CONNECTIVITY PROPERTIES OF THE SET OF STABILIZING STATIC DECENTRALIZED CONTROLLERS*

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Abstract. The complexity of the optimal decentralized control (ODC) problem is reflected in the properties of its feasible set. We study the complexity of the ODC problem through an analysis of the set of stabilizing static decentralized controllers, and show that there is no polynomial upperbound on its number of connected components. In particular, it is proved that this number is exponential in the order of the system for a subclass of problems. Since every point in each of these components is the unique solution of the ODC problem for some quadratic objective functional, the results of this work prove that, without prior knowledge for initialization, local search algorithms cannot solve the ODC problem to global optimality for all decentralized control structures. In an effort to understand the connection between the geometric properties of the feasible set of the ODC problem and the control structure, we further identify decentralized structures that admit tractable connectivity properties, using a combination of the Routh-Hurwitz criterion and Lyapunov stability theory.

Key words. Decentralized control, stability, connectivity

AMS subject classifications. 90C26, 90C30, 90C60, 93A14, 93B25, 93B27, 93B60, 93C05, 93C41, 93D05, 93D15

1. Introduction.

Classical state-space solutions to optimal centralized control problems do not scale well as the dimension of the system increases [10]. Real-world controllers also have structural constraints, such as locality and delay. The optimal decentralized control (ODC) problem has been proposed in the literature to bridge this gap. On the one hand, ODC can have nonlinear optimal solutions even for linear systems and is NP-hard in the worst case [31, 6]. On the other hand, the existence of dynamic structured feedback laws is completely captured by the notion of fixed modes [26]. Furthermore, several works have discovered structural conditions on the system and/or the controller under which the ODC problem admits tractable solutions. The conditions include spatially invariance [2], partially nestedness [25], positiveness [24], and quadratic invariance [19]. More recently, the System Level Approach [28] has convexified structural constraints at the expense of working with many impulse response matrices. Promising approximation [12, 1, 21] and convex relaxation techniques [27, 8, 14, 9] also exist in the literature.

A recent line of research, initiated in the machine learning community, suggests using nonlinear programming methods based on local search for the optimal control problems [13]. Local search methods are well-studied for convex problems, and they normally come with optimality guarantees [8]. However, when the problem is non-convex, these methods may converge to a saddle point or a non-global local minimum [5]. Local search algorithms are effective: (i) when they are initialized at a point close enough to a globally optimal solution, or (ii) when there is no spurious local optimum and it is possible to escape saddle points [15, 17, 32, 16]. They have also been applied to instances of ODC to obtain approximate solutions [20, 29], but a question arises as to whether local search is effective for ODC.

In this paper, we prove that the chances of success for the global convergence

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of local search methods applied to a general ODC problem are theoretically slim. Specifically, we prove that the feasible set of the ODC problem in the static case, which includes all structured static controllers that stabilize the system, can be not only non-convex but also disconnected where the number of connected components grows exponentially in the order of the system. Since any point in the feasible set is the unique globally optimal solution of ODC for some quadratic objective functional, this result implies that there is no reformulation of the problem with a smooth change of variables that could convexify the problem. Therefore, one would need to resort to computationally expensive convex hull approaches. Moreover, if one seeks to solve a hard instance of the ODC problem through local search, the algorithm needs to be initialized an exponential number of times unless prior information about the location of the solution is available in order to start in the correct connected component. This result contrasts with the recent findings in [13], arguing that local search could be useful for optimal control problems because it is guaranteed to work for the optimal centralized control problem.

Although the number of connected components is shown to be exponential in this work, we also demonstrate that favorably structured systems can have a single connected component. In particular, it is proved that the set of static stabilizing controllers is connected for damped systems no matter what the control structure is. Moreover, a bound on the number of connected components is provided in the scalar case. For block structured systems with a sufficient number of free elements, we develop a series of equivalence relations that describe the exact number of connected components of structured stable matrices.

This work is related to several papers in the literature. The set of stabilizing controllers has been studied from many angles. The work [23] parametrizes the set of stable state-feedback controllers under no structural constraints. The paper [22] studies the connectivity of stable linear systems and concludes that single-input single-output systems of order n have at most $n + 1$ connected components, while stable multi-input multi-output systems have only one connected component. The work [3] investigates what types of sparse patterns can sustain stable dynamics, using graph theory. To the authors' best knowledge, the connectivity of decentralized stabilizing controllers has not been studied before.

The remainder of this paper is organized as follows. Notations and problem formulations are given in Section 2. We derive elementary connectivity properties of the set of stabilizing controllers and bound the number of connected components for scalar controllers in Section 3. Section 4 examines a subclass of decentralized control problems where the number of connected components is exponential. We also show that highly damped systems admit a connected set of decentralized controllers. Section 5 describes the connectivity properties of structured stable matrices with zero blocks. Concluding remarks are drawn in Section 6.

2. Problem Formulation. Consider the linear time-invariant system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t),\\ y(t) &= Cx(t),\end{aligned}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$ are real matrices of compatible sizes. The vector $x(t)$ is the state of the system and $y(t)$ is the output. We focus on the static case, where the control input $u(t)$ is to be determined via a static output-feedback law $u(t) = Ky(t)$ with the gain $K \in \mathbb{R}^{m\times p}$ such that some measure of performance is optimized. Since the analysis to follow is on the feasible set, the initial state (being deterministic or stochastic) and the objective function (being quadratic or some other function of the system's signals) are unimportant. With no loss of generality, we consider the case with a given initial state $x(0) = x_0$ and a quadratic performance measure

$$J(K, x_0) = \int_0^{\infty} \left[x^T(t)Qx(t) + 2x^T(t)Du(t) + u^T(t)Ru(t)\right] dt \tag{2.1}$$

where the matrix $L = \left[\begin{smallmatrix} Q & D \\ D^T & R \end{smallmatrix}\right]$ is positive semi-definite and R is positive definite. We use the notations $L \succeq 0$ and $R \succ 0$ to denote positive semi-definiteness and positive definiteness, respectively. The closed-loop system is

$$\dot{x}(t) = (A + BKC)x(t).$$

A matrix is stable, or equivalently Hurwitz, if all its eigenvalues lie in the open left half plane. K is said to stabilize the system if $A + BKC$ is stable. All the matrices considered in this work are real-valued unless otherwise noted. The objective is to study the set of structured stabilizing controllers

$$\mathcal{K}_{\mathcal{S}} = \{K : A + BKC \text{ is stable}, K \in \mathcal{S}\}, \tag{2.2}$$

where $\mathcal{S} \subseteq \mathbb{R}^{m\times p}$ is a linear subspace of matrices, often specified by fixing certain entries of the matrix to zero. Decentralized and distributed controllers could be specified by the set $\mathcal{S}$ with a prescribed sparsity pattern. The set of sparse stable matrices

$$\mathcal{A}_{\mathcal{T}} = \{A : A \text{ stable and } A \in \mathcal{T}\} \tag{2.3}$$

is a special case of (2.2), where $\mathcal{T} \subseteq \mathbb{R}^{n\times n}$ is a linear subspace of matrices. When $\mathcal{T}$ is the set of sparse matrices, we represent $\mathcal{T}$ with a sparsity pattern where $*$ denotes the non-zero entries. The connectivity properties of $\mathcal{K}_{\mathcal{S}}$ and $\mathcal{A}_{\mathcal{T}}$ will be studied under Euclidean topology. The notation $\text{diag}(a_1, \dots, a_n)$ denotes the n-by-n diagonal matrix with diagonal entries $a_1, \dots, a_n$. The notation $\mathbb{E}[X|Y]$ denotes the expectation of the random variable X conditioned on the random variable Y.

Geometrically, the set of stable matrices is an open non-convex cone with the origin removed. The sets $\mathcal{K}_{\mathcal{S}}$ and $\mathcal{A}_{\mathcal{T}}$ are obtained by slicing this open cone of stable matrices along an affine subspace and a linear subspace, respectively. The slicing affects the number of connected components for each of these sets and thereby reflects the tractability of the optimal decentralized control problem.

3. Connectivity Properties in Special Cases. In this section, we prove global geometric properties of the stabilizing set $\mathcal{K}_{\mathcal{S}}$ for certain choices of B, C and $\mathcal{S}$ using elementary arguments.

Recall that one can characterize the stability of matrices in different ways. Lyapunov's characterization [11, §4.1] states that a matrix M is stable if and only if there is a solution $P \succ 0$ to the equation $MP + PM^T + I = 0$. Another algebraic characterization is the Routh-Hurwitz criterion [4, §11.17], which states that a matrix is stable if the coefficients of its characteristic polynomial satisfy a set of polynomial inequalities. These techniques make possible a study of the set of stabilizing static controllers in the case with no structural constraints and full state measurements.

LEMMA 3.1. *Assume that $\mathcal{S} = \mathbb{R}^{m\times p}$ and $C = I$. The set $\mathcal{K}_{\mathcal{S}}$ is connected, but generally non-convex.*

Proof. Observe that $\mathcal{K}_\mathcal{S}$ is the continuous image of the set

$$\mathcal{H} = \{(R,P) : AP + BR + PA^T + R^T B^T = -I, P \succ 0\}$$

through the map $(R,P) \to RP^{-1}$. Moreover, $\mathcal{H}$ is connected since it is the intersection of a linear space and a convex cone. The map is well-defined as P is positive definite; it is also surjective from the Lyapunov's characterization: whenever $A+BK$ is stable, there is a matrix $P \succ 0$ such that $(A+BK)P + P(A+BK)^T = -I$ and the tuple (R,P) can be mapped to the desired K under the formula $KP = R$.

