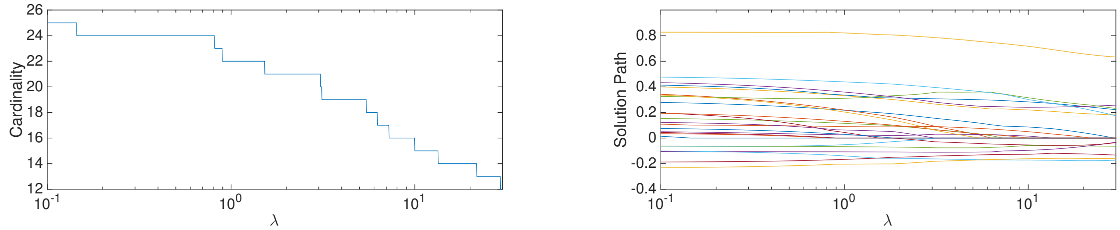
As proved in the section III, if either R or Σ_v is sufficiently large, the SODC problem is convex. A question arises as to whether we can still address the non-convexity of SODC if the above condition is not satisfied. To this end, one approach is to resort to a penalization method, where the objective is to design a penalty term $C(K)$ such that $J_c(K) = J(K) + C(K)$ becomes convex in the region $\mathcal{K} \cap \mathcal{S}$. Indeed, the added penalty will not effect the optimal solution of SODC problem if $C(K)$ is designed in such a way that $J_c(K)$ becomes the lower convex envelope of $J(K)$. However, obtaining the lower convex envelope of $J(K)$ is a daunting task due to the highly non-convex behavior of $J(K)$. Therefore, one can resort to simpler penalty functions (acting as a price of non-convexity) at the expense of losing some optimality guarantee. Consider a penalty function of the form $C(K) = \text{vec}\{K\}^T T \text{vec}\{K\}$. The problem reduces to finding a matrix T such that $\nabla^2 J + T \succeq 0$ for every $K \in \mathcal{K} \cap \mathcal{S}$. A simple choice for the matrix T is aI, where a is equal to the minimum eigenvalue of $\nabla^2 J$ over all matrices $K \in \mathcal{K} \cap \mathcal{S}$. However, this sort of penalization is not usually efficient because all elements of K are penalized with the same weight or, equivalently, all eigenvalues of the Hessian (including the positive ones) are shifted by the same number a. A more efficient way of designing T is as follows:

$$\min \quad \text{trace}\{T\} \tag{34a}$$
$$\text{s.t.} \quad \nabla^2 J|_K + T \succeq 0, \qquad \forall K \in \mathcal{K} \cap \mathcal{S} \tag{34b}$$
$$T \succeq 0 \tag{34c}$$

The reasoning behind the objective function of the above problem is that it aims to indirectly reduce the rank of T. Although (34) is a convex program, it is infinite-dimensional. As future work, we aim to find a more tractable method to design appropriate penalty functions in order to convexify the SODC problem.

V. Conclusions

This paper studies an important special case of the stochastic optimal decentralized control (SODC) problem,



(a) Cardinality

(b) Solution Path (each curve shows one element of the controller)

Fig. 3: Solution path and Cardinality of the optimal controller for a randomly generated system

where the goal is to design a static structurally constrained controller for a stable system. First, we prove that if either the noise covariance or the input weighting matrix is not too small, then the design of an optimal decentralized controller subject to any arbitrary sparsity pattern is naturally a convex problem, provided that the controller is sought within a convex stability region around the origin. We also investigate the SODC problem in the case where the goal is to design a sparse controller whose structure is not pre-specified but softly penalized via a regularization term. It is shown that the cardinality of the controller as a function of the regularization coefficient changes by one at each breakpoint, under some genericity conditions. If the noise covariance and input weighting matrix are not sufficiently large, a convex penalty can be added to the objective to convexify the control design problem. The results of this work can be readily generalized to the design of a dynamic controller for an unstable system, provided that an initial stabilizing controller with a desirable sparsity pattern is available.

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