On the Connectivity Properties of Feasible Regions of Optimal Decentralized Control Problems

Yingjie Bi and Javad Lavaei

***Abstract*—The optimal decentralized control (ODC) is an NP-hard problem with many applications in real-world systems. There is a recent trend of using local search algorithms for solving optimal control problems. However, the effectiveness of these methods depends on the connectivity property of the feasible region of the underlying optimization problem. In this paper, for ODC problems with static controllers, we develop a novel criterion for certifying the connectivity of the feasible region in the case where the input and output matrices of the system dynamics are identity. This criterion can be checked via an efficient algorithm, and it is used to prove that the number of communication networks leading to connected feasible regions is greater than a square root of the exponential number of possible communication networks (named patterns). For ODC problems with dynamic controllers, we prove that under certain mild conditions the closure of the feasible region is always connected after some parameterization, for general communication networks and system dynamics.**

***Index Terms*—Decentralized control, optimal control, nonconvexity.**

I. INTRODUCTION

The field of optimal decentralized control (ODC) has emerged in response to the prevalence of communication constraints among agents in many real-world interconnected or multi-agent systems, including power grids [2], computer networks [3] and robotics [4]. Being a nonconvex optimization problem, the general ODC problem has been proved to be computationally intractable [5], [6]. Many techniques have been proposed in the literature to convexify or solve special cases of the ODC problem [7]–[13].

Inspired by the learning algorithms in the field of machine learning, the recent work [14] has advocated for using local search methods to solve the optimal control problems. Local search methods have several advantages, such as having low computational and memory complexities and the ability to be implemented without explicitly establishing the underlying model. The main issue with these methods is that they are not guaranteed to find the global optimal solution if the problem does not have a convex structure. However, for the classical (centralized) LQR optimal control problem, [14] has proved that the gradient descent method converges to the globally optimal solution despite the non-convexity of the problem. Given this surprising result, it is natural to ask whether local search methods are also effective for ODC problems. The paper [15] shows the global convergence of local search methods under the quadratic invariance condition.

The effectiveness of local search methods depends on the connectivity properties of the feasible region. If the feasible region is connected, a local search method only needs to take feasible directions. As being successful in many machine learning problems, stochastic gradient methods are able to find near-globally optimal solutions of nonconvex problems even in the presence of some types of spurious local minima [16]–[19]. However, if the feasible region is disconnected, then there is a local minimum in each connected component, which significantly increases the computational burden and is the underlying reason for the NP-hardness of many problems.

The recent work [20] has found a class of ODC problems with n state variables whose feasible regions have $O(2^n)$ connected components. This negative result shows that local search methods are not effective for general ODC problems, since there could be an exponential number of local minima that are far away from each other. The follow-up paper [21] characterizes the connectivity property for single-input-single-output systems. However, for general multiple-input-multiple-output systems, there are only a few cases in which the connectivity of the feasible region has been determined.

In this paper, we investigate the ODC problem in two scenarios of static controllers and dynamic controllers, and the goal is to derive conditions under which the feasible region of the ODC problem in each scenario is connected. For static ODC problems, we focus on the cases where the input and output matrices of the system dynamics are identity. To this end, a new criterion for the connectivity of the feasible region is developed, which can be verified by an efficient algorithm. Furthermore, based on the new tool, the following two results are developed: (i) The feasible region is connected for most dense communication networks; (ii) There are an exponential number of communication networks with connected feasible regions. These networks constitute a set of easier ODC problems that can be used as approximations for other ODC problems, which is in the same vein as the common approach of using convex functions to approximate nonconvex functions. For ODC problems with dynamic controllers, we first parametrize all possible controllers in some metric space, and then prove that the closure of the feasible region in this space is always connected under some mild conditions.

This paper mainly focuses on providing sufficient conditions to certify the connectivity of the feasible region, rather than finding conditions for the disconnectivity of the region as done

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This work was supported by grants from ARO, ONR, AFOSR and NSF. A preliminary version has appeared in [1]. Compared with the conference paper, we have extended the analysis to static ODC problems with nonidentity input and output matrices and developed a major result on the connectivity properties of the feasible regions of dynamic ODC problems.

in [20]. Proving disconnectivity properties requires different mathematical techniques than the ones used in this paper to discover connectivity properties. In particular, the common way to prove that a set is connected is to show that any two points in the set can be connected by a path, while proving disconnectivity and counting the number of connected components usually involve constructing a partition of the set into disjoint open subsets. The former technique is used in the proof of Proposition 1 and Theorem 11, and the latter one is used in the proof of Proposition 6.

II. NOTATION AND PROBLEM FORMULATION

We summarize the common notations used in the paper:

- Normal letters such as A refer to matrices, while bold letters such as $\mathbf{H}$ refer to controllers or systems.
- I_n is the $n \times n$ identity matrix, and 0_n is the zero matrix.
- $\operatorname{diag}(a_1, a_2, \ldots, a_n)$ is the diagonal matrix whose diagonal entries are $a_1, a_2, \ldots, a_n$.
- $\mathcal{S}_n$ is the set of $n \times n$ symmetric matrices.
- $\mathcal{P}_n$ is the set of $n \times n$ positive definite matrices.
- $\mathcal{K}_n$ is the set of $n \times n$ stable matrices, i.e., matrices whose eigenvalues have a negative real part.
- $\|M\|_\infty$ is the elementwise infinity norm of the matrix M.

A. ODC Problems with Static Controllers

Consider the continuous-time linear system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \end{aligned} \tag{1}$$

in which $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input and $y(t) \in \mathbb{R}^p$ is the output. The optimal static output-feedback control problem is to design a feedback controller $u(t) = -Ky(t)$ with $K \in \mathbb{R}^{m \times p}$ while minimizing certain cost functional. For example, in the classical infinite-horizon LQR problem, the objective is to minimize

$$J = \int_0^{+\infty} (x^T(t)Qx(t) + u^T(t)Ru(t) + 2x^T(t)Nu(t))\mathrm{d}t$$

subject to the constraint that the closed-loop system

$$\dot{x}(t) = (A - BKC)x(t)$$

must be stable. Consider the ODC problem of designing an optimal static controller minimizing an arbitrary cost functional (not necessarily a quadratic one), where there are some communication constraints enforcing certain entries of K to be zero. Let

$$\mathscr{P} \subseteq \{(i,j) | 1 \le i \le m,\ 1 \le j \le p\}$$

be the set of indices of the free variables K_{ij} whose values are not restricted by the communication constraints. The set $\mathscr{P}$ will be referred to as a *pattern* in this paper. After substituting

$$x(t) = \mathrm{e}^{(A-BKC)t}x(0), \quad u(t) = -KC\mathrm{e}^{(A-BKC)t}x(0)$$

into the cost functional, the ODC problem can be formulated as the minimization of a cost function with the only variable K over the feasible region

$$\mathcal{D} = \{K \in \mathcal{L}(\mathscr{P}) | A - BKC \in \mathcal{K}_n\},$$

where the linear subspace $\mathcal{L}(\mathscr{P})$ is given by

$$\mathcal{L}(\mathscr{P}) = \{K \in \mathbb{R}^{m \times p} | K_{ij} = 0,\ \forall (i,j) \notin \mathscr{P}\}. \tag{2}$$

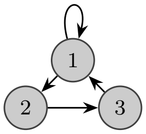


Fig. 1. Graph representation of the pattern (3).

The performance of local search methods for solving ODC through this formulation (or other reformulation of the problem) is directly related to the geometric properties of the feasible region $\mathcal{D}$. In Sections III and IV, we will study the connectivity of $\mathcal{D}$ under the usual Euclidean topology. For the special case when $m = n = p$ and B and C are identity matrices, similar to the notations used in [22], the pattern $\mathscr{P}$ can be represented by both a matrix and a directed graph with possible self-loops. For instance, the pattern

$$\{(1,1), (1,2), (2,3), (3,1)\} \tag{3}$$

can be described in the matrix form

$$\begin{bmatrix} * & * & 0 \\ 0 & 0 & * \\ * & 0 & 0 \end{bmatrix}$$

or equivalently in the graph form given in Fig. 1. In addition, for a pattern $\mathscr{P}$ viewed as a graph, we denote its complement graph as $\mathscr{P}^c$ and the number of edges in $\mathscr{P}^c$, i.e., the number of “0”s in $\mathscr{P}$, as $|\mathscr{P}^c|$.

In this work, we will use the fact that an arbitrary pattern $\mathscr{P}$ for a given system can be converted to a simple diagonal pattern for an augmented system. More precisely, one can order all the pairs $(i,j) \in \mathscr{P}$ into a list $(i_1, j_1), \ldots, (i_r, j_r)$, and define two matrices B' and C' of sizes $m \times r$ and $r \times p$, respectively, by setting

$$B'_{i_k k} = 1, \quad C'_{k j_k} = 1, \quad \forall k = 1, \ldots, r$$

and setting the remaining entries to zero. Then, any feedback gain K satisfying the pattern $\mathscr{P}$ can be decomposed as

$$K = B' \operatorname{diag}(K_{i_1 j_1}, K_{i_2 j_2}, \ldots, K_{i_r j_r}) C'.$$

Hence, there is a one-to-one mapping between the feasible region of the original ODC problem with the system matrices (A, B, C) under the pattern $\mathscr{P}$ and that of the ODC problem with the system matrices $(A, BB', C'C)$ under a diagonal pattern. This observation suggests that one can limit the pattern $\mathscr{P}$ to be a diagonal pattern when studying general ODC problems with arbitrary B and C matrices.