To show that $\mathcal{K}_\mathcal{S}$ is generally non-convex, consider the second-order system

$$A = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}, B = \begin{bmatrix} 0 & b_0 \\ 1 & b_1 \end{bmatrix}, K = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$$

where A and the first column of B are in the canonical form to ensure controllability. The closed-loop matrix is equal to

$$A + BK = \begin{bmatrix} b_0 k_{21} & 1 + b_0 k_{22} \\ -a_0 + k_{11} + b_1 k_{21} & -a_1 + k_{12} + b_1 k_{22} \end{bmatrix}.$$

To analyze the stability, we use the Routh-Hurwitz criterion and write

$$\mathcal{K}_\mathcal{S} = \{K : \text{trace}(A+BK) < 0, \det(A+BK) > 0\}.$$

Notice that $\mathcal{K}_\mathcal{S}$ is not convex in general since its intersection with the lower dimensional subspace $k_{21} = 0$ is given by

$$\left\{ K = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} : -a_1 + k_{12} + b_1 k_{22} < 0, (1 + b_0 k_{22})(-a_0 + k_{11}) < 0 \right\},$$

which turns out to be the union of two disjoint polyhedrons if $b_0 \neq 0$ (due to the product in the second condition). ∎

An implication of Lemma 3.1 is that the feasible set of the linear-quadratic optimal centralized control problem is connected, which justifies the success of the local search algorithm proven in [13] for centralized controllers. Another insightful, but impractical, scenario is the case with $B = C = I$ and a mostly arbitrary $\mathcal{S}$. This is studied below.

LEMMA 3.2. *Assume that $B = C = I$ and that $\mathcal{S}$ contains $-I$. Then, the set $\mathcal{K}_\mathcal{S}$ is connected.*

Proof. Since $\mathcal{S}$ is a linear subspace, we have $-\lambda I \in \mathcal{S}$ for every $\lambda \in \mathbb{R}$. Given two arbitrary matrices $K_1, K_2 \in \mathcal{K}_\mathcal{S}$, consider the following connected path from $A + K_1$ to $A + K_2$:

$$\begin{aligned} A + K_1 &\overset{\text{increase } \lambda}{\rightarrow} A + K_1 - \lambda I \\ &\overset{K_1 \to K_2}{\rightarrow} A + K_2 - \lambda I \\ &\overset{\text{decrease } \lambda}{\rightarrow} A + K_2, \end{aligned}$$

where

- $\lambda \geq 0$ is first increased to a large value;
- we move from $A + K_1 - \lambda I$ to $A + K_2 - \lambda I$ via an arbitrary continuous path between K_1 and K_2 in $\mathcal{S}$;

- λ is decreased eventually.

The parameter λ can be made so large that all matrices on the path from $A+K_1-\lambda I$ to $A+K_2-\lambda I$ could be regarded as a small (on the order of K_2-K_1) perturbation of the large matrix $A+K_1-\lambda I$. Such small perturbation preserves the stability condition of $A+K_1-\lambda I$. The proof is completed by noting that the designed path, which connects K_1 and K_2, involves only controllers in $\mathcal{S}$ and passes through only stabilizing matrices continuously. ∎

If the measurement matrix C is not the identity matrix, the set could become disconnected even in the simplest case $K=k\in\mathbb{R}$. This is demonstrated in the example below. To differentiate vectors from matrices, we rewrite B as b and C as c^T, where b and c are column vectors in $\mathbb{R}^n$.

Example 1. Assume that (A,b) is controllable and $c\neq 0$, where $A\in\mathbb{R}^{3\times 3}$. Then, the set $\mathcal{K}$ can have at most two connected components. To prove this statement, with no loss of generality we write the system in the controllable canonical form, i.e.,

$$A=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -a_0 & -a_1 & -a_2\end{bmatrix},\; b=\begin{bmatrix}0\\0\\1\end{bmatrix},\; c^T=\begin{bmatrix}c_0 & c_1 & c_2\end{bmatrix}.$$

The Routh-Hurwitz criterion characterizes stability with the set of inequalities

$$\begin{aligned}a_0-kc_0&>0,\\ a_1-kc_1&>0,\\ a_2-kc_2&>0,\\ (a_0-kc_0)&<(a_2-kc_2)(a_1-kc_1).\end{aligned}$$

Consider the quadratic function $f(k)=(a_2-kc_2)(a_1-kc_1)$, which can have at most two branches that lie above the line a_0-kc_0. The intersection of these branches with the interval defined by the first three linear inequalities leads to at most 2 connected components. An example with exactly two components can be produced by the parameters

$$(a_0,a_1,a_2)=(-5,-1,1),\quad (c_0,c_1,c_2)=(0.85,0.2,0.2).$$

Figure 1 verifies the above result by plotting the maximum real part of the closed-loop eigenvalues versus k.

It can be inferred from Example 1 that the coordinates of the set of stabilizing controllers are "one-sided". This is not surprising since when $A+BKC$ is stable, it holds that $\operatorname{trace}(A+BKC)<0$. We elaborate on this result in Lemma 3.3.

Lemma 3.3. *Consider the case $m=p=1$. Suppose that (A,b) is controllable and $c\neq 0$. Then, the scalar set $\mathcal{K}_{\mathcal{S}}$ cannot extend to infinity on both sides.*

Proof. As before, with no loss of generality consider the canonical form

$$A=\begin{bmatrix}0 & & I\\ -a_0 & \cdots & -a_{n-1}\end{bmatrix}, b=\begin{bmatrix}0\\1\end{bmatrix}, c^T=[c_0,\ldots,c_{n-1}].$$

The matrix $A+bkc^T$ has the characteristic polynomial

$$(a_0-c_0k)+(a_1-c_1k)x+\ldots,+(a_{n-1}-c_{n-1}k)x^{n-1}+x^n=0.$$

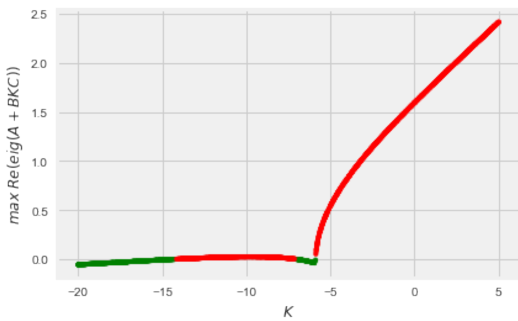


FIG. 1. *As discussed in Example 1, the set of stabilizing controllers can have two connected components for a third-order system. Observe that there are two intervals for k that produce eigenvalues in the left-half complex plane.*

It follows from the Routh-Hurwitz criterion that the coefficients of this polynomial must be positive. Since $c \neq 0$, there is some entry $c_{i_0} \neq 0$ and, as a result, k is prevented from extending to infinity on one side due to the inequality $a_{i_0} - c_{i_0} k > 0$.∎

In what follows, we will bound the number of connected components for scalar controllers.

THEOREM 3.4. *Consider the case $m = p = 1$. Suppose that (A, b) is controllable and $c \neq 0$. The scalar set $\mathcal{K}_{\mathcal{S}}$ can have at most $\lceil \frac{n}{2} \rceil$ connected components.*

Proof. If there is no stabilizing controller in $\mathcal{S}$, then $\mathcal{K}_{\mathcal{S}} = \emptyset$; otherwise one can first stabilize A with some controller k_0 and then analyze the set of shifted controllers $k - k_0$. As a result, without loss of generality one can assume that A is stable. We call a controller k *critical* when it is on the boundary of the set stabilizing controllers, implying the presence of a closed-loop eigenvalue on the imaginary axis. Consider the solution to the equation

$$\begin{aligned} 0 &= \det(\mathbf{j}wI - A - kbc^T) \\ &= \det(\mathbf{j}wI - A)\det(1 - kc^T(\mathbf{j}wI - A)^{-1}b) \qquad (3.1) \end{aligned}$$

(the symbol $\mathbf{j}$ denotes the imaginary unit). Since A is stable, the first term in the second line of (3.1) is not zero and therefore the second term must be zero. Taking its real and imaginary part yields

$$1 - k \times \mathrm{Re}\{c^T(\mathbf{j}wI - A)^{-1}b\} = 0, \qquad (3.2)$$

$$\mathrm{Im}\{c^T(\mathbf{j}wI - A)^{-1}b\} = 0. \qquad (3.3)$$

Equation (3.3) is of the form $\mathrm{Im}\left\{\frac{f(\mathbf{j}w)}{g(\mathbf{j}w)}\right\} = 0$ with $g(\mathbf{j}w) = \det(\mathbf{j}wI - A) \neq 0$; equivalently, one can write $\mathrm{Im}\{f(\mathbf{j}w)\overline{g(\mathbf{j}w)}\} = 0$ where $f(\mathbf{j}w)$ is a polynomial of degree at most $n-1$, $g(\mathbf{j}w) = \det(\mathbf{j}wI - A)$ is a polynomial of degree n, and overline denotes the complex conjugate. $\mathrm{Im}\{f(\mathbf{j}w)\overline{g(\mathbf{j}w)}\}$ is a polynomial of degree $2n-1$ in w with only odd degree terms; it can have at most $2n-1$ real roots that are symmetric around 0. Because $\mathrm{Re}\{f(\mathbf{j}w)\overline{g(\mathbf{j}w)}\}$ has only even degree terms, at most n distinct pairs of the symmetric roots of (3.3) can be plugged into (3.2). This leads to at most n critical values for the scalar k and divides the real line into at most $n+1$ intervals of interlacing stable-unstable controller regions. At most $\lceil \frac{n+1}{2} \rceil$ of them are stable.

Note that when $n+1$ is odd, Lemma 3.3 rules out one interval that extends to infinity. As a result, the upper bound can be sharpened to $\lfloor \frac{n+1}{2} \rfloor = \lceil \frac{n}{2} \rceil$. ∎

Theorem 3.4 states that the number of connected components would grow with the dimension of the system even in the special case $m = p = 1$. Our bound is *tight* when $n = 3$ in light of Example 1.

4. Exponential Subclass.

One of the main results of this paper is stated below.

THEOREM 4.1. *There is no polynomial function with respect to the order of the system that can serve as an upper-bound on the number of connected components of the set of decentralized stabilizing controllers.*

To prove the theorem, it suffices to show the existence of a subclass of decentralized control problems whose set of stabilizing controllers has an exponential number of connected components. Our proof requires a lemma that characterizes the stability of tri-diagonal matrices whose diagonal elements are mostly purely imaginary complex numbers. Define the inertia $\mathrm{In}(G)$ of an $n \times n$ matrix G as the triplet $\mathrm{In}(G) = (\pi(G), \nu(G), \delta(G))$, where $\pi(G)$, $\nu(G)$ and $\delta(G)$ count the eigenvalues of G with positive, negative and zero real parts, respectively.

LEMMA 4.2 (From [30]). *Consider the tri-diagonal matrix*

$$G = \begin{bmatrix} f_1 + \mathbf{j}g_1 & f_2 & 0 & \cdots & \cdots & 0 \\ -h_2 & \mathbf{j}g_2 & f_3 & \ddots & & \vdots \\ 0 & -h_3 & \mathbf{j}g_3 & f_4 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & -h_{n-1} & \mathbf{j}g_{n-1} & f_n \\ 0 & \cdots & \cdots & 0 & -h_n & \mathbf{j}g_n \end{bmatrix},$$

where f_i, g_i and h_i are real for $i = 1, ..., n$, $f_1 \neq 0$, and $f_i h_i \neq 0$ for $i = 2, \ldots, n$. Then,

$$\mathit{In}(G) = \mathit{In}(D),$$

where

$$D = \mathrm{diag}(f_1, f_1 f_2 h_2, f_1 f_2 f_3 h_2 h_3, \ldots, f_1 \cdots f_n h_2 \cdots h_n).$$

A corollary of Lemma 4.2 for the stability of real tri-diagonal matrices is given below.

COROLLARY 4.3. *Given the tri-diagonal real matrix A of the form*

$$A = \begin{bmatrix} f_1 & f_2 & 0 & \cdots & \cdots & 0 \\ -h_2 & 0 & f_3 & 0 & & \vdots \\ 0 & -h_3 & 0 & f_4 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & -h_{n-1} & 0 & f_n \\ 0 & \cdots & \cdots & 0 & -h_n & 0 \end{bmatrix}, \tag{4.1}$$

it holds that

- *If* $f_1 < 0$ *and* $f_i h_i > 0$ *for all* $i \in \{2, \ldots, n\}$, *then* A *is stable.*
- *If* $f_i h_i < 0$ *for some index* $i \in \{2, \ldots, n\}$, *then* A *is unstable.*

Remark 4.4. Sparse stable matrices theory [3] states that the graph associated with the sparsity pattern of the matrix in (4.1) is a chain and has nested Hamiltonian sub-graphs. The graph is sufficient to sustain stable dynamics. Moreover, the sparse matrix space is minimally stable because: (i) if f_1 is set to zero, then the trace of the matrix becomes zero and therefore at least one eigenvalue should be unstable, (ii) if any non-diagonal element is set to zero, then the matrix decomposes into a block triangular form where the lower diagonal block has a zero trace, leading to instability.

Due to Remark 4.4, Corollary 4.3 gives necessary and sufficient conditions for the stability of a class of matrices, which can be used to analyze both connected components and separating hyper-surfaces. In what follows, we will first show the possibility of 2^{n-1} connected components in the case with a non-identify C and then develop a similar result for $C = I$.

THEOREM 4.5. *Let* $A \in \mathbb{R}^{n\times n}$ *be in the form of* (4.1), *and set* $B \in \mathbb{R}^{n\times(2n-2)}$, $C \in \mathbb{R}^{(2n-2)\times n}$ *and* $K \in \mathbb{R}^{(2n-2)\times(2n-2)}$ *to*

$$B = \left[\begin{array}{cccc|cccc} 0 & \cdots & \cdots & 0 & +1 & 0 & \cdots & 0 \\ -1 & \ddots & & \vdots & 0 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & \vdots & & \ddots & +1 \\ 0 & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 \end{array}\right],$$

$$C = \left[\begin{array}{ccccc} 1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{array}\right],$$

$$K = \mathrm{diag}(k_2, \ldots, k_n, k_2, \ldots, k_n).$$

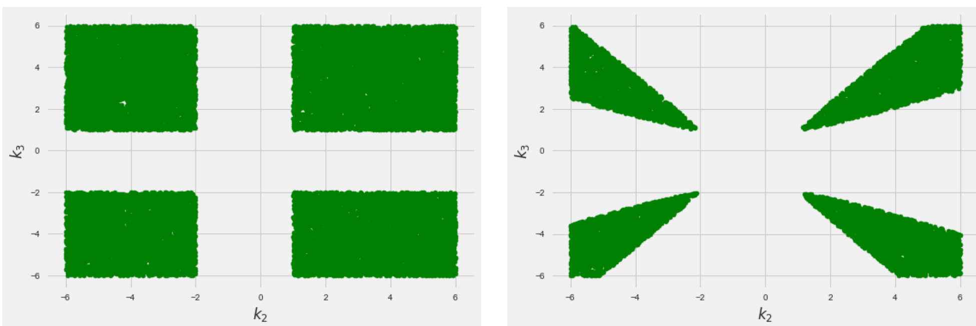
Suppose that $f_1 < 0$ *and* $f_i \neq h_i$ *for* $i = 2, \ldots, n$. *Then, the set* $\mathcal{K}$ *has at least* 2^{n-1} *connected components.*

Proof. The closed-loop matrix $A + BKC$ can be expressed as

$$\begin{bmatrix} f_1 & f_2 + k_2 & 0 & \cdots & \cdots & 0 \\ -h_2 - k_2 & 0 & f_3 + k_3 & \ddots & & \vdots \\ 0 & -h_3 - k_3 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 0 & f_n + k_n \\ 0 & \cdots & \cdots & 0 & -h_n - k_n & 0 \end{bmatrix}.$$

It results from Corollary 4.3 and Remark 4.4 that the closed-loop stability is equivalent to the conditions $(h_i + k_i)(f_i + k_i) > 0$ for $i = 2, \dots, n$. Equivalently, either $k_i < \min(-h_i, -f_i)$ or $k_i > \max(-h_i, -f_i)$ holds for $i = 2, \dots, n$. Therefore, the region of stabilizing K, parametrized in $(k_2, \dots, k_n) \in \mathbb{R}^{n-1}$, is separated by $n-1$ hyperplanes $k_i = -(f_i + h_i)/2$ for $i = 2, \dots, n$. Since there are stable regions on both sides of each of those hyperplanes, the overall number of connected components becomes at least 2^{n-1}. ∎

The result of Theorem 4.5 is demonstrated in the left plot of Figure 2 for $n = 3$. Note that the "one-sided" result of Lemma 3.3 does not hold here since K is not a scalar.



(a) $\epsilon = 0$ (b) $\epsilon = 0.2$

FIG. 2. *We randomly sample K and check the closed-loop stability for an instance of the system in Theorem 4.5. The controller is parametrized in terms of (k_2, k_3) where $n = 3$, with $f_i = -1$ and $h_i = 2$ for $i = 1, 2, 3$. The projection of the set $\mathcal{K}$ onto the 2-dimensional space corresponding to (k_2, k_3) is shown in green. The left figure shows that there are $2^{n-1} = 4$ connected components, where each coordinate takes values in $(-\infty, -2)$ or $(1, \infty)$ to be stable. The right figure shows the connected components when the number 0.2 is added to each diagonal entry of A.*

Remark 4.6. Note that eigenvalues are continuous functions of the entries of a matrix and that the connected components studied in the proof of Theorem 4.5 are separated by a positive margin. Therefore, one may speculate that a small perturbation of A will not change the number of connected components. This is not the case in general since the eigenvalues of $A + BKC$ can become arbitrarily close to the imaginary axis when $\|K\|$ is large, as illustrated in Figure 3. However, one part of every connected component is resistant to perturbations. For example, with $\epsilon > 0$, the set $\{K : (A + \epsilon I) + BKC \text{ stable } \}$ is a subset of $\{K : A + BKC \text{ stable } \}$, the former contains only those controllers that make the closed-loop eigenvalues at least ϵ away from the imaginary axis. The number ϵ can be set so small that at least one point from each component remains stable. In other words, a new matrix A obtained by adding ϵ to the diagonal of the matrix in (4.1) gives rise of an exponential number of connected components where the number cannot change with a very small perturbation of its elements. This is illustrated in the right plot of Figure 2.

The subclass of problems studied in Theorem 4.5 may be unsatisfactory as it requires that the free elements of K repeat themselves and that $C \neq I$. The next theorem addresses these issues.

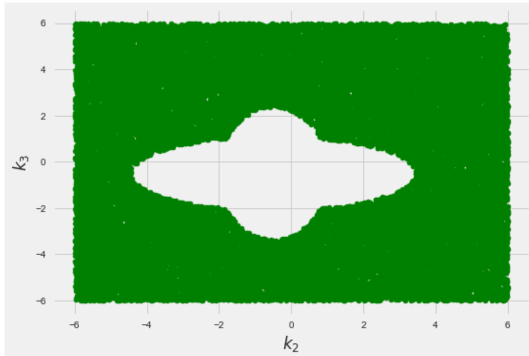


FIG. 3. *If the diagonal of A are reduced by 0.2, then the set $\mathcal{K}$ becomes connected. The projection of the set $\mathcal{K}$ onto the 2-dimensional space corresponding to (k_2, k_3) is shown in green.*

THEOREM 4.7. *Let A be in the form*

$$A = \begin{bmatrix} f_1+\epsilon & f_2 & 0 & \cdots & \cdots & 0 \\ -h_2 & \epsilon & f_3 & \ddots & & \vdots \\ 0 & -h_3 & \epsilon & f_4 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & -h_{n-1} & \epsilon & f_n \\ 0 & \cdots & \cdots & 0 & -h_n & \epsilon \end{bmatrix}, \tag{4.2}$$

where $\epsilon > 0$, $f_1 < 0$, and $(-1)^i(f_i - h_{i+1}) > 0$ for $i = 2, \ldots, n$. Consider $B \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{n\times n}$ and $K \in \mathbb{R}^{n\times n}$ to be

$$B = \begin{bmatrix} 0 & 1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & 0 & 1 \\ & & -1 & 0 \end{bmatrix}, \quad C = I,$$

$$K = \operatorname{diag}(k_1, k_2, \ldots, k_n).$$

For a small enough ϵ, the set $\mathcal{K}$ has at least F_n connected components, where $F_0 = 1, F_1 = 1, F_{i+2} = F_{i+1} + F_i$ for $i = 0, 1, \ldots$ is the Fibonacci sequence, which is on the order of $\left(\frac{1+\sqrt{5}}{2}\right)^n$.