B. ODC Problems with Dynamic Controllers

In contrast to static controllers whose communication constraints can be represented by a single pattern $\mathscr{P}$, dynamic controllers are much harder to characterize. A dynamic controller contains multiple local controllers, each with some internal state. Using the technique discussed at the end of

Section II-A, one can convert a dynamic ODC problem with an arbitrary sparsity pattern to another ODC problem with a diagonal pattern, which can be directly formulated as follows:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + \sum_{i=1}^{r} B_i u_i(t), \\ y_i(t) &= C_i x(t), \quad \forall i = 1, \dots, r, \end{aligned} \tag{4}$$

where $u_i(t)$ and $y_i(t)$ are the input and output of subsystem i. The goal is to design a decentralized dynamic controller $\mathbf{H}$ consisting of r local controllers, each associated to one subsystem. The local controller for subsystem i, named $\mathbf{H}_i$, is modeled as

$$\begin{aligned} \dot{z}_i(t) &= \tilde{A}_i z_i(t) + \tilde{B}_i y_i(t), \\ u_i(t) &= \tilde{C}_i z_i(t) + \tilde{D}_i y_i(t). \end{aligned}$$

The local controller $\mathbf{H}_i$ can be described by its matrices as

$$\mathbf{H}_i = (\tilde{A}_i, \tilde{B}_i, \tilde{C}_i, \tilde{D}_i).$$

The size of $\tilde{A}_i$ is called the *degree* of local controller $\mathbf{H}_i$, and $\mathbf{H}_i$ is said to be *stable* if $\tilde{A}_i$ is a stable matrix.

The decentralized controller $\mathbf{H}$ itself will be denoted as

$$\mathbf{H} = (\mathbf{H}_1, \dots, \mathbf{H}_r).$$

Let $\mathcal{H}$ denote the set of all decentralized controllers for the system (4). Similar to the case of static controllers, a controller $\mathbf{H} \in \mathcal{H}$ is said to (internally) stabilize the system (4) if the closed-loop system is stable, and the set of all stabilizing controllers will be denoted as $\mathcal{H}_s$. The existence of such stabilizing decentralized controllers can be characterized by the notions of decentralized fixed modes [23] or decentralized overlapping fixed modes [24]. The connectivity properties of the dynamic ODC problems will be studied in Section V.

III. CONNECTIVITY PROPERTIES OF STATIC ODC PROBLEMS

In this section, the connectivity properties of ODC problems with static controllers will be explored. First, we limit ourselves to problems with B and C matrices being identity and develop a powerful criterion for the connectivity of their feasible regions. An accompanying algorithm will then be devised based on the criterion, and its implications will be deferred to Section IV. Then, we show that the developed results also hold true for the case when B and C are not identity but structurally keep the pattern invariant. At the end of this section, the difficulties of investigating the connectivity properties for general ODC problems with arbitrary B and C will be explained, and we will discuss how the introduction of dynamic controllers would simplify the problem.

A. The Connectivity Criterion

To enable the mathematical analysis of the feasible region of ODC problems corresponding to a pattern $\mathscr{P}$, it is beneficial to introduce a new notion as stated below.

Definition 1: A pattern $\mathscr{P}$ is said to be *stably expandable* if the following property holds: for every two stable matrices $K^0 \in \mathcal{K}_n$ and $K^1 \in \mathcal{K}_n$, together with arbitrary continuous functions $\gamma_{ij}(\tau) : [0,1] \to \mathbb{R}$ defined for all $(i,j) \notin \mathscr{P}$ with the endpoint conditions

$$K^0_{ij} = \gamma_{ij}(0), \quad K^1_{ij} = \gamma_{ij}(1),$$

there exists a (continuous) path $K(\tau) : [0,1] \to \mathcal{K}_n$ with $K(0) = K^0$, $K(1) = K^1$ expanding the functions $\gamma_{ij}(\tau)$, i.e.,

$$K_{ij}(\tau) = \gamma_{ij}(\tau), \quad \forall \tau \in [0,1],\ \forall (i,j) \notin \mathscr{P}.$$

Since the feasible region of an ODC problem is always an open set, for a given pattern $\mathscr{P}$, if in the above definition the functions $\gamma_{ij}(\tau)$ are further restricted to be constant functions, then $\mathscr{P}$ satisfies this modified definition if and only if the feasible region $\mathcal{D}$ is connected for all ODC problems with pattern $\mathscr{P}$, arbitrary matrix A, and identity matrices B and C. However, because $\gamma_{ij}(\tau)$ are arbitrary in Definition 1, the stable expandability is generally a stronger property than being connected. In this section, we work on the stable expandability instead of the original definition of connectivity, because the stable expandability of a pattern $\mathcal{P}$ can be reduced to that of a subpattern of $\mathcal{P}$. As a result, the to-be-developed results will be sufficient conditions only. We believe that finding a polynomial-time verifiable condition that is equivalent to connectivity is similar to finding the boundary of P and NP for a class of problems, which is by itself an NP-hard problem. The above discussion can be summarized in the following proposition.

Proposition 1: If a pattern $\mathscr{P}$ is stably expandable, then the corresponding feasible region $\mathcal{D}$ is connected for ODC problems with arbitrary $A \in \mathbb{R}^{n \times n}$ and $B = C = I_n$. ■

We emphasize that the property of stable expandability implies the connectivity of the feasible region for all values of the matrix A. In contrast, [20, Theorem 6.1] shows that the feasible region is always connected under an arbitrary pattern $\mathscr{P}$, arbitrary matrix A whose diagonal entries are sufficiently negative, and $B = C = I_n$.

The remaining goal is to further develop sufficient conditions in several steps to imply the stable expandability for a pattern $\mathscr{P}$. The conditions should be efficiently verifiable and hold for numerous patterns. First, Theorem 3 provides sufficient conditions for the stable expandability based on the reduction idea mentioned above. Then, Corollary 4 provides further sufficient conditions implying the ones in Theorem 3. Finally, Algorithm 1 gives an efficient algorithm to certify a pattern based on the conditions in Corollary 4. We start with some basic properties of the stable expandability:

1) If the pattern $\mathscr{P}$ is stably expandable and $\mathscr{P} \subseteq \mathscr{Q}$, then $\mathscr{Q}$ is also stably expandable.
2) If the pattern $\mathscr{P}$ is stably expandable, its transpose $\mathscr{P}^T$ is also stably expandable, since any matrix is similar to its transpose.
3) If the pattern $\mathscr{P}$ is stably expandable and $\mathscr{Q}$ is isomorphic (as a graph) to $\mathscr{P}$, then $\mathscr{Q}$ is also stably expandable, since any matrix is similar to the matrix obtained by simultaneously applying the same permutation on its rows and columns.

For ODC problems with identity B and C, it is shown in [20] that the feasible region $\mathcal{D}$ is connected if the diagonal elements of K are free. The following proposition shows that such patterns also satisfy the stronger property of stable expandability.

Proposition 2: A pattern $\mathscr{P}$ is stably expandable if its complement graph $\mathscr{P}^c$ does not have self-loops.

Proof: Given stable matrices K^0, K^1 and functions $\gamma_{ij}(\tau)$, consider an arbitrary path $K'(\tau)$ satisfying all the requirements in Definition 1 except the stability of $K'(\tau)$ for all $\tau \in [0, 1]$. Define the function $\sigma(\tau)$ to be the largest real part of the eigenvalues of $K'(\tau)$. Then, $\sigma(\tau)$ is a continuous function and the path

$$K(\tau) = K'(\tau) - (\max\{\sigma(\tau), 0\} + \tau(1-\tau))I_n,$$

is guaranteed to be contained in $\mathcal{K}_n$. Since $K(0) = K'(0)$, $K(1) = K'(1)$ and the two paths have the same off-diagonal elements, $K(\tau)$ is a path satisfying all the requirements in Definition 1. ■

It is desirable to show that the stable expandability of a pattern may be checked by analyzing smaller subpatterns.

Theorem 3: The $n \times n$ pattern

$$\begin{array}{r} n-q\left\{\vphantom{\mathscr{A}}\right. \\ q\left\{\vphantom{\mathscr{C}}\right. \end{array}\left[\begin{array}{c:c} \mathscr{A} & \mathscr{B} \\ \hdashline \underbrace{\mathscr{C}}_{n-q} & \underbrace{\mathscr{D}}_{q} \end{array}\right]$$

is stably expandable if

1) The pattern $\mathscr{A}$ is stably expandable.
2) The number of rows in $\mathscr{B}$ without "0"s is at least $q+1$.
3) The pattern $\mathscr{C}$ does not contain "0"s.

Proof: Please see the Appendix. ■

To check the connectivity of the feasible region $\mathcal{D}$ associated with a pattern $\mathscr{P}$, one can partition the vertices of $\mathscr{P}$ appropriately and then apply Theorem 3 multiple times. This will be formalized below.

Corollary 4: A pattern $\mathscr{P}$ is stably expandable if there exists a partition $\{S_1, S_2, \ldots, S_w\}$ of the vertices such that

1) For every $1 \leq k < l \leq w$, there is no edge from S_l to S_k in the complement graph $\mathscr{P}^c$.
2) The subpattern with the vertex set S_1 is stably expandable.
3) For every $k > 1$, if d_k denotes the number of vertices i with the property

$$i \notin S_k \text{ and } \exists j \in S_k \text{ s.t. } (i, j) \in \mathscr{P}^c$$

and r_k denotes the number of vertices in S_k, then

$$\sum_{l=1}^{k-1} r_l > d_k + r_k. \tag{5}$$

Proof: Let $\mathscr{P}_k$ be the subpattern of $\mathscr{P}$ with the vertex set $\cup_{l=1}^{k} S_l$. We prove by induction that $\mathscr{P}_k$ is stably expandable. The base step $k = 1$ is obviously true. Now, assume that $\mathscr{P}_{k-1}$ is stably expandable. After ordering the vertices, one can write the subpattern $\mathscr{P}_k$ in matrix form as follows:

$$\begin{bmatrix} \mathscr{P}_{k-1} & \mathscr{B} \\ \mathscr{C} & \mathscr{D} \end{bmatrix}.$$

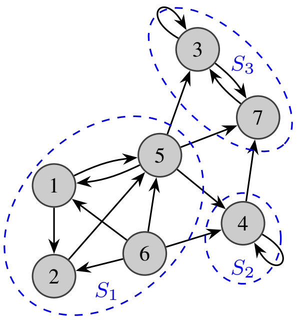


Fig. 2. The complement graph of the pattern studied in Example 1.

In the above pattern, $\mathscr{P}_{k-1}$ is a stably expandable pattern of size $\sum_{l=1}^{k-1} r_l$ and $\mathscr{D}$ is the subpattern of $\mathscr{P}$ with the vertex set S_k. By Condition 1, $\mathscr{C}$ does not have "0"s. Moreover, the number of rows in $\mathscr{B}$ containing "0"s is exactly d_k. Therefore, in light of Condition 3, the number of rows in $\mathscr{B}$ without "0"s can be computed as

$$\sum_{l=1}^{k-1} r_l - d_k > r_k.$$

By Theorem 3, the subpattern $\mathscr{P}_k$ is stably expandable. This completes the proof. ■

It is worth mentioning that Corollary 4 is equivalent to Theorem 3 in the sense that the conditions in Theorem 3 are a special case of those in Corollary 4 if we choose the partition to be $\{S_1, S_2\}$ with S_1 corresponding to the subpattern $\mathscr{A}$ and S_2 corresponding to the subpattern $\mathscr{D}$ in Theorem 3. Moreover, both results can be combined with other ways of showing the stable expandability such as using the property that the stable expandability is invariant under transposition.

Example 1: To illustrate the application of Corollary 4, consider the 7×7 pattern $\mathscr{P}$ whose complement graph $\mathscr{P}^c$ is given in Fig. 2. One can partition the pattern into three parts S_1, S_2 and S_3 as shown in Fig. 2, where

$$r_1 = 4, \quad r_2 = 1, \quad r_3 = 2, \quad d_2 = 2, \quad d_3 = 2.$$

For this partition, the subpattern corresponding to S_1 is stably expandable due to Proposition 2, and the other conditions in Corollary 4 can be directly verified. As a result, the pattern $\mathscr{P}$ is stably expandable and the feasible region of the ODC problem is connected if B and C are identity matrices.

B. The Connectivity Detection Algorithm

The objective of this part is to develop an algorithm that finds a suitable partition for an arbitrary pattern $\mathscr{P}$ to reason about its stable expandability based on Corollary 4. To design the algorithm, the first step is to choose S_1. Since the only stably expandable patterns known initially are the ones satisfying Proposition 2, we select S_1 to be the set of all vertices not reachable in the complement graph $\mathscr{P}^c$ from any vertex with a self-loop in $\mathscr{P}^c$.

Next, since the partition in Corollary 4 has an acyclic structure, it is natural to consider the strongly connected components of the complement graph $\mathscr{P}^c$. By the definition of S_1, there is no edge from the vertices in $\{1,\dots,n\}-S_1$ to the vertices in S_1. Now, we further divide the prior set into strongly connected components $S_2,\dots,S_w$ of $\mathscr{P}^c$. The remaining task is to find an ordering for the sets $S_2,\dots,S_w$ such that Conditions 1 and 3 in Corollary 4 are satisfied.

The above ordering problem is analogous to the task scheduling problem studied in [25]. If each set S_k is regarded as a task that requires r_k amount of time to complete, then the goal is to find an ordering of all tasks satisfying the precedence constraints in such a way that S_1 becomes the first task and the starting time for each remaining task S_k becomes strictly later than d_k+r_k. We propose Algorithm 1 based on the above ideas. If the algorithm returns "succeeded", then the feasible region $\mathcal{D}$ associated with the given pattern has been proved to be connected, while "failed" means that it cannot determine whether the feasible region is connected.

Algorithm 1 (Checking Connectivity for Pattern $\mathscr{P}$):

```
Compute $S_1$ through a breadth-first search.
if $S_1 = \emptyset$ then
    return failed
end if
Divide $\{1,\dots,n\}-S_1$ into strongly connected components $S_2,\dots,S_w$ of $\mathscr{P}^c$.
Compute $d_k$ and $r_k$ for each $S_k$.
Remove $S_1$ from the graph.
$T \leftarrow r_1$
while $T < n$ do
    Find an unprocessed set $S_k$ with no incoming edges in $\mathscr{P}^c$ and $T > d_k + r_k$.
    if not found then
        return failed
    end if
    Remove $S_k$ from the graph.
    $T \leftarrow T + r_k$
end while
return succeeded
```

C. Numerical Examples

To demonstrate the performance of Algorithm 1, it is desirable to provide some numerical examples with randomly generated patterns. Let each entry in the pattern be chosen as "0" independently with a fixed probability. Two types of random instances are considered: (i) the *dense* case where

$$\Pr(\mathscr{P}_{ij}=0)=\frac{c}{n} \tag{6}$$

with some parameter c, (ii) and the *sparse* case where

$$\Pr(\mathscr{P}_{ij}=0)=c.$$

For each of the two cases, we generate 1000 random samples according to the above probability distribution, run Algorithm 1 on these samples and calculate the empirical probability that the algorithm finds patterns with connected feasible regions.

The result for the dense case is given in Fig. 3(a). It can be observed that most patterns generated by $c<1$, i.e., patterns whose number of "0" is approximately less than n, are associated with connected feasible regions. This observation will be mathematically supported in Section IV.

The result for the sparse case is given in Fig. 3(b). The feasible region for a sparse pattern such as the counterexample in [20, Theorem 7.6] may have an exponential number of connected components, while the set is connected in the full matrix case associated with the centralized problem. Thus, the number of connected components goes from an exponential number to one as we go from the sparse case to the dense case. This implies that the sparse case is harder than the dense case. On the other hand, it will be shown in Section IV that it is still possible to construct an exponential number of patterns, with approximately up to half of the entries being "0", which can all be certified via Theorem 3.

D. General Static ODC Problems

In the following, we turn to general static ODC problems in which B or C is not necessarily identity. In the cases when the structure of B and C keeps the pattern $\mathscr{P}$ invariant, the results developed before all hold.

Proposition 5: If a pattern $\mathscr{P}$ is stably expandable, then the corresponding feasible region $\mathcal{D}$ is connected for ODC problems with arbitrary matrices $A,B,C\in\mathbb{R}^{n\times n}$ satisfying

$$\mathcal{L}(\mathscr{P})=\{BKC|K\in\mathcal{L}(\mathscr{P})\},$$

where $\mathcal{L}(\mathscr{P})$ is the linear subspace defined in (2).

Proof: By the assumption, the linear map $f:\mathcal{L}(\mathscr{P})\to\mathcal{L}(\mathscr{P})$ given by $f(K)=BKC$ is invertible, which implies that it is also a homeomorphism between the feasible regions

$$\begin{aligned}\mathcal{D}_1&=\{K_1\in\mathcal{L}(\mathscr{P})|A-BK_1C\in\mathcal{K}_n\},\\ \mathcal{D}_2&=\{K_2\in\mathcal{L}(\mathscr{P})|A-K_2\in\mathcal{K}_n\}.\end{aligned}$$

Proposition 1 implies the connectivity of $\mathcal{D}_2$, and so is $\mathcal{D}_1$. ■

For example, when B and C are diagonal matrices with nonzero diagonal entries, the assumption in Proposition 5 is automatically satisfied, and thus the feasible region is connected if the pattern $\mathscr{P}$ is stably expandable. However, based on the following two observations, the above result cannot be easily extended to static ODC problems with general matrices B and C. First, the following result shows that connectivity is not a robust property under perturbation of B and C matrices.

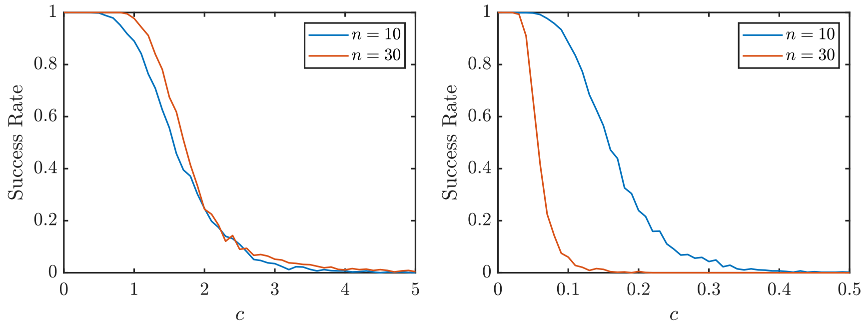
Proposition 6: There is a static ODC problem with system matrices $(A,B=I_n,C=I_n)$ and some pattern $\mathscr{P}$ such that its feasible region is connected but for any $\epsilon>0$ one can always find matrices $B'\in\mathbb{R}^{n\times n}$ and $C'\in\mathbb{R}^{n\times n}$ satisfying

$$\|B'-B\|_\infty<\epsilon,\quad \|C'-C\|_\infty<\epsilon,$$

for which the new ODC problem with the same pattern $\mathscr{P}$ and system matrices (A,B',C') has a disconnected feasible region.

Proof: Consider the $n\times n$ pattern

$$\mathscr{P}=\begin{bmatrix}* & * & \dots & * & 0\\ * & * & \dots & * & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ * & * & \dots & * & 0\\ * & * & \dots & * & *\end{bmatrix}.$$



(a) The dense case

(b) The sparse case

Fig. 3. The success rate of Algorithm 1 on random examples with different values for n and c.

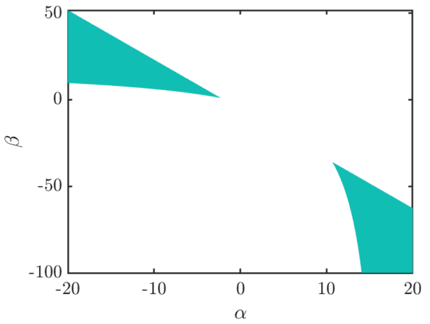


Fig. 4. The feasible region of the ODC problem in Example 2.