Proof. First, assume that $\epsilon = 0$ and consider the closed-loop matrix $A + BKC$:

$$\begin{bmatrix} f_1 & f_2+k_2 & 0 & \cdots & \cdots & 0 \\ -h_2-k_1 & 0 & f_3+k_3 & \ddots & & \vdots \\ 0 & -h_3-k_2 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 0 & f_n+k_n \\ 0 & \cdots & \cdots & 0 & -h_n-k_{n-1} & 0 \end{bmatrix}.$$

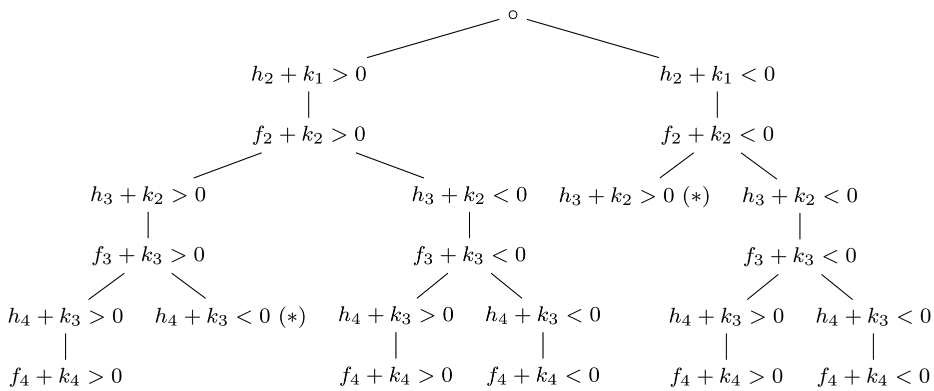


FIG. 4. *This tree shows the enumerating signs of the closed-loop matrix entries for $n = 4$. The branch marked with (∗) has contradictory inequalities.*

In light of Corollary 4.3 and Remark 4.4, the necessary and sufficient conditions for the closed-loop stability are $(h_i + k_{i-1})(f_i + k_i) > 0$ for $i = 2, ..., n$. As a result, if $h_2 + k_1 > 0$, then $f_2 + k_2 > 0$. Now, because $h_3 < f_2$, the term $h_3 + k_2$ can be positive or negative. If it is positive, then $f_3 + k_3$ must be positive, and we can move on to study the sign of $h_4 + k_3$. As we proceed, note that not all sign assignments for $h_i + k_{i-1}$ and $f_i + k_i$ are possible due to the assumptions on f_i and h_i. The enumeration procedure is illustrated in Figure 4. Any path from the root to the bottom level leaf passes through a set of linear inequalities that together enclose an open polyhedron of stable regions. These stable regions are separated by the hyperplanes $h_{i+1} + k_i = 0$ for $i = 1, 2, \ldots, n-1$ and $f_i + k_i = 0$ for $i = 2, 3, \ldots, n$.

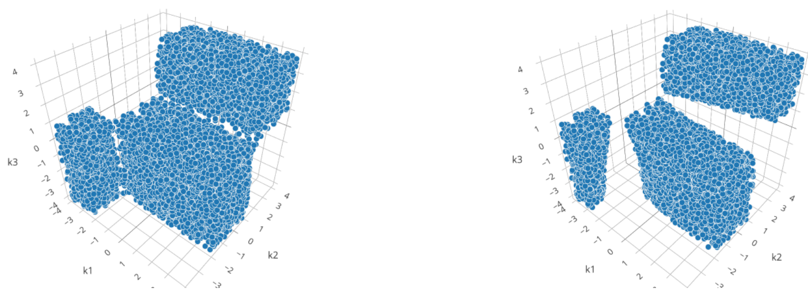
Next, we count the number of branches. If $h_i + k_{i-1} > 0$ (or equivalently $f_i + k_i > 0$) appears m_i times and $h_i + k_{i-1} < 0$ (or equivalently $f_i + k_i < 0$) appears n_i times, assuming $m_i \geq n_i$, the next level will have at most $(m_i + n_i) + \max(m_i, n_i) = 2m_i + n_i$ branches. This number is achievable if $f_i < h_{i+1}$, which means keeping all the children of the inequalities $f_i + k_i > 0$ and pruning one child from each inequality $f_i + k_i < 0$. Then, $m_{i+1} = m_i$, $n_{i+1} = m_i + n_i$, and $n_{i+1} \geq m_{i+1}$, which reverses the order of m_i and n_i. It can be verified that the total number of connected regions $m_i + n_i$ satisfies the iteration of the Fibonacci sequence.

The connected regions are separated by the hyperplanes $k_i = -f_i$ or $k_i = -h_{i+1}$ with no margin. However, when $\epsilon > 0$, the connected components are separated. More precisely, whenever $k_i = -f_i$ or $k_i = -h_{i+1}$, the matrix $A + BKC$ decomposes into a block triangular form where the lower diagonal block has a positive trace, which means that the matrix cannot be stable. When ϵ is small enough, the original connected regions described by linear inequalities do not shrink abruptly — in fact, at least one point from every polyhedron remains stable. As a result, these stable regions are the true connected components of the stabilizing controller set. ∎

To illustrate Theorem 4.7, consider the matrix

$$A = \begin{bmatrix} -1+\epsilon & 2 & 0 & & & & \\ -2 & \epsilon & 1 & 0 & & & \\ 0 & -1 & \epsilon & 2 & 0 & & \\ & 0 & -2 & \epsilon & 1 & 0 & \\ & & 0 & -1 & \epsilon & 2 & 0 \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}. \tag{4.3}$$

The corresponding set $\mathcal{K}$ obtained by sampling random matrices K and checking the closed-loop stability is provided in Figure 5 for $n = 3$.



(a) $\epsilon = 0$ (b) $\epsilon = 0.02$

FIG. 5. *We randomly sample K and check the closed-loop stability for an instance of the system in Theorem 4.7 with $n = 3$, the matrix A given in (4.3), and $K = \text{diag}(k_1, k_2, k_3)$. The projection of the set $\mathcal{K}$ onto the 3-dimensional space corresponding to (k_1, k_2, k_3) is shown in blue.*

Our exponential examples are based on specific settings of the parameters f_i and h_i in the matrix A that maximize the number of connected components. We next show that even if the parameters f_i and h_i are considered random, the expected number of connected components is still exponential.

THEOREM 4.8. *Consider the matrices A, B, C, and K defined in Theorem 4.7, and let f_i and h_j be independent random variables whose distribution are standard normal for $i = 1, \ldots, n$ and $j = 2, \ldots, n$. If ϵ is small enough, the expected number of connected component of $\mathcal{K}_\mathcal{S}$ is at least $\left(\frac{3}{2}\right)^{n-2}$.*

Proof. With the assumed distribution, $f_i < h_{i+1}$ and $f_i > h_{i+1}$ occur equally likely, while $f_i = h_{i+1}$ happens with zero probability. Our enumeration tree is random, and we count the number of leaves as follows. If $f_i + k_i > 0$ appears m_i times and $f_i + k_i < 0$ appears n_i times for $i \geq 2$, the next level has two possibilities:

(i) $f_i < h_{i+1}$, which keeps all the children of the inequalities $f_i + k_i > 0$ and prunes one child from each inequality $f_i + k_i < 0$. Therefore, $m_{i+1} = m_i$ and $n_{i+1} = m_i + n_i$.

(ii) $f_i > h_{i+1}$, which keeps all the children of the inequalities $f_i + k_i < 0$ and prunes one child from each inequality $f_i + k_i > 0$. Therefore, $m_{i+1} = m_i + n_i$ and $n_{i+1} = n_i$.

Combining the two cases, we can calculate the expected number of children $m_{i+1} +$

n_{i+1} conditioned on m_i and n_i in the previous level:

$$\begin{aligned}\mathbb{E}[m_{i+1}+n_{i+1}|m_i,n_i] &= \mathbb{E}[m_{i+1}+n_{i+1}|m_i,n_i,f_{i+1}<h_{i+2}]\mathbb{P}(f_{i+1}<h_{i+2})\\ &\quad+\mathbb{E}[m_{i+1}+n_{i+1}|m_i,n_i,f_{i+1}>h_{i+2}]\mathbb{P}(f_{i+1}>h_{i+2})\\ &= (2m_i+n_i)\frac{1}{2}+(2n_i+m_i)\frac{1}{2}=\frac{3}{2}(m_i+n_i).\end{aligned}$$

With the initial conditions $\mathbb{E}[m_2+n_2|f_1>0]=0$ and $\mathbb{E}[m_2+n_2|f_1<0]=2$, we have $\mathbb{E}[m_2+n_2]=1$. Using induction, it can be concluded that $\mathbb{E}[m_n+n_n]=\left(\frac{3}{2}\right)^{n-2}$. ∎

By adopting a randomized setting, we are able to analyze the change of connected components when one element k_{i_0} is fixed to zero for some index $i_0\in\{1,2,\ldots,n-1\}$. The proof is based on a careful counting of branches and is provided in the Appendix.

PROPOSITION 4.9. *With the same setting as in Theorem 4.8, assume that $K=\mathrm{diag}(k_1,\ldots,k_n)$ and k_{i_0} is fixed to zero for some index $i_0\in\{1,\ldots,n\}$. Then, the expected number of connected components of $\mathcal{K}_\mathcal{S}$ for a small enough ϵ is at least*

$$\begin{cases}\frac{1}{6}(\frac{3}{2})^{n-2}, & \text{if } 2\le i_0\le n-1.\\ \frac{1}{2}(\frac{3}{2})^{n-2}, & \text{if } i_0=1 \text{ or } i_0=n.\end{cases}$$

All previous results suggest that the diagonal entries of A being positive contribute to the complexity of the feasible set $\mathcal{K}$. Theorem 4.10 below shows that the diagonal of A being negative is a desirable structure in the sense that if A is highly dampened, the feasible set is connected independent of control structures.

THEOREM 4.10. *Given arbitrary matrices A, B and C of compatible dimensions and a linear subspace of matrices $\mathcal{S}$, the set*

$$\mathcal{K}_{\mathcal{S},\lambda}=\{K: A-\lambda I+BKC \text{ is stable },K\in\mathcal{S}\}$$

is connected when $\lambda>0$ is large enough.

Proof. Consider a number μ and let λ be a parameter that increases from μ toward ∞. Since $\lambda\ge\mu$, we have $\mathcal{K}_{\mathcal{S},\lambda}\supseteq\mathcal{K}_{\mathcal{S},\mu}$, and therefore $\mathcal{K}_{\mathcal{S},\lambda}$ contains all components of K_μ but could possibly connect them or add new components. The addition of new components with the increase of λ could occur only a finite number of times. Because the Routh-Hurwitz criterion describes $\mathcal{K}_{\mathcal{S},\lambda}$ by polynomial inequalities in the entries of $A-\lambda I+BKC$, the set $\mathcal{K}_{\mathcal{S},\lambda}$ is semi-algebraic with a finite number of connected components given the order of the system [7]. To connect all those components, we first increase λ until no new connected component appears, then select a controller from each connected component, and cover all those controllers with a ball $\mathcal{B}\subseteq\mathcal{S}$. By making λ so large that all controllers in $\mathcal{B}$ become stable, we glue all of the connected components. ∎

The interpretation of the result of Theorem 4.10 is that if the open-loop matrix of the system can be written as $A-\lambda I$ for a large λ, then the feasible set of ODC is connected. This corresponds to highly damped systems.

Remark 4.11. It is noted in [18] that if we consider the discounted cost

$$\int_0^\infty e^{-2\lambda t}(x^TQx+2u^TDx+u^TRu)dt,$$

or equivalently make a change of variables $\hat{x}(t)=e^{-\lambda t}x(t)$ and $\hat{u}(t)=e^{-\lambda t}u(t)$, then the closed-loop dynamics become equal to $\dot{\hat{x}}(t)=(A-\lambda I+BKC)\hat{x}(t)$. Therefore, it follows from Theorem 4.10 that the feasible set of the ODC problem is connected for discounted costs with a large forgetting factor.

Remark 4.12. It is known in the context of inverse optimal control [18] that any static state-feedback gain K is the unique minimizer of some quadratic performance measure (2.1) for all initial states. One such measure is

$$\int_0^\infty (u(t) - Kx(t))^T R\,(u(t) - Kx(t))\,dt.$$

where R is a positive definite matrix. As a result, every point in any connected component is an optimal solution to some ODC problem. Since there is an exponential number of connected components in certain cases, random initialization is unlikely to successfully locate the optimal component unless prior information is available or the system is favorably structured. Local search algorithms, therefore, fail for general ODC problems.

5. Stable Matrices with Block Patterns. In this section, we analyze the connectivity of the set of sparse stable matrices $\mathcal{A}_\mathcal{T}$, defined in (2.3). It follows from Lemma 3.2 that only in matrices with constrained diagonal entries do nontrivial connectivity properties emerge, and we study sparse stable matrices with zero blocks in the diagonal.

5.1. Two-by-two block. Below is the main theorem.

THEOREM 5.1. *Consider the matrix subspace*

$$\mathcal{T} = \left\{ \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & 0_{(n-r)\times(n-r)} \end{bmatrix} \middle| A_{21} \in \mathcal{Z} \right\},$$

where $\mathcal{Z}$ is any subspace of matrices in $\mathbb{R}^{(n-r)\times r}$. Then, the sets $\mathcal{A}_\mathcal{T}$ and

$$\{A_{21} : A_{21} \text{ has full row rank}, A_{21} \in \mathcal{Z}\}$$

have the same number of connected components.

Proof. For clarity the proof is first stated without the constraint $A_{21} \in \mathcal{Z}$; this incurs no loss of generality. A is stable if and only if there is a matrix $P = \left[\begin{smallmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{smallmatrix}\right] \succ 0$ partitioned accordingly that satisfies the Lyapunov equation

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & 0 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} \begin{bmatrix} A_{11}^T & A_{21}^T \\ A_{12}^T & 0 \end{bmatrix} = \begin{bmatrix} -I & 0 \\ 0 & -I \end{bmatrix}. \tag{5.1}$$

Note that P is unique and depends continuously on A whenever A is stable [11, §4.1]. We solve the partitioned equation

$$A_{11}P_{11} + A_{12}P_{12}^T + P_{11}A_{11}^T + P_{12}A_{12}^T = -I \tag{5.2}$$

$$A_{11}P_{12} + A_{12}P_{22} + P_{11}A_{21}^T = 0 \tag{5.3}$$

$$A_{21}P_{12} + P_{12}^T A_{21}^T = -I. \tag{5.4}$$

Since $P_{22} \succ 0$, (5.3) uniquely determines the unconstrained block

$$A_{12} = -(A_{11}P_{12} + P_{11}A_{21}^T)P_{22}^{-1}.$$

Substituting it back to (5.2) yields

$$A_{11}P_{11}+P_{11}A_{11}^T-(A_{11}P_{12}+P_{11}A_{21}^T)P_{22}^{-1}P_{12}^T-P_{12}P_{22}^{-T}(A_{21}P_{11}+P_{12}^TA_{11}^T)=-I,$$

or equivalently

$$\begin{aligned}(5.5)\quad A_{11}(P_{11}-P_{12}P_{22}^{-1}P_{12}^T)+(P_{11}-P_{12}P_{22}^{-1}P_{12}^T)A_{11}^T=&\\ -I+P_{11}A_{21}^TP_{22}^{-1}P_{12}^T+P_{12}P_{22}^{-T}A_{21}P_{11}.&\end{aligned}$$

The equation above can be simplified using the Schur complement $\tilde{P}_{11}=P_{11}-P_{12}P_{22}^{-1}P_{12}^T$, which is an arbitrary positive definite matrix. One can write

$$\begin{aligned}A_{11}\tilde{P}_{11}+\tilde{P}_{11}A_{11}^T=-I+\tilde{P}_{11}A_{21}^TP_{22}^{-1}P_{12}^T+P_{12}P_{22}^{-T}A_{21}\tilde{P}_{11}+P_{12}P_{22}^{-1}P_{12}^TA_{21}^TP_{22}^{-1}P_{12}^T&\\ +P_{12}P_{22}^{-T}A_{21}P_{12}P_{22}^{-1}P_{12}^T.&\end{aligned}$$

In light of (5.4), this is equivalent to

$$(5.6)\qquad A_{11}\tilde{P}_{11}+\tilde{P}_{11}A_{11}^T=-I+\tilde{P}_{11}A_{21}^TP_{22}^{-1}P_{12}^T+P_{12}P_{22}^{-T}A_{21}\tilde{P}_{11}-P_{12}P_{22}^{-2}P_{12}^T.$$

Given A_{21}, P_{12}, $\tilde{P}_{11}\succ 0$, and $P_{22}\succ 0$, the eigenvalues of $\tilde{P}_{11}$ do not sum to zero. Therefore, (5.6) can be regarded as a Lyapunov equation where the unknown block A_{11} has a unique symmetric solution $A_{11}=A_{11}^T$; all other solutions A_{11} lie in a linear subspace that contains this symmetric solution. The symmetric solution, moreover, depends continuously on $\tilde{P}_{11}$ as long as $\tilde{P}_{11}$ remains in the positive semi-definite cone, which is connected. As a result, not only are all A_{11} connected to a symmetric A_{11}, all symmetric A_{11} given $\tilde{P}_{11}$ are connected to the symmetric solution A_{11} given $\tilde{P}_{11}=I$, which we denote by $\phi(A_{12},P_{12},P_{22})$:

$$\phi(A_{12},P_{12},P_{22})=\frac{1}{2}\left(-I+A_{21}^TP_{22}^{-1}P_{12}^T+P_{12}P_{22}^{-T}A_{21}-P_{12}P_{22}^{-2}P_{12}^T\right).$$

The above argument retracts the solutions of (5.2)-(5.4) while maintaining the topological property of connectivity. Using $\sim$ to denote the equivalence of connected components, we state the retraction procedure

$$\begin{aligned}
(5.7)\qquad \mathcal{A}_\mathcal{T}&\sim\left\{\left(\begin{bmatrix}A_{11}&A_{12}\\A_{21}&0\end{bmatrix},\begin{bmatrix}P_{11}&P_{12}\\P_{12}^T&P_{22}\end{bmatrix}\right):(5.1),\begin{bmatrix}P_{11}&P_{12}\\P_{12}^T&P_{22}\end{bmatrix}\succ 0\right\}\\
(5.8)\qquad &\sim\{(A_{11},A_{21},P_{11},P_{12},P_{22}):(5.4),(5.5),P_{11}\succ P_{12}P_{22}^{-1}P_{12}^T,P_{22}\succ 0\}\\
(5.9)\qquad &\sim\{\left(A_{11},A_{21},\tilde{P}_{11},P_{12},P_{22}\right):(5.4),(5.6),\tilde{P}_{11}\succ 0,P_{22}\succ 0\}\\
(5.10)\qquad &\sim\{(A_{11},A_{21},P_{12},P_{22}):(5.4),A_{11}=\phi(A_{12},P_{12},P_{22}),P_{22}\succ 0\}\\
(5.11)\qquad &\sim\{(A_{21},P_{12},P_{22}):(5.4),P_{22}\succ 0\}\\
(5.12)\qquad &\sim\{(A_{21},P_{12}):(5.4)\}.
\end{aligned}$$

The first equivalence (5.7) follows from the fact that for any stable matrix A, the formula

$$P=\int_0^\infty e^{A\tau}e^{A^T\tau}d\tau,$$

gives the unique solution to the Lyapunov equation and the solution depends continuously on the matrix A. (5.8) follows from the unique solution of A_{12} and the characterization of partitioned positive definite matrices with Schur complements:

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} \succ 0 \iff P_{11} \succ P_{12} P_{22}^{-1} P_{12}^T \text{ and } P_{22} \succ 0.$$

(5.9) follows from the simplification of Lyapunov equation, and the one-one correspondence between $\tilde{P}_{11}$ and P_{11} given (P_{12}, P_{22}). (5.10) follows from the retraction of the solutions to (5.6); (5.11) follows from the continuity of function ϕ, and finally (5.12) throws away the free variable P_{22} because it does not appear in the relationship between A_{21} and P_{12}.

(5.12) can be further simplified. We first show that (5.4) has a solution if and only if A_{21} has full rank. If there is a vector $x \in \mathbb{R}^s$ such that $x^T A_{21} = 0$, pre-multiply and post-multiply (5.4) by x yields

$$0 = x^T (A_{21} P_{12} + P_{12}^T A_{21}^T) x = -x^T x,$$

or equivalently, $x = 0$. Therefore, A_{21} has full row rank and similarly, P_{12} has full column rank. On the other hand, given any full row rank matrix A_{21}, (5.4) has a full rank solution $P_{12} = -1/2 A_{21}^+$, where A_{21}^+ is the Moore-Penrose inverse. This completes the proof for the first equivalence in

$$\begin{aligned} \{(A_{21}, P_{12}) : (5.4)\} &\sim \{(A_{21}, P_{12}) : (5.4), A_{21} \text{ has full row rank}\} \\ &\sim \{\left(A_{21}, -1/2 A_{21}^+\right) : A_{21} \text{ has full row rank}\} \\ &\sim \{A_{21} : A_{21} \text{ has full row rank}\}. \end{aligned}$$

The second equivalence follows from the fact that, given A_{21} has full row rank, a solution $P_{12} = -1/2 A_{21}^+$ to (5.4) always exists and all solutions lie in a subspace that can be retracted to that solution. The final equivalence comes from dropping the redundant second coordinate, since the Moore-Penrose inverse is continuous over full rank matrices.

The above proof imposes no restriction on A_{21}; it holds even if A_{21} is restricted to a subspace $\mathcal{Z}$. ∎

In the special case where $\mathcal{Z}$ is the whole space and A_{21} has more columns than rows, the set is connected.

COROLLARY 5.2. *Assume that $\mathcal{Z} = \mathbb{R}^{(n-r) \times r}$, where $2r > n$. Then, the set $\mathcal{A}_{\mathcal{T}}$ is connected.*

Proof. From Theorem 5.1, if suffices to show the connectivity of

$$\left\{ A_{21} \in \mathbb{R}^{(n-r) \times r} : A_{21} \text{ has full row rank} \right\}.$$

This set is the image of the continuous map $(U, D, V) \to UDV$ from the connected set $\mathcal{U} \times \mathcal{D} \times \mathcal{V}$, where

$$\begin{aligned} \mathcal{U} &= \left\{ U \in \mathbb{R}^{(n-r) \times (n-r)} : U \text{ is a orthogonal matrix with determinant } 1 \right\} \\ \mathcal{D} &= \left\{ D \in \mathbb{R}^{(n-r) \times r} : D_{ii} > 0 \text{ for } i = 1, \ldots, r \text{ and all other entries are } 0 \right\} \\ \mathcal{V} &= \left\{ V \in \mathbb{R}^{r \times r} : V \text{ is a orthogonal matrix with determinant } 1 \right\} \end{aligned}$$

$\mathcal{U}$ and $\mathcal{V}$ are connected because the set of orthogonal matrices with positive determinant is connected. The map is surjective, because every full rank matrix A_{21} has a singular value decomposition $A_{21} = UDV$, where $D_{ii} > 0$ for $i = 1, \ldots, r$. If $\det(U) = -1$, we can flip the sign of the first column of U and the first row of V to ensure that $\det(U) = 1$ while preserving the product. If $\det(V) = -1$, we can flip the sign of the last row of V, and since $n - r < r$, the last row does not affect the product UDV. ∎

Corollary 5.3. *Suppose* $2r \geq n$ *and* $\mathcal{Z} = \{A_{21} \in \mathbb{R}^{(n-r)\times r} : A_{ij} = 0 \text{ for } j \neq i\}$. *Then, the set* $\mathcal{A}_{\mathcal{T}}$ *has* 2^{n-r} *connected components.*

Proof. We invoke Theorem 5.1. For a diagonal matrix to have full rank, all its diagonal entries must be nonzero, and therefore, every diagonal entry of A_{21} can be either positive or negative. Those $(n - r)$ diagonal entries give rise to 2^{n-r} connected components. ∎

5.2. More Complicated Block Patterns. We generalize the results in the previous section to the case where the space of matrices $\mathcal{T}$ has a block structure as in

$$\mathcal{T} = \left\{ \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & 0_{r\times r} & 0_{r\times(n-2r)} \\ 0_{(n-2r)\times r} & A_{32} & 0_{(n-2r)\times(n-2r)} \end{bmatrix} \middle| A_{21} \in \mathcal{Z}_1, A_{32} \in \mathcal{Z}_2 \right\}, \tag{5.13}$$

where $\mathcal{Z}_1 \subseteq \mathbb{R}^{r\times r}$ and $\mathcal{Z}_2 \subseteq \mathbb{R}^{(n-2r)\times r}$ are arbitrary subsets of matrices.

Theorem 5.4. *The set* $\mathcal{A}_{\mathcal{T}}$ *with* $\mathcal{T}$ *defined in* (5.13) *has the same number of connected components as the set*

$$\{(A_{21}, A_{32}) : A_{21} \in \mathcal{Z}_1, A_{32} \in \mathcal{Z}_2, A_{21} \text{ and } A_{32} \text{ have full row rank}\}.$$

We provide the proof in the Appendix. The result of Theorem 5.4 is verified for $n = 3$ in Figure 6, where 4 connected components are found. In order to strictly separate the components, we plot the samples of sparse stable matrices whose eigenvalues are away from the imaginary axis by a fixed margin.

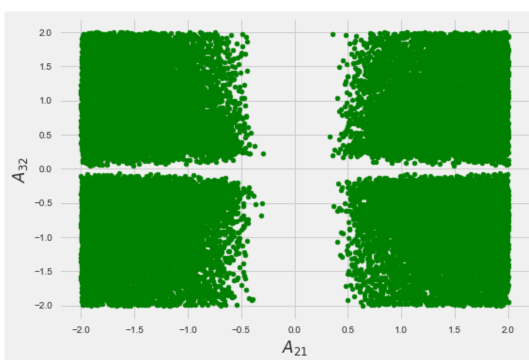


Fig. 6. *Verifying the result of Theorem 5.4 in the case* $n = 3$ *and* $r = 1$, *we plot the projection of* A *onto* (A_{21}, A_{32}). *The entries of the matrix* A *are sampled uniformly over* $[-2, 2]$. *The green points marked those matrix* A *such that* $0.2I + A$ *is stable.*

Remark 5.5. The result of Theorem 5.4 can be generalized to n-by-n block matrices if the blocks are square and the first row and the lower diagonal blocks of A are nonzero. The square block assumption on the sub-diagonals of A ensures that, for any full rank sub-diagonals, the first row of A and the upper-triangular entries of P can always be solved from the Lyapunov equation. Specially, in case of scalar blocks, the set of stable matrices with the following pattern has 2^{n-1} connected components:

$$\begin{bmatrix} * & * & \cdots & \cdots & * \\ * & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & * & 0 \end{bmatrix}$$

This relaxes the condition $2r \leq n$ of Corollary 5.3.

The sparsity pattern discussed in Remark 5.5 seems to suggest that the sparsity of the matrix space directly contributes to the number of connected components. The connection between sparsity and connectivity is complicated in that the number of connected components may remain exponential even when half of the matrix entries are free (such matrices are often regarded as dense).

THEOREM 5.6. *The set $\mathcal{A}_{\mathcal{T}}$ has 2^{n-1} connected components, where $\mathcal{T}$ is the subset of matrices with the sparsity pattern:*

$$\begin{bmatrix} * & * & * & \cdots & \cdots & * \\ * & 0 & * & \cdots & \cdots & * \\ 0 & * & 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & * \\ 0 & \cdots & \cdots & 0 & * & 0 \end{bmatrix}$$

The theorem can be proved in a same manner as Theorem 5.4 with a different reduction order. The proof is provided in the Appendix.

6. Conclusion. In this paper, we studied the connectivity properties of the set of static stabilizing decentralized controllers. We demonstrated through a subclass of problems that the NP-hardness of optimal decentralized control could be attributed to a large number of connected components. In particular, we proved that the number of connected components for chain subsystems would follow a Fibonacci sequence. Even if the elements of the system matrix are random, the expected number of connected components is still exponential. The fact that the structure of the decentralized control problem can cause intractability leads to our study of specific system and controller properties that have connectivity guarantees. We bound the number of connected components for the scalar control case. We showed that connectivity would not be an issue for highly damped systems independent of the control structures. In case the system matrix has a certain block structure, we fully characterized the number of connected components. Our results qualified the applicability of local search algorithms to optimal decentralized control problems and emphasized structural considerations.

One future research direction is the analysis of the connectivity properties of dynamic controllers. Dynamic controllers have more flexibility in the choice of parameters and therefore we expect better connectivity properties to hold. On the constructive side, it is important to identify system or control structural properties that guarantee the connectivity of the feasible set. The connectivity result, combined with an analysis of the absence of saddle points, will shed light on the possibility of applying local search algorithms to decentralized control problems.

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Appendix A. Proof of Proposition 4.9.

Proof. We adopt the same notation of m_i and n_i in Theorem 4.8. Let m'_{i+1} and n'_{i+1} denote the number of appearances of $h_{i+1}+k_i > 0$ and $h_{i+1}+k_i < 0$, respectively. In Theorem 4.8, $m'_{i+1} = m_{i+1}$ and $n'_{i+1} = n_{i+1}$. The situation is different when some k_{i_0} is set to zero. We first consider the case $2 \leq i_0 \leq n-1$.

The random variable $m_i + n_i$ evolves from $i = 1$ to $i = i_0 - 1$ in the same manner as Theorem 4.8. Therefore, given m_{i_0-1} copies of the inequality $f_{i_0-1} + k_{i_0-1} > 0$ and n_{i_0-1} copies of the inequality $f_{i_0-1} + k_{i_0-1} < 0$, conditioned on m_{i_0-1} and n_{i_0-1}, we have

$$(m'_{i_0}, n'_{i_0}) = \begin{cases} (m_{i_0-1}, m_{i_0-1} + n_{i_0-1}), & \text{with probability } \frac{1}{2} \\ (m_{i_0-1} + n_{i_0-1}, n_{i_0-1}), & \text{with probability } \frac{1}{2} \end{cases}.$$

Since k_{i_0} is fixed to zero, when $f_{i_0} > 0$, all inequalities $f_{i_0} + k_{i_0} < 0$ are pruned, and when $f_{i_0} < 0$, all inequalities $f_{i_0} + k_{i_0} > 0$ are pruned. Therefore, conditioned on m'_{i_0} and n'_{i_0},

$$(m_{i_0}, n_{i_0}) = \begin{cases} (m'_{i_0}, 0), & \text{with probability } \frac{1}{2} \\ (0, n'_{i_0}), & \text{with probability } \frac{1}{2} \end{cases}.$$

Count similarly m'_{i_0+1} and n'_{i_0+1}, we account for the loss of freedom in $h_{i_0+1} + k_{i_0}$:

$$(m'_{i_0+1}, n'_{i_0+1}) = \begin{cases} (m_{i_0}, 0), & \text{with probability } \frac{1}{2} \\ (0, n_{i_0}), & \text{with probability } \frac{1}{2} \end{cases}.$$

After this, the evolution of (m_i, n_i) from i to $i+1$ is the same as *Theorem* 4.8. It holds that $m_{i_0+1} = m'_{i_0+1}$ and $n_{i_0+1} = n'_{i_0+1}$. In sum,

$$\begin{aligned} \mathbb{E}[m_{i_0+1} + n_{i_0+1} | m_{i_0-1}, n_{i_0-1}] &= \mathbb{E}[m'_{i_0+1} + n'_{i_0+1} | m_{i_0-1}, n_{i_0-1}] \\ &= \frac{1}{2}\mathbb{E}[m_{i_0} + n_{i_0} | m_{i_0-1}, n_{i_0-1}] \\ &= \frac{1}{4}\mathbb{E}[m'_{i_0} + n'_{i_0} | m_{i_0-1}, n_{i_0-1}] \\ &= \frac{3}{8}(m_{i_0-1} + n_{i_0-1}). \end{aligned}$$

Hence, after fixing $k_{i_0} = 0$, the number of children is smaller by a factor of $\frac{1}{6}$ compared with Theorem 4.8.

When $i_0 = 1$, $h_2 + k_1$ appears only once in the tree, and the expected number is cut by one half, because after fixing $k_1 = 0$, either $h_2 > 0$ or $h_2 < 0$ is kept. In the same vein, when $i_0 = n$, only half of the leaves are kept. ∎

Appendix B. Proof of Theorem 5.4.

Proof. Similar to Theorem 5.1, we first ignore the constraints $A_{21} \in \mathcal{Z}_1$ and $A_{32} \in \mathcal{Z}_2$. A is stable if and only if there is a matrix $P \succ 0$ partitioned accordingly that satisfies the Lyapunov equation

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & 0 & 0 \\ 0 & A_{32} & 0 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \begin{bmatrix} A_{11}^T & A_{21}^T & 0 \\ A_{12}^T & 0 & A_{32}^T \\ A_{13}^T & 0 & 0 \end{bmatrix} = -I. \tag{B.1}$$

The solution P is unique whenever A is stable.

We first show that

$$A_{21} \text{ and } A_{32} \text{ have full row rank.} \tag{B.2}$$

Consider the $(2,2)$ and $(3,3)$ blocks of (B.1):

$$A_{21}P_{12} + P_{21}A_{21}^T = -I \tag{B.3}$$

$$A_{32}P_{23} + P_{32}A_{32}^T = -I. \tag{B.4}$$

If $x^T A_{32} = 0$, conjugate (B.4) with x to obtain

$$0 = x^T(A_{32}P_{23} + P_{32}A_{32}^T)x = -x^T x,$$

or equivalently, $x = 0$, which means that A_{32} has full row rank. Similarly, A_{21} has full row rank.

Next we consider the $(1,3)$ and $(2,3)$ blocks of (B.1):

$$A_{11}P_{13} + A_{12}P_{23} + A_{13}P_{33} + P_{12}A_{32}^T = 0 \tag{B.5}$$

$$A_{21}P_{13} + P_{22}A_{32}^T = 0. \tag{B.6}$$

Because P_{33} is invertible, A_{13} can be uniquely determined from (B.5). Because A_{21} is full row rank and square, P_{13} can be uniquely determined from (B.6). The equation corresponding to the remaining blocks after eliminating A_{13} can be extracted by pre-multiply (B.1) by

$$W = \begin{bmatrix} I & 0 & -P_{13}P_{33}^{-1} \\ 0 & I & -P_{23}P_{33}^{-1} \end{bmatrix},$$

and post-multiply (B.1) by W^T, which yields

$$\begin{aligned} &\begin{bmatrix} A_{11} & A_{12} - P_{13}P_{33}^{-1}A_{32} \\ A_{21} & -P_{23}P_{33}^{-1}A_{32} \end{bmatrix} \begin{bmatrix} \bar{P}_{11} & \bar{P}_{12} \\ \bar{P}_{21} & \bar{P}_{22} \end{bmatrix} + \begin{bmatrix} \bar{P}_{11} & \bar{P}_{12} \\ \bar{P}_{21} & \bar{P}_{22} \end{bmatrix} \begin{bmatrix} A_{11}^T & A_{21}^T \\ A_{12}^T - A_{32}^T P_{33}^{-1} P_{32} & -A_{32}^T P_{33}^{-1} P_{32} \end{bmatrix} \\ &= \begin{bmatrix} -I - P_{13}P_{33}^{-2}P_{31} & -P_{13}P_{33}^{-2}P_{32} \\ -P_{23}P_{33}^{-2}P_{31} & -I - P_{23}P_{33}^{-2}P_{32} \end{bmatrix}. \end{aligned} \tag{B.7}$$

where the partitioned Schur complement $\bar{P}_{ij}$ is equal to $P_{ij} - P_{i3}P_{33}^{-1}P_{3j}$ for $i, j = 1, 2$. The $(1,2)$ and $(2,2)$ blocks of (B.7) are

$$A_{11}\bar{P}_{12} + (A_{12} - P_{13}P_{33}^{-1}A_{32})\bar{P}_{22} + \bar{P}_{11}A_{21}^T = -P_{13}P_{33}^{-2}P_{32} \tag{B.8}$$

$$A_{21}\bar{P}_{12} + \bar{P}_{21}A_{21}^T = -I - P_{23}P_{33}^{-2}P_{32} + P_{23}P_{33}^{-1}A_{32}\bar{P}_{22} + \bar{P}_{22}A_{32}^T P_{33}^{-1}P_{32}. \tag{B.9}$$

Since $\bar{P}_{22}$ is invertible, A_{12} can be uniquely determined from (B.8). (B.9) is the same as (B.3) given (B.4) and (B.6). Eliminate A_{12} similarly by conjugating (B.7) with $\begin{bmatrix} I & \bar{P}_{12}\bar{P}_{22}^{-1} \end{bmatrix}$, which yields

$$(A_{11} - \bar{P}_{12}\bar{P}_{22}^{-1}A_{21})\tilde{P}_{11} + \tilde{P}_{11}(A_{11}^T - A_{21}^T\bar{P}_{22}^{-1}\bar{P}_{21}) = *, \tag{B.10}$$

where $\tilde{P}_{11} = \bar{P}_{11} - \bar{P}_{12}\bar{P}_{22}^{-1}\bar{P}_{21}$, and the right hand side is a negative definite matrix determined by P. Since $\tilde{P}_{11}$ is positive definite, its eigenvalue do not sum up to zero; therefore, the solution A_{11} always exists and can be shrunk to a symmetric solution that depends continuously on P, as explained in Theorem 5.1. Using $\sim$ to denote the equivalence of connected components,

$$\mathcal{A}_{\mathcal{T}} \sim \left\{ \left(\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & 0 & 0 \\ 0 & A_{32} & 0 \end{bmatrix}, \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \right) : \text{(B.1)}, \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \succ 0, \text{(B.2)} \right\} \tag{B.11}$$

$$\begin{aligned} \sim \Bigg\{ \Bigg(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & 0 \end{bmatrix}, A_{32}, P_{23}, P_{33}, \begin{bmatrix} \bar{P}_{11} & \bar{P}_{12} \\ \bar{P}_{21} & \bar{P}_{22} \end{bmatrix} \Bigg) : \text{(B.4)}, \text{(B.7)}, P_{33} \succ 0, \\ \begin{bmatrix} \bar{P}_{11} & \bar{P}_{12} \\ \bar{P}_{21} & \bar{P}_{22} \end{bmatrix} \succ 0, \text{(B.2)} \Bigg\} \end{aligned} \tag{B.12}$$

$$\begin{aligned} \sim \Big\{ \Big(A_{11}, A_{21}, A_{32}, P_{23}, P_{33}, \bar{P}_{12}, \bar{P}_{22}, \tilde{P}_{11} \Big) : \text{(B.4)}, \text{(B.9)}, \text{(B.10)}, \\ P_{33} \succ 0, \bar{P}_{22} \succ 0, \tilde{P}_{11} \succ 0, \text{(B.2)} \Big\} \end{aligned} \tag{B.13}$$

$$\sim \left\{ \left(A_{21}, A_{32}, P_{23}, P_{33}, \bar{P}_{12}, \bar{P}_{22} \right) : \text{(B.4)}, \text{(B.9)}, P_{33} \succ 0, \bar{P}_{22} \succ 0, \text{(B.2)} \right\} \tag{B.14}$$

$$\sim \left\{ \left(A_{21}, A_{32}, P_{33}, \bar{P}_{22} \right) : P_{33} \succ 0, \bar{P}_{22} \succ 0, \text{(B.2)} \right\} \tag{B.15}$$

$$\sim \{ (A_{21}, A_{32}) : \text{(B.2)} \}. \tag{B.16}$$

The first equivalence (B.11) is justified as in (5.7), with the additional condition that A_{21} and A_{32} must have full row rank. (B.12) follows from the unique continuous solution of A_{13} and P_{13} in (B.5)-(B.6). (B.13) follows from the unique solution of A_{12} in (B.8). (B.14) follows from the retraction of the solutions to (B.10). Since A_{32} has full row rank, (B.4) is always solvable in P_{23}, and the solution subspace can be retracted to the pseudo-inverse solution $P_{23} = 1/2A_{32}^{+}$, which is a continuous function over the full-rank matrix A_{32}. The same argument applies to (B.9), where the solution $\bar{P}_{12}$ always exists and can be continuously retracted to the pseudo-inverse solution. This arrives at (B.15). (B.16) discards the redundant coordinates.

The proof above imposes no restriction on A_{21} and A_{32}; it holds with any additional subspace constraint on them. ∎

Appendix C. Proof of Theorem 5.6.

Proof. We show the proof for the case $n = 3$; the proof carries over to the general case. The idea is the same as Theorem 5.4, with minor differences in the reduction order and in the justification for full-rank blocks. Consider the solution pair (A, P) to the Lyapunov equation

(C.1)
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & 0 & a_{23} \\ 0 & a_{32} & 0 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & 0 \\ a_{12} & 0 & a_{32} \\ a_{13} & a_{23} & 0 \end{bmatrix} = -I.$$

where $P \succ 0$ is unique whenever $A = \left[\begin{smallmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & 0 & a_{23} \\ 0 & a_{32} & 0 \end{smallmatrix}\right]$ is stable. Consider the $(1,3)$, $(2,3)$ and $(3,3)$ blocks of (C.1),

$$a_{11}p_{13} + a_{12}p_{23} + a_{13}p_{33} + p_{12}a_{32} = 0 \tag{C.2}$$
$$a_{21}p_{13} + a_{23}p_{33} + p_{22}a_{32} = 0 \tag{C.3}$$
$$a_{32}p_{23} + p_{32}a_{32} = -1. \tag{C.4}$$

Since p_{33} is invertible, a_{13} and a_{23} are uniquely determined from (C.2) and (C.3). The equation in the remaining blocks after eliminating a_{13} and a_{23} can be extracted by pre-multiply (C.1) by

$$W = \begin{bmatrix} 1 & 0 & -p_{13}p_{33}^{-1} \\ 0 & 1 & -p_{23}p_{33}^{-1} \end{bmatrix}$$

and post-multiply (C.1) by W^T:

(C.5)
$$\begin{aligned} &\begin{bmatrix} a_{11} & a_{12} - p_{13}p_{33}^{-1}a_{32} \\ a_{21} & -p_{23}p_{33}^{-1}a_{32} \end{bmatrix} \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} \\ \bar{p}_{21} & \bar{p}_{22} \end{bmatrix} + \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} \\ \bar{p}_{21} & \bar{p}_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} \\ a_{12} - a_{32}p_{33}^{-1}p_{32} & -a_{32}p_{33}^{-1}p_{32} \end{bmatrix} \\ &\qquad = \begin{bmatrix} -1 - p_{13}p_{33}^{-2}p_{31} & -p_{13}p_{33}^{-2}p_{32} \\ -p_{23}p_{33}^{-2}p_{31} & -1 - p_{23}p_{33}^{-2}p_{32} \end{bmatrix}, \end{aligned}$$

where the partitioned Schur complement $\bar{p}_{ij}$ is equal to $p_{ij} - p_{i3}p_{33}^{-1}p_{3j}$ for $i, j = 1, 2$. The $(1,2)$ and $(2,2)$ blocks of (C.5) are

$$a_{11}\bar{p}_{12} + (a_{12} - p_{13}p_{33}^{-1}a_{32})\bar{p}_{22} + \bar{p}_{11}a_{21} = -p_{13}p_{33}^{-2}p_{32} \tag{C.6}$$
$$a_{21}\bar{p}_{12} + \bar{p}_{21}a_{21} = -1 - p_{23}p_{33}^{-2}p_{32} + p_{23}p_{33}^{-1}a_{32}\bar{p}_{22} + \bar{p}_{22}a_{32}p_{33}^{-1}p_{32}. \tag{C.7}$$

Similarly, since $\bar{p}_{22}$ is invertible, a_{12} can uniquely solved from (C.6). Eliminating a_{12} similarly by conjugating (C.5) with $\left[\begin{smallmatrix}1 & \bar{p}_{12}\bar{p}_{22}^{-1}\end{smallmatrix}\right]$ gives

$$(a_{11} - \bar{p}_{12}\bar{p}_{22}^{-1}a_{21})\tilde{p}_{11} + \tilde{p}_{11}(a_{11} - a_{21}\bar{p}_{22}^{-1}\bar{p}_{21}) = * \tag{C.8}$$

where $\tilde{p}_{11} = \bar{p}_{11} - \bar{p}_{12}\bar{p}_{22}^{-1}\bar{p}_{21}$ and the right hand side is a negative definite matrix determined by P. Because $\tilde{p}_{11}$ is positive definite, its eigenvalues do not sum up to zero. As a result, the solution a_{11} always exists and can be shrunk to a symmetric solution that depends continuously on P. We retract the solution set, where $\sim$ denotes the equivalence of connected components:

$$\begin{aligned}
\mathcal{A}_{\mathcal{T}} &\sim \left\{ \left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & 0 & a_{23} \\ 0 & a_{32} & 0 \end{bmatrix}, \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} \right) : \text{(C.1)}, \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} \succ 0 \right\} \\
&\sim \left\{ \left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & 0 \end{bmatrix}, a_{32}, p_{13}, p_{23}, p_{33}, \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} \\ \bar{p}_{21} & \bar{p}_{22} \end{bmatrix} \right) : \text{(C.4)}, \text{(C.5)}, p_{33} \succ 0, \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} \\ \bar{p}_{21} & \bar{p}_{22} \end{bmatrix} \succ 0 \right\} \\
&\sim \{(a_{11}, a_{21}, a_{32}, p_{13}, p_{23}, p_{33}, \bar{p}_{12}, \bar{p}_{22}, \tilde{p}_{11}) : \text{(C.4)}, \text{(C.7)}, \text{(C.8)}, \\
&\qquad\qquad\qquad\qquad\qquad\qquad p_{33} \succ 0, \bar{p}_{22} \succ 0, \tilde{p}_{11} \succ 0\} \\
&\sim \{(a_{21}, a_{32}, p_{13}, p_{23}, p_{33}, \bar{p}_{12}, \bar{p}_{22}) : \text{(C.4)}, \text{(C.7)}, p_{33} \succ 0, \bar{p}_{22} \succ 0\}\,.
\end{aligned}$$

The equivalence is justified similarly. We first add an additional the Lyapunov matrix P and then repeatedly discard the upper-triangular entires of A, which are uniquely solved, while transforming the representation of P with the Schur complement until we reach (C.8), which is always solvable in a_{11}. This discarding procedure produces a series of equations in the form of (C.7) and (C.4). Since scalar multiplication commutes, we substitute (C.4) to (C.7) and find that the right hand side of (C.7) is strictly less than zero, hence $a_{21} \neq 0$. In the same vein, (C.4) implies $a_{32} \neq 0$. We have proved that all lower sub-diagonal entries of A cannot be zero. With nonzero a_{21} and a_{32}, the remaining equations uniquely determine the sub-diagonal entries $(\bar{p}_{12}, p_{23})$, we arrive at the final series equivalences:

$$\begin{aligned}
\mathcal{A}_{\mathcal{T}} &\sim \{(a_{21}, a_{32}, p_{13}, p_{23}, p_{33}, \bar{p}_{12}, \bar{p}_{22}) : \text{(C.4)}, \text{(C.7)}, p_{33} > 0, \bar{p}_{22} > 0, a_{32} \neq 0, a_{21} \neq 0\} \\
&\sim \{(a_{21}, a_{32}, p_{13}, p_{33}, \bar{p}_{22}) : p_{33} > 0, \bar{p}_{22} > 0, a_{32} \neq 0, a_{21} \neq 0\} \\
&\sim \{(a_{21}, a_{32}) : a_{32} \neq 0, a_{21} \neq 0\}\,.
\end{aligned}$$

After discarding the redundant coordinates, we are left with $n-1$ nonzero conditions on the sub-diagonals of A, which give rise to 2^{n-1} connected components. ∎