The feasible region of the ODC problem with arbitrary A and $B = C = I_n$ is connected because the diagonal entries in $\mathcal{P}$ are all free. However, for the ODC problem with the same pattern $\mathscr{P}$ but for the system with matrices $A = 0_n, C' = I_n$ and

$$B' = \begin{bmatrix} 1 & 0 & \dots & 0 & \delta \\ 0 & 1 & \dots & 0 & \delta \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & \delta \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix},$$

the corresponding feasible region

$$\mathcal{D} = \{K \in \mathcal{L}(\mathscr{P}) | -B'K \in \mathcal{K}_n\}$$

will be disconnected if $\delta \neq 0$. To prove this, note that the linear subspace $\mathcal{E} = \{B'K | K \in \mathcal{L}(\mathscr{P})\}$ is the set of all matrices $A \in \mathbb{R}^{n \times n}$ satisfying

$$A_{in} = \delta A_{nn}, \quad i = 1, \dots, n-1. \tag{7}$$

The set $\mathcal{E} \cap \mathcal{K}_n$ is disconnected since

$$\begin{bmatrix} -1 & \dots & 0 & 0 & -\delta \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & -1 & 0 & -\delta \\ 0 & \dots & 0 & -1 & -\delta \\ 0 & \dots & 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & \dots & 0 & 0 & \delta \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & -1 & 0 & \delta \\ 0 & \dots & 0 & -2 & \delta \\ 0 & \dots & 0 & -3/\delta & 1 \end{bmatrix}$$

are two stable matrices in $\mathcal{E}$ but any path in $\mathcal{E}$ connecting the above two matrices must pass through some matrix $M \in \mathcal{E}$ with $M_{nn} = 0$. The entries in the last column of M are all zero due to (7) and thus M is unstable. Since the connectivity of the feasible region $\mathcal{D}$ would imply the connectivity of the set $\mathcal{E} \cap \mathcal{K}_n$, $\mathcal{D}$ cannot be a connected set. ■

The situation would become even more complex if the B and C matrices are allowed to be far away from identity. Disconnectivity can occur in the feasible region of these problems due to the instabilizability or undetectability of the partially closed-loop system, as illustrated by the following example.

Example 2: Consider the following ODC problem in which

$$A = \begin{bmatrix} 3 & -0.2 \\ 0.4 & 0.8 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & -1 \\ -1.5 & -0.7 \end{bmatrix}, \quad C = I_2,$$

and the pattern is diagonal. Let $K = \text{diag}(\alpha, \beta)$, where α and β are the free parameters of the controller gain. As shown in Fig. 4, the feasible region has two connected components.

Consider the partially closed-loop system obtained by fixing the parameter α at some value and treating β as a free parameter. This new system with the input $u_2(t)$ and the output $y_2(t)$ is not stabilizable when $\alpha = 10.42$ and not detectable when $\alpha = -0.27$. Therefore, the closed-loop system is not stable for these two particular values of α no matter what the value of β is. On the other hand, when $-2.21 \leq \alpha \leq 10.66$ except for the above two α values, the partially closed-loop system is stabilizable and detectable, which can be stabilized by certain dynamic local controller from $y_2(t)$ to $u_2(t)$ instead of the static local controller $u_2(t) = \beta y_2(t)$. As will be shown in Section V, the introduction of dynamic controllers into this problem will reduce the gap between the connected components in Fig. 4 from a finite gap to an infinitesimal one. Note that the connectivity feature in this example is typical, and it is straightforward to generate many other systems with a similar feature by random search.

IV. APPLICATIONS OF THE CONNECTIVITY CRITERION

The connectivity criterion proposed in Section III has two important usages[1]. First, one can apply Theorem 3 to certify the connectivity for certain classes of patterns. ODC problems with these patterns are well suited for local search methods due to the connectivity of the feasible region. Second, one can construct patterns that make the feasible region $\mathcal{D}$ connected by exploiting the conditions in Theorem 3. The designed patterns can be used to relax an ODC problem with an unfavorable pattern $\mathscr{P}$ to another ODC problem with a favorable pattern $\mathscr{Q}$ such that $\mathscr{P} \subseteq \mathscr{Q}$. For example, such a pattern $\mathscr{Q}$ can be found by removing some self-loops in $\mathscr{P}^c$ to enlarge the initial vertex set S_1 in Algorithm 1 or by removing the original incoming edges to the set S_k that violate the inequality (5) minimally at the step when Algorithm 1 cannot proceed. The relaxed problem may be solved by local search methods, and its solution provides a lower bound to the original problem. This is mainly a numerical approach, and bounding the gap

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[1] In this section, all ODC problems are assumed to be static and with identity B and C matrices.

theoretically between the actual optimal value and the lower bound requires a further study.

A. Proving Connectivity

As an application of Theorem 3, it is desirable to prove that most dense patterns with a small number of ‘0”s lead to a connected feasible region $\mathcal{D}$.

Theorem 7: Let r denote the number of vertices in the largest strongly connected component of the complement graph $\mathscr{P}^c$. If

$$|\mathscr{P}^c| \leq n - \max\{r, 2\},$$

then $\mathscr{P}$ is stably expandable.

Proof: Following the argument used in Algorithm 1, let S_1 be the set of all vertices not reachable in $\mathscr{P}^c$ from any vertex with a self-loop in $\mathscr{P}^c$. Observe that each vertex not in S_1 has either a self-loop or an incoming edge in $\mathscr{P}^c$. Since there are at most $n-2$ edges in $\mathscr{P}^c$, the set S_1 cannot be empty. We partition the remaining vertices $\{1, \ldots, n\} - S_1$ into strongly connected components of $\mathscr{P}^c$ and perform a topological sorting over these components. The result is a list of strongly connected components $\{S_2, \ldots, S_w\}$ for which there is no edge from S_l to S_k in $\mathscr{P}^c$ for $k < l$.

For any subset S of vertices, let $E(S)$ denote the number of edges (including self-loops) in $\mathscr{P}^c$ whose destinations belong to S. Using this notation, it can be concluded that

$$E(S_l) \geq r_l, \quad \forall l = 2, \ldots, w. \tag{8}$$

This obviously holds true for a non-singleton S_l since S_l is strongly connected in $\mathscr{P}^c$. On the other hand, if $S_l = \{i\}$ is a singleton, then $i \notin S_1$. The observation made at the beginning of the proof implies that $E(S_l) \geq 1 = r_l$.

We claim that the partition $\{S_1, \ldots, S_w\}$ satisfies all the conditions in Corollary 4. If not, then there exists some $k > 1$ such that

$$d_k + r_k \geq \sum_{l=1}^{k-1} r_l. \tag{9}$$

In this case, there exists at least one vertex j in $\cup_{l=2}^{k} S_l$ with a self-loop in $\mathscr{P}^c$; otherwise, by the definition all these vertices in $\cup_{l=2}^{k} S_l$ should belong to S_1, which is not possible.

Now, we investigate two scenarios. If the strongly connected component S_k is a singleton, then

$$\sum_{l=2}^{k} E(S_l) \geq d_k + 1,$$

where the additional “1” above counts for the self-loop of vertex j. By (8), (9) and the above inequality, one can write

$$\begin{aligned} |\mathscr{P}^c| &\geq \sum_{l=2}^{w} E(S_l) \geq d_k + 1 + \sum_{l=k+1}^{w} r_l \\ &\geq \sum_{l=1}^{k-1} r_l + \sum_{l=k+1}^{w} r_l = n - 1 \\ &> n - \max\{r, 2\}, \end{aligned}$$

which is a contradiction.

For the scenario in which S_k is not a singleton, since S_k is strongly connected, it holds that

$$\sum_{l=2}^{k} E(S_l) \geq d_k + r_k + 1.$$

Similar to the previous scenario, one can write

$$\begin{aligned} |\mathscr{P}^c| &\geq \sum_{l=2}^{w} E(S_l) \geq d_k + r_k + 1 + \sum_{l=k+1}^{w} r_l \\ &\geq \sum_{l=1}^{k-1} r_l + 1 + \sum_{l=k+1}^{w} r_l = n - r_k + 1 \\ &> n - \max\{r, 2\}, \end{aligned}$$

which is also a contradiction. ■

Remark 1: For each pattern $\mathscr{P}$ whose complement graph $\mathscr{P}^c$ is acyclic[2], Theorem 7 implies that $\mathscr{P}$ is stably expandable as long as $|\mathscr{P}^c| \leq n-2$. This bound is tight. The $n \times n$ pattern

$$\mathscr{P} = \begin{bmatrix} * & \ldots & * & * & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ * & \ldots & * & * & 0 \\ * & \ldots & * & * & * \\ * & \ldots & * & * & 0 \end{bmatrix}$$

has an acyclic $\mathscr{P}^c$ with $|\mathscr{P}^c| = n - 1$, but the corresponding feasible region $\mathcal{D}$ for the case $A = 0$ is not connected. The reason is that the stable matrices

$$\begin{bmatrix} -1 & & & & \\ & \ddots & & & \\ & & -1 & & \\ & & & -1 & -1 \\ & & & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} -1 & & & & \\ & \ddots & & & \\ & & -1 & & \\ & & & -1 & 1 \\ & & & -1 & 0 \end{bmatrix}$$

conform with the pattern $\mathscr{P}$, while any path within $\mathscr{P}$ between these matrices must pass through an unstable matrix whose last column is all zero.

Remark 2: For random patterns subject to the distribution given by (6) with a parameter $c < 1$, [26] has proven that with high probability the largest strongly connected component of $\mathscr{P}^c$ is of size $O(\log n)$. Therefore, as long as n is sufficiently large such that

$$|\mathscr{P}^c| \approx cn < n - O(\log n),$$

Theorem 7 implies that the feasible region $\mathcal{D}$ associated with the pattern is connected. This explains why most patterns in Fig. 3(a) with $c < 1$ are certified.

B. Constructing Patterns with Connected Feasible Regions

According to the numerical result in Fig. 3(b), Algorithm 1 cannot certify the connectivity for many sparse patterns. This is not surprising since the complement graphs of most of these patterns are strongly connected and cannot be decomposed as required in Corollary 4. However, using Theorem 3, one can still construct an exponential class of desirable patterns with

---

[2] Here, we allow acyclic graphs to contain self-loops.

up to approximately half of entries being “0”. The construction procedure is provided in Algorithm 2 below.

Algorithm 2 (Generating Patterns with Connected Feasible Regions):

```
$\mathscr{P} \leftarrow \mathrm{diag}(*, *)$.
for $i \leftarrow 3$ to $n$ do
    Add one row and column at the bottom and right side of $\mathscr{P}$.
    Fill the newly added entries with “$*$”.
    Choose at most $i - 3$ elements from $\{1, \dots, i - 1\}$.
    For each chosen element $j$, set $\mathscr{P}_{ji} \leftarrow 0$.
    Optionally set $\mathscr{P}_{ii} \leftarrow 0$.
    Optionally set $\mathscr{P} \leftarrow \mathscr{P}^T$.
end for
```

The next theorem shows that the number of favorable patterns generated by Algorithm 2 is roughly the square root of $2^{n\times n}$ or the total number of possible patterns. Although the constructed patterns only account for some proportion of all patterns, they are abundant enough to be used for approximations of other ODC problems.

Theorem 8: Given a system with an arbitrary matrix $A \in \mathbb{R}^{n\times n}$ and $B = C = I_n$, there are at least $2^{n(n-1)/2-1}$ patterns whose corresponding feasible regions $\mathcal{D}$ are connected.

Proof: Let $f(n)$ be the number of different $n \times n$ patterns that can be generated by Algorithm 2. Note that $f(2) = 1$. For $n \geq 3$, the algorithm essentially allows us to arbitrarily choose “0” or “$*$” for the entries $\mathscr{P}_{1n}, \dots, \mathscr{P}_{(n-1)n}$, except the cases that all these entries are “0” or only one of them is “$*$”. Based on this observation, one can write

$$f(n) = f(n-1) \times (2^{n-1} - 1 - (n-1)) \times 2 \times 2,$$

where the second last “2” counts for the choice of $\mathscr{P}_{nn}$ and the last “2” counts for the choice of using a transpose. Since

$$2^{n-1} - 1 - (n-1) \geq 2^{n-3}, \quad \forall n \geq 3,$$

it holds that

$$f(n) \geq f(n-1) \times 2^{n-1}, \quad \forall n \geq 3.$$

By induction, $f(n) \geq 2^{n(n-1)/2-1}$. ■

V. CONNECTIVITY PROPERTIES OF DYNAMIC ODC PROBLEMS

As discussed in the preceding sections, the feasible region of static ODC problems may or may not be connected. It turns out that this is due to using static controllers. In this section, we aim to show that dynamic ODC problems have a highly desirable connectivity property. However, such connectivity property is not directly possessed by the original feasible region of the dynamic ODC problem, i.e., the set $\mathcal{H}_s$ of all stabilizing decentralized controllers. Instead, we first parametrize the feasible region into a parameter space on which optimal control problems can be equivalently recast. The formal definition of a parameterization is stated in Definition 2, and the constructed parameterization is given in Proposition 10 which is based on the Youla parameterization. Finally, we embed the parameter space into a larger metric space and prove in Theorem 11 and Theorem 13 that its closure is connected under mild conditions. This unique feature of dynamic ODC problems directly affects the performance of local search methods.

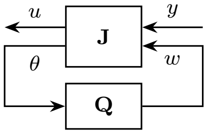


Fig. 5. The Youla parameterization.

A. Parameterization of Dynamic Stabilizing Controllers

The constraint on the feasible region $\mathcal{H}_s$ is that the decentralized controller must stabilize the system, which is difficult to analyze because it depends on both the controller and the system. To simplify the structure of $\mathcal{H}_s$, it is beneficial to first parametrize this set based on the Youla parameterization, which transforms the original constraint to a new one on the (internal) stability of some controller in the parameter space. The latter only depends on the controller itself and is easier to handle. We start with a general definition.

Definition 2: A set $\mathcal{S}$ together with a mapping

$$f : \mathcal{S} \to \mathcal{H}_s$$

is called a parameterization of $\mathcal{H}_s$ if f has the following property: for any controller $\mathbf{H} \in \mathcal{H}_s$, there exists a point $\mathbf{Q} \in \mathcal{S}$ such that $f(\mathbf{Q})$ and $\mathbf{H}$ have the same transfer function.

In other words, every controller in $\mathcal{H}_s$ is the image of at least one point in $\mathcal{S}$ under the parameterization mapping f (up to the equivalence of having the same transfer function). It is not important that multiple points in $\mathcal{S}$ can be mapped to the same controller in $\mathcal{H}_s$ since, for example, any transfer function could have many time-domain representations obtained via a similarity transformation. The intuition behind the above definition is that when solving an ODC problem one can optimize over the parameter set $\mathcal{S}$ instead of directly optimizing over the set $\mathcal{H}_s$ of controllers.

Theorem 9 (Youla parameterization): Assume the system (1) is stabilizable and detectable. Let F and L be arbitrary matrices such that $A + BF$ and $A + LC$ are stable. For any controller $\mathbf{Q}$ for the system (1), define $g(\mathbf{Q}, F, L)$ to be the controller described by the block diagram in Fig. 5, in which the block $\mathbf{J}$ is given by

$$\begin{aligned}
\dot{\hat{x}}(t) &= (A + BF + LC)\hat{x}(t) - Ly(t) + Bw(t), \\
u(t) &= F\hat{x}(t) + w(t), \\
\theta(t) &= -C\hat{x}(t) + y(t).
\end{aligned}$$

Then, the mapping $g(\cdot, F, L)$ has the following properties:

1) For any stable controller $\mathbf{Q}$, $g(\mathbf{Q}, F, L)$ stabilizes the system (1).
2) For any stabilizing controller $\mathbf{H}$ for the system (1), there exists some stable controller $\mathbf{Q}$ such that $\mathbf{H}$ and $g(\mathbf{Q}, F, L)$ have the same transfer function.

Proof: See [27, Thm. 12.8]. ■

In the following, we write a decentralized controller $\mathbf{H} \in \mathcal{H}$ as $\mathbf{H} = (\bar{\mathbf{H}}, \mathbf{H}_r)$ with the partial controller $\bar{\mathbf{H}}$ defined as

$$\bar{\mathbf{H}} = (\mathbf{H}_1, \ldots, \mathbf{H}_{r-1}).$$

Similar to Example 2, after applying the partial controller $\bar{\mathbf{H}}$ to the original system (4), one can obtain a partially closed-loop system with the input $u_r(t)$ and the output $y_r(t)$, which will be denoted as $\mathbf{G}(\bar{\mathbf{H}})$, and the corresponding matrices of this system will be called $A(\bar{\mathbf{H}})$, $B(\bar{\mathbf{H}})$ and $C(\bar{\mathbf{H}})$, respectively. Note that a decentralized controller $\mathbf{H}$ stabilizes the original system if and only if $\mathbf{G}(\bar{\mathbf{H}})$ is stabilizable and detectable while the last local controller $\mathbf{H}_r$ stabilizes $\mathbf{G}(\bar{\mathbf{H}})$. In the next proposition, the Youla parameterization is applied based on the latter viewpoint instead of using the Youla parameterization on the entire decentralized controller, since otherwise the diagonal structure constraint on the decentralized controller cannot be nicely transferred to an equivalent constraint in a simple form on the parameter space.

Proposition 10: Let $\mathcal{Q}$ be the set of all decentralized controllers $(\bar{\mathbf{H}}, \mathbf{Q}) \in \mathcal{H}$ such that $\mathbf{Q}$ is a stable controller and the partially closed-loop system $\mathbf{G}(\bar{\mathbf{H}})$ is both stabilizable and detectable. Then, there exists a mapping $f : \mathcal{Q} \to \mathcal{H}_s$ being a parameterization for the set $\mathcal{H}_s$.

Proof: For each $\bar{\mathbf{H}}$ whose corresponding partially closed-loop system $\mathbf{G}(\bar{\mathbf{H}})$ is stabilizable and detectable, choose $F(\bar{\mathbf{H}})$ and $L(\bar{\mathbf{H}})$ such that both $A(\bar{\mathbf{H}}) + B(\bar{\mathbf{H}})F(\bar{\mathbf{H}})$ and $A(\bar{\mathbf{H}}) + L(\bar{\mathbf{H}})C(\bar{\mathbf{H}})$ are stable. For each $(\bar{\mathbf{H}}, \mathbf{Q}) \in \mathcal{Q}$, define

$$f(\bar{\mathbf{H}}, \mathbf{Q}) = (\bar{\mathbf{H}}, g(\mathbf{Q}, F(\bar{\mathbf{H}}), L(\bar{\mathbf{H}}))).$$

By Theorem 9, $g(\mathbf{Q}, F(\bar{\mathbf{H}}), L(\bar{\mathbf{H}}))$ stabilizes the system $\mathbf{G}(\bar{\mathbf{H}})$ and thus $f(\bar{\mathbf{H}}, \mathbf{Q}) \in \mathcal{H}_s$.

To show that the mapping f is indeed a parameterization, consider any $(\bar{\mathbf{H}}, \mathbf{H}_r) \in \mathcal{H}_s$. Then, $\mathbf{H}_r$ stabilizes the system $\mathbf{G}(\bar{\mathbf{H}})$ and thus $\mathbf{G}(\bar{\mathbf{H}})$ is both stabilizable and detectable. Moreover, by Theorem 9 again, there exists some stable controller $\mathbf{Q}$ such that $\mathbf{H}_r$ and $g(\mathbf{Q}, F(\bar{\mathbf{H}}), L(\bar{\mathbf{H}}))$ have the same transfer function, which further implies that $(\bar{\mathbf{H}}, \mathbf{H}_r)$ and $f(\bar{\mathbf{H}}, \mathbf{Q})$ also have the same transfer function. ∎

After Proposition 10 is applied, a dynamic ODC problem can be equivalently formulated as an optimization problem over the parameter set $\mathcal{Q}$. Similar to the ODC problem with static controllers, the converted optimization problem may be solved effectively using local search methods if $\mathcal{Q}$ is a connected set.

To study the connectivity properties of the set $\mathcal{Q}$, it is necessary to introduce some topology on this set. In the following, we will first define a metric on the superset $\mathcal{H}$ and view $\mathcal{Q}$ as a subspace. Because $\mathcal{H}$ contains decentralized controllers of arbitrarily high degrees, the metric must be defined in some special way to handle controllers of different degrees. For two decentralized controllers $\mathbf{H}, \mathbf{H}' \in \mathcal{H}$ with

$$\mathbf{H} = (\mathbf{H}_1, \ldots, \mathbf{H}_r), \quad \mathbf{H}_i = (\tilde{A}_i, \tilde{B}_i, \tilde{C}_i, \tilde{D}_i),$$
$$\mathbf{H}' = (\mathbf{H}'_1, \ldots, \mathbf{H}'_r), \quad \mathbf{H}'_i = (\tilde{A}'_i, \tilde{B}'_i, \tilde{C}'_i, \tilde{D}'_i),$$

define

$$\rho(\mathbf{H}, \mathbf{H}') = \max_{i=1,\ldots,r} \max(\|\hat{A}_i - \hat{A}'_i\|_\infty, \|\hat{B}_i - \hat{B}'_i\|_\infty, \|\hat{C}_i - \hat{C}'_i\|_\infty, \|\hat{D}_i - \hat{D}'_i\|_\infty),$$

where $\hat{A}_i$, $\hat{A}'_i$, $\hat{B}_i$, $\hat{B}'_i$, $\hat{C}_i$, $\hat{C}'_i$, $\hat{D}_i$, $\hat{D}'_i$ are infinite matrices obtained from padding zeros on the right and below sides of A_i, A'_i, B_i, B'_i, C_i, C'_i, D_i, D'_i, respectively. This enables computing the norm between two matrices of different sizes. Here, other norms can be also used, and the choice of norm will not affect the connectivity properties.

B. Connectivity of the Parameter Space

As mentioned above, it is favorable to have a dynamic ODC problem with a connected feasible set $\mathcal{Q}$ in the parametrized space. Unfortunately, the set $\mathcal{Q}$ is not always connected due to the existence of some partial controller $\bar{\mathbf{H}}$ whose corresponding partially closed-loop system $\mathbf{G}(\bar{\mathbf{H}})$ is not stabilizable or detectable. However, under some mild conditions on the system (4), one can prove that the closure of $\mathcal{Q}$ is always connected. This weaker connectivity property is still useful for many local search methods, such as the stochastic gradient method, because in this case the infeasible region is infinitesimally small and the introduced noise in the algorithm would allow the trajectory to cross different connected components freely.

Theorem 11: Assume that for any nonnegative integer n_0 there exists a partial controller $\bar{\mathbf{H}}^* = (\mathbf{H}^*_1, \ldots, \mathbf{H}^*_{r-1})$ such that the degree of each local controller $\mathbf{H}^*_i$ is at least n_0 and that the partially closed-loop system $\mathbf{G}(\bar{\mathbf{H}}^*)$ is both controllable and observable. Then, the closure of the feasible set $\mathcal{Q}$ of the dynamic ODC problem is connected.

Before proceeding with the proof, we need to state a basic topological property.

Lemma 12: Given a topological space $\mathcal{T}$ and one of its dense subspace $\mathcal{S}$, if any two points in $\mathcal{S}$ can be connected by a path in $\mathcal{T}$, then $\mathcal{T}$ is connected.

Proof: Select $x \in \mathcal{S}$ and let $\mathcal{S}'$ be the set of points in $\mathcal{T}$ reachable from x using paths in $\mathcal{T}$. By definition, $\mathcal{S} \subseteq \mathcal{S}'$ and $\mathcal{S}'$ is path-connected. Being a closure of $\mathcal{S}'$, $\mathcal{T}$ must be connected. ∎

Proof of Theorem 11: By Lemma 12, we only need to prove that any two controllers

$$\mathbf{H} = (\mathbf{H}_1, \ldots, \mathbf{H}_{r-1}, \mathbf{Q}) \in \mathcal{Q},$$
$$\mathbf{H}' = (\mathbf{H}'_1, \ldots, \mathbf{H}'_{r-1}, \mathbf{Q}') \in \mathcal{Q}$$

can be connected by a path in the closure of $\mathcal{Q}$. Let n_0 be the largest degree of all the local controllers $\mathbf{H}_i$ and $\mathbf{H}'_i$ for $i = 1, \ldots, r-1$, and $\bar{\mathbf{H}}^*$ be the corresponding partial controller given by the assumption of the theorem. First, we consider the special case in which

1) $\mathbf{H}_i$, $\mathbf{H}'_i$ and $\mathbf{H}^*_i$ have the same degree for all $i = 1, \ldots, r-1$,
2) $\mathbf{Q}$ and $\mathbf{Q}'$ have the same degree.

Let

$$\mathbf{Q} = (\tilde{A}_r, \tilde{B}_r, \tilde{C}_r, \tilde{D}_r), \quad \mathbf{Q}' = (\tilde{A}'_r, \tilde{B}'_r, \tilde{C}'_r, \tilde{D}'_r).$$

Since the set of stable matrices is connected by Proposition 2, a path $\tilde{A}_r(\tau)$ between $\tilde{A}_r$ and $\tilde{A}'_r$ can be found such that $\tilde{A}_r(\tau)$ is a stable matrix when $0 < \tau < 1$. We connect the remaining matrices $\tilde{B}_r$ and $\tilde{B}'_r$, $\tilde{C}_r$ and $\tilde{C}'_r$, $\tilde{D}_r$ and $\tilde{D}'_r$ arbitrarily to obtain the paths $\tilde{B}_r(\tau)$, $\tilde{C}_r(\tau)$, $\tilde{D}_r(\tau)$, respectively. Then,

$$\mathbf{Q}(\tau) = (\tilde{A}_r(\tau), \tilde{B}_r(\tau), \tilde{C}_r(\tau), \tilde{D}_r(\tau))$$

is a path of stable local controllers between $\mathbf{Q}$ and $\mathbf{Q}'$.

Similarly, one can expand

$$\mathbf{H}_i = (\tilde{A}_i, \tilde{B}_i, \tilde{C}_i, \tilde{D}_i), \quad \mathbf{H}'_i = (\tilde{A}'_i, \tilde{B}'_i, \tilde{C}'_i, \tilde{D}'_i),$$
$$\mathbf{H}^*_i = (\tilde{A}^*_i, \tilde{B}^*_i, \tilde{C}^*_i, \tilde{D}^*_i).$$

For each $i = 1, \ldots, r-1$, define a path $\tilde{A}_i(\tau)$ of matrices passing through the three matrices $\tilde{A}_i$, $\tilde{A}^*_i$ and $\tilde{A}'_i$ as follows:

$$\tilde{A}_i(\tau) = 2(1-\tau)(1/2-\tau)\tilde{A}_i + 4\tau(1-\tau)\tilde{A}^*_i + 2\tau(\tau - 1/2)\tilde{A}'_i.$$

Note that the above path is allowed to contain unstable matrices. Define $\tilde{B}_i(\tau)$, $\tilde{C}_i(\tau)$ and $\tilde{D}_i(\tau)$ similarly, which give rise to the path of local controllers

$$\mathbf{H}_i(\tau) = (\tilde{A}_i(\tau), \tilde{B}_i(\tau), \tilde{C}_i(\tau), \tilde{D}_i(\tau))$$

with $\mathbf{H}_i(1/2) = \mathbf{H}^*_i$. Let $\bar{\mathbf{H}}(\tau) = (\mathbf{H}_1(\tau), \ldots, \mathbf{H}_{r-1}(\tau))$. The next step is to prove that the partially closed-loop system $\mathbf{G}(\bar{\mathbf{H}}(\tau))$ is controllable and observable except at a finite number of points τ.

Let $A(\bar{\mathbf{H}}^*), B(\bar{\mathbf{H}}^*), C(\bar{\mathbf{H}}^*)$ be the matrices describing the system $\mathbf{G}(\bar{\mathbf{H}}^*)$, and $A(\bar{\mathbf{H}}(\tau)), B(\bar{\mathbf{H}}(\tau)), C(\bar{\mathbf{H}}(\tau))$ be the matrices for the system $\mathbf{G}(\bar{\mathbf{H}}(\tau))$. By assumption, the pair $(A(\bar{\mathbf{H}}^*), B(\bar{\mathbf{H}}^*))$ is controllable, and therefore there exist some matrix M and vector v such that the single-input system $(A(\bar{\mathbf{H}}^*) + B(\bar{\mathbf{H}}^*)M, B(\bar{\mathbf{H}}^*)v)$ is controllable [27, Sec. 3.4]. Note that the pair $(A(\bar{\mathbf{H}}(\tau)) + B(\bar{\mathbf{H}}(\tau))M, B(\bar{\mathbf{H}}(\tau))v)$ is controllable if and only if the determinant of the corresponding controllability matrix is nonzero. Since this determinant is a polynomial of τ that is nonzero at $\tau = 1/2$, it is always nonzero except at a finite number of points $\tau_1, \ldots, \tau_s$. Hence, the pair $(A(\bar{\mathbf{H}}(\tau)) + B(\bar{\mathbf{H}}(\tau))M, B(\bar{\mathbf{H}}(\tau))v)$ and thus the pair $(A(\bar{\mathbf{H}}(\tau)), B(\bar{\mathbf{H}}(\tau)))$ are controllable except at the above points. By the same argument, one can also prove that the pair $(A(\bar{\mathbf{H}}(\tau)), C(\bar{\mathbf{H}}(\tau)))$ is observable except at a finite number of points $\tau_{s+1}, \ldots, \tau_t$.

Finally, define $\mathbf{H}(\tau) = (\bar{\mathbf{H}}(\tau), \mathbf{Q}(\tau))$. This is a path from $\mathbf{H}$ to $\mathbf{H}'$ within the set $\mathcal{H}$. Furthermore, since $\mathbf{H}(\tau) \in \mathcal{Q}$ for all $0 \le \tau \le 1$ except when τ equals to $\tau_1, \ldots, \tau_t$, $\mathbf{H}(\tau)$ must belong to the closure of $\mathcal{Q}$ for every τ.

In the case when the degree of some local controller $\mathbf{H}_i$ is smaller than the degree of $\mathbf{H}^*_i$ or the degree of $\mathbf{Q}$ is smaller than the degree of $\mathbf{Q}'$, one can first promote the degree of $\mathbf{H}_i$ or $\mathbf{Q}$ by padding zeros in the corresponding matrices such that the new local controller has the same degree of $\mathbf{H}^*_i$ or $\mathbf{Q}'$. Let $\hat{\mathbf{H}}$ be the controller obtained by replacing the local controllers in $\mathbf{H}$ with these new local controllers with promoted degrees, and let $\hat{\mathbf{H}}(\tau)$ be the path starting from $\hat{\mathbf{H}}$ constructed by the procedure above. Then,

$$\mathbf{H}(\tau) = \begin{cases} \mathbf{H}, & \text{if } \tau = 0, \\ \hat{\mathbf{H}}(\tau), & \text{if } 0 < \tau \le 1 \end{cases}$$

is continuous and thus the desired path from the original controller $\mathbf{H}$ since $\rho(\mathbf{H}, \hat{\mathbf{H}}) = 0$. The cases when the degrees of $\mathbf{H}'_i$ or $\mathbf{Q}'$ are smaller can be handled similarly. ■

Remark 3: In practice, the degree of controllers in the parameter space $\mathcal{Q}$ should be limited, say at most m_0, to make the search space finite-dimensional. Assume that $\mathbf{H}$ is the initial point in the search space and $\mathbf{H}'$ is the optimal one. Since the degree of each controller in the path between $\mathbf{H}$ and $\mathbf{H}'$ constructed in the proof of Theorem 11 is upper bounded, this path is still within the limited finite-dimensional search space as long as m_0 is sufficiently large, which means that local search algorithms can still possibly find the optimal solution without leaving the closure of the feasible region.

Theorem 11 presumes the existence of partial controllers with arbitrarily high degrees that make the partially closed-loop system controllable and observable. The next theorem shows that such an assumption will be further implied by certain mild conditions. Here, we define the structural graph of the system to be a direct graph with r vertices that has edge (i, j) if the transfer function from input u_i to output y_j is not constantly zero. The system is said to be *strongly connected* if its structural graph is strongly connected [28].

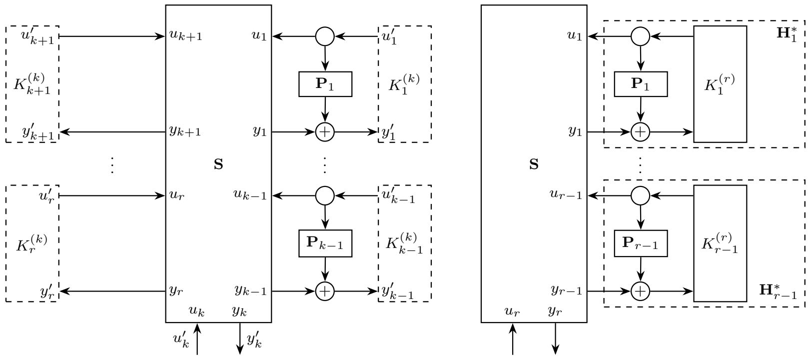
Theorem 13: The assumption in Theorem 11 holds if the system is strongly connected and does not have decentralized fixed modes.

For systems satisfying the conditions in Theorem 13, [29, Theorem 4] proves that it is relatively easy to find a partial controller $(\mathbf{H}^*_1, \ldots, \mathbf{H}^*_{r-1})$ guaranteeing the controllability and observability of the partially closed-loop system with each local controller $\mathbf{H}^*_i$ being a static feedback controller. Our Theorem 13 can be regarded as a generalization of this result to higher degrees. The proof of Theorem 13 also depends on the aforementioned result, which is stated below for convenience.

Lemma 14 ([29, Theorem 4]): Assume that the system is strongly connected and does not have decentralized fixed modes. Given a partial controller $\bar{\mathbf{H}}^* = (\mathbf{H}^*_1, \ldots, \mathbf{H}^*_{r-1})$, if each local controller $\mathbf{H}^*_i$ is a static controller described by a matrix K_i, then the partially closed-loop system $\mathbf{G}(\bar{\mathbf{H}}^*)$ is controllable and observable unless the matrices K_i belong to some low-dimensional hypersurfaces.

Proof of Theorem 13: To construct a partial controller $(\mathbf{H}^*_1, \ldots, \mathbf{H}^*_{r-1})$ that makes the partially closed-loop system controllable and observable with the degree of $\mathbf{H}^*_i$ being n_i, we will design a suitable sub-controller $\mathbf{P}_k$ for $k = 1, \ldots, r-1$. Define $\mathbf{S}_k$ for $k = 1, \ldots, r$ to be the system obtained by parallelly connecting the corresponding input and output of the original system $\mathbf{S}$ and the sub-controllers $\mathbf{P}_1, \ldots, \mathbf{P}_{k-1}$ as shown in Fig. 6(a). In what follows, we will prove by induction that each system $\mathbf{S}_k$ is strongly connected and does not have decentralized fixed modes. The choice of $\mathbf{P}_k$ will also be described in the corresponding inductive step.

The base step $k = 1$ is directly from the assumption since $\mathbf{S}_1$ is the same as the original system $\mathbf{S}$. For the inductive step, given a system $\mathbf{S}_k$ that is already strongly connected and without decentralized fixed modes, one can apply Lemma 14 (on the subsystem with the input u'_k and the output y'_k instead of u'_r and y'_r as said in the statement of Lemma 14) to obtain



(a) (b)

Fig. 6. (a) The system $\mathbf{S}_k$ with inputs $u'_1, \ldots, u'_r$ and outputs $y'_1, \ldots, y'_r$ (drawn solely in solid lines) and the partially closed-loop system $\mathbf{S}_k(\mathbf{K}^{(k)})$ (drawn in both solid and dashed lines) used in the proof of Theorem 13. (b) The final partial controller $\bar{\mathbf{H}}^* = (\mathbf{H}^*_1, \ldots, \mathbf{H}^*_{r-1})$ for the original system $\mathbf{S}$ constructed by the proof of Theorem 13.

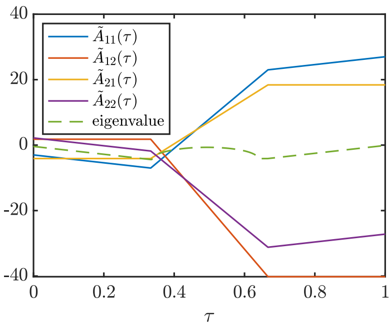


Fig. 7. A stable path between the two matrices $\tilde{A}(0)$ and $\tilde{A}(1)$ in Example 3. The curve labelled "eigenvalue" shows the maximum real part of the eigenvalues for the corresponding matrices on the path.

a partial controller

$$\mathbf{K}^{(k)} = (K^{(k)}_1, \ldots, K^{(k)}_{k-1}, K^{(k)}_{k+1}, \ldots, K^{(k)}_r)$$

such that each local controller $K^{(k)}_i$ is static and the corresponding partially closed-loop system $\mathbf{S}_k(\mathbf{K}^{(k)})$ with the input u'_k and the output y'_k, as shown in Fig. 6(a) again, is both controllable and observable. Next, choose an arbitrary strictly proper sub-controller $\mathbf{P}_k$ of degree n_k, whose input has the same dimension as u'_k and output has the same dimension as y'_k, such that:

1) $\mathbf{P}_k$ is both controllable and observable.
2) $\mathbf{S}_k(\mathbf{K}^{(k)})$ and $\mathbf{P}_k$ do not have common modes.

Now, consider the partially closed-loop system $\mathbf{S}_{k+1}(\mathbf{K}^{(k)})$ obtained by applying the above partial controller $\mathbf{K}^{(k)}$ to the system $\mathbf{S}_{k+1}$, which can be viewed as the parallel combination of $\mathbf{S}_k(\mathbf{K}^{(k)})$ and $\mathbf{P}_k$. A standard application of the Popov–Belevitch–Hautus test shows that the parallel combination of two controllable and observable systems without common modes is still controllable and observable. Therefore, one can find a dynamic controller $\mathbf{Q}$ for the partially closed-loop system $\mathbf{S}_{k+1}(\mathbf{K}^{(k)})$, or equivalently find a decentralized controller

$$(K^{(k)}_1, \ldots, K^{(k)}_{k-1}, \mathbf{Q}, K^{(k)}_{k+1}, \ldots, K^{(k)}_r)$$

for the system $\mathbf{S}_{k+1}$ to freely assign the poles of the result closed-loop system. This means that the system $\mathbf{S}_{k+1}$ does not have decentralized fixed modes. Moreover, since the off-diagonal entries of the transfer matrix of $\mathbf{S}_{k+1}$ and those of the original system $\mathbf{S}$ are the same, $\mathbf{S}_{k+1}$ is also strongly connected, which completes the induction.

Finally, we arrive at the system $\mathbf{S}_r$ satisfying the assumption of Lemma 14. We then apply Lemma 14 to $\mathbf{S}_r$ again to find a static partial controller

$$\mathbf{K}^{(r)} = (K^{(r)}_1, \ldots, K^{(r)}_{r-1})$$

ensuring the controllability and observability of the partially closed-loop system $\mathbf{S}_r(\mathbf{K}^{(r)})$. For every $i = 1, \ldots, r-1$, define a local controller $\mathbf{H}^*_i$ by combining the sub-controller $\mathbf{P}_i$ of degree n_i and the static controller $K^{(r)}_i$ as shown in Fig. 6(b). Then, $\bar{\mathbf{H}}^* = (\mathbf{H}^*_1, \ldots, \mathbf{H}^*_{r-1})$ is the desired partial controller. Each local controller $\mathbf{H}^*_i$ has the desired degree n_i, while the partially closed-loop system $\mathbf{S}(\bar{\mathbf{H}}^*)$ is the same as $\mathbf{S}_r(\mathbf{K}^{(r)})$ that is known to be controllable and observable. ■

Example 3: To demonstrate the ideas developed in this section, we will revisit Example 2 and illustrate how the introduction of dynamic controllers can solve the connectivity issue in the static ODC problems. Assume that the goal is to find a path between the static controllers

$$K(0) = \mathrm{diag}(-3, 2), \quad K(1) = \mathrm{diag}(12, -40).$$

As shown in Fig. 4, this is impossible because the above two static controllers are in different connected components of the feasible region of the static ODC problem. In the dynamic ODC problem, we can first compute the Youla parameterization for two endpoints and then convert them to the parameter space. For the controller $K(0)$, consider the partially closed-loop system with the input u_2 and the output y_2, and denote its matrices as $(\bar{A}(0), \bar{B}(0), \bar{C}(0))$. According to Theorem 9, we need to find the stabilizing matrices $F(0)$ and $L(0)$ for this system. For example, the following ones can be used:

$$F(0) = \begin{bmatrix} 4 & -4 \end{bmatrix}, \quad L(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

After the Youla parameterization, the corresponding controller in the parameter space is given by $\mathbf{H}(0) = (\mathbf{H}_1(0), \mathbf{Q}(0))$, where $\mathbf{H}_1(0) = -K(0)_{11} = 3$ is a static feedback controller with a scalar gain (the sign difference here is due to the convention of using negative feedback in static ODC problems) and $\mathbf{Q}(0) = (\tilde{A}(0), \tilde{B}(0), \tilde{C}(0), \tilde{D}(0))$ is a dynamic controller with the matrices

$$\begin{aligned} \tilde{A}(0) &= \bar{A}(0) - \bar{B}(0)K(0)_{22}\bar{C}(0) = \begin{bmatrix} -3 & 1.8 \\ -4.1 & 2.2 \end{bmatrix}, \\ \tilde{D}(0) &= -K(0)_{22} = -2, \\ \tilde{B}(0) &= \bar{B}(0)D(0) - L(0) = \begin{bmatrix} 1 \\ 2.4 \end{bmatrix}, \\ \tilde{C}(0) &= D(0)\bar{C}(0) - F(0) = \begin{bmatrix} -4 & 2 \end{bmatrix}. \end{aligned}$$

Note that $\tilde{A}(0)$ is exactly the matrix of the closed-loop system under the controller $K(0)$. Similar computation can also be worked out for $K(1)$, which gives

$$\begin{aligned} &F(1) = \begin{bmatrix} 20 & 12 \end{bmatrix}, \quad L(1) = \begin{bmatrix} -55 \\ -38 \end{bmatrix}, \quad \mathbf{H}_1(1) = -12, \\ &\tilde{A}(1) = \begin{bmatrix} 27 & -40.2 \\ 18.4 & -27.2 \end{bmatrix}, \quad \tilde{B}(1) = \begin{bmatrix} 15 \\ 10 \end{bmatrix}, \\ &\tilde{C}(1) = \begin{bmatrix} -20 & 28 \end{bmatrix}, \quad \tilde{D}(1) = 40. \end{aligned}$$

To find a path between $\mathbf{H}(0)$ and $\mathbf{H}(1)$, one can first connect $\tilde{A}(0)$ and $\tilde{A}(1)$ by a path within the set of stable matrices (Fig. 7 illustrates one possible choice for such a path), and then connect $\mathbf{H}_1, \tilde{B}, \tilde{C}, \tilde{D}$ by arbitrary paths such as linear functions. The final path $\mathbf{H}(\tau)$ is within the parameter space except when $\mathbf{H}_1(\tau)$ is -10.42 or 0.27 corresponding to the cases the partially closed-loop system is not controllable or observable as computed in Example 2.

VI. CONCLUSION

In this paper, we studied the connectivity properties of the feasible regions of ODC problems. For problems with static controllers and identity B and C matrices, after introducing the notion of stable expandability, we developed a novel criterion together with an efficient algorithm to certify the connectivity of the feasible region of a given ODC problem. Subsequently, we proved that the feasible region of the ODC problem is connected for most dense communication patterns as well as an exponential class of patterns. In the presence of dynamic controllers, we proved that the closure of the feasible region is connected in some metric space for dynamic ODC problems under some mild conditions.

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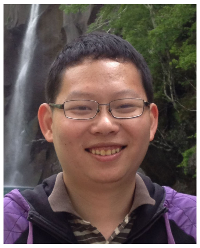

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APPENDIX
PROOF OF THE CONNECTIVITY CRITERION

We first recall some fundamental facts about stable matrices. Assume that $A \in \mathcal{K}_n$ is a stable matrix and $Q \in \mathbb{R}^{n\times n}$ is an arbitrary matrix. The standard Lyapunov equation

$$PA + A^T P = -Q \tag{10}$$

is equivalent to

$$(I_n \otimes A^T + A^T \otimes I_n)\operatorname{vec} P = -\operatorname{vec} Q,$$

where $\operatorname{vec} X$ is the vector resulted from stacking all the columns of a matrix X, and $\otimes$ is the Kronecker product. Since A is stable,

$$I_n \otimes A^T + A^T \otimes I_n$$

is invertible (see [30, Sec. 12.11]) and thus the Lyapunov equation (10) has a unique solution $P = \Phi(A, Q)$ such that

$$\operatorname{vec}(\Phi(A, Q)) = -(I_n \otimes A^T + A^T \otimes I_n)^{-1}\operatorname{vec} Q.$$

The above equation implies that $\Phi(A, Q)$ continuously depends on the matrices A and Q. Furthermore, if Q is positive definite, then this unique solution $\Phi(A, Q)$ is also positive definite. Conversely, given a matrix $A \in \mathbb{R}^{n\times n}$, if there exist two positive definite matrices $P, Q \in \mathcal{P}_n$ satisfying the Lyapunov equation (10), then A is stable.

Lemma 15: A block matrix

$$\begin{matrix} n\left\{\vphantom{\begin{matrix}A\\A\end{matrix}}\right. \\ m\left\{\vphantom{\begin{matrix}C\\C\end{matrix}}\right. \\ \end{matrix}\begin{matrix}\left[\begin{array}{c:c} A & B \\ \hdashline C & D \end{array}\right] \\ \underbrace{\hphantom{AA}}_{n}\underbrace{\hphantom{BB}}_{m}\end{matrix}$$

is stable if and only if there exist $Q \in \mathbb{R}^{m\times n}$ and $R \in \mathcal{P}_m$ such that

1) $A' = A - BR^{-1}Q$ is stable.
2) $QB + RD + B^T Q^T + D^T R = -I_m$.
3) $C = \Gamma(A', B, D, Q, R)$, where

$$\begin{aligned} \Gamma(A', B, D, Q, R) = -R^{-1}(QA' - R^{-1}Q \\ - RDR^{-1}Q + B^T\Phi(A', I_n + Q^T(R^{-1})^2 Q)). \end{aligned} \tag{11}$$

Proof: The original matrix is stable if and only if there exists a positive definite matrix

$$\begin{bmatrix} P & Q^T \\ Q & R \end{bmatrix},$$

blocked in the same way as the original matrix, satisfying the Lyapunov equation:

$$\begin{bmatrix} P & Q^T \\ Q & R \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix} + \begin{bmatrix} A^T & C^T \\ B^T & D^T \end{bmatrix}\begin{bmatrix} P & Q^T \\ Q & R \end{bmatrix} = -I_{n+m}.$$

Using the property of the Schur complement, the above condition is equivalent to the existence of $P \in \mathcal{S}_n$, $Q \in \mathbb{R}^{m\times n}$ and $R \in \mathcal{P}_m$ satisfying

$$PA + Q^T C + A^T P + C^T Q = -I_n, \tag{12a}$$

$$QB + RD + B^T Q^T + D^T R = -I_m, \tag{12b}$$

$$QA + RC + B^T P + D^T Q = 0, \tag{12c}$$

$$P - Q^T R^{-1} Q \in \mathcal{P}_n. \tag{12d}$$

The equation (12b) is the same as Condition 2 in the statement of the lemma. Let $A' = A - BR^{-1}Q$ and $P' = P - Q^T R^{-1} Q$. Then,

$$\begin{aligned} P'A' + (A')^T P' &= PA + A^T P \\ &+ Q^T C + C^T Q - Q^T (R^{-1})^2 Q \\ &- Q^T R^{-1}(QA + RC + B^T P + D^T Q) \\ &- (A^T Q^T + C^T R + PB + Q^T D)R^{-1}Q \\ &+ Q^T R^{-1}(I_m + QB + RD + B^T Q^T + D^T R)R^{-1}Q. \end{aligned}$$

Therefore, given (12b) and (12c), equation (12a) is equivalent to

$$P'A' + (A')^T P' = -I_n - Q^T(R^{-1})^2 Q.$$

The above equation and (12d) are further equivalent to the stability of A' and

$$P' = \Phi(A', I_n + Q^T(R^{-1})^2 Q).$$

In addition, equation (12c) can be rewritten as

$$\begin{aligned} C &= -R^{-1}(QA + B^T P + D^T Q) \\ &= -R^{-1}(QA' + QBR^{-1}Q + B^T P' \\ &\quad + B^T Q^T R^{-1} Q + D^T Q) \\ &= -R^{-1}(QA' - R^{-1}Q - RDR^{-1}Q + B^T P') \\ &= -R^{-1}(QA' - R^{-1}Q - RDR^{-1}Q \\ &\quad + B^T \Phi(A', I_n + Q^T(R^{-1})^2 Q)), \end{aligned}$$

which leads to Condition 3 of the lemma. ■

Proof of Theorem 3: Without loss of generality, assume that the first $q + 1$ rows of $\mathscr{B}$ do not contain "0"s while the remaining rows of $\mathscr{B}$ and the pattern $\mathscr{D}$ are all "0"s.

Given stable matrices

$$K^0 = \begin{bmatrix} A^0 & B^0 \\ C^0 & D^0 \end{bmatrix}, \quad K^1 = \begin{bmatrix} A^1 & B^1 \\ C^1 & D^1 \end{bmatrix}$$

and functions $\gamma_{ij}(\tau)$ as in Definition 1, there exist $Q^0, Q^1 \in \mathbb{R}^{q\times(n-q)}$ and $R^0, R^1 \in \mathcal{P}_q$ satisfying the corresponding conditions in Lemma 15. It is desirable to expand $\gamma_{ij}(\tau)$ into a path $K(\tau)$ from K^0 to K^1 and construct additional paths $Q(\tau)$ and $R(\tau)$ with given endpoints at the same time such that the conditions in Lemma 15 are satisfied for $K(\tau)$, $Q(\tau)$ and $R(\tau)$ at every $\tau \in [0, 1]$.

Let $D(\tau)$ be the path from D^0 to D^1 whose components are completely determined by the functions $\gamma_{ij}(\tau)$ above, and let $R(\tau)$ be a path of positive definite matrices from R^0 to R^1 whose existence is guaranteed by the convexity of $\mathcal{P}_q$. Define

$$\varphi(\tau) = -I_q - R(\tau)D(\tau) - D^T(\tau)R(\tau).$$

Then, $\varphi(\tau)$ is a path of symmetric matrices with

$$\begin{aligned} \varphi(0) &= Q^0 B^0 + (B^0)^T (Q^0)^T, \\ \varphi(1) &= Q^1 B^1 + (B^1)^T (Q^1)^T. \end{aligned}$$

Let

$$\begin{aligned} \psi(\tau) = \frac{1}{2}(\varphi(\tau) + (1-\tau)Q^0 B^0 - (1-\tau)(B^0)^T (Q^0)^T \\ + \tau Q^1 B^1 - \tau (B^1)^T (Q^1)^T). \end{aligned}$$

The above path $\psi(\tau)$ satisfies

$$\psi(\tau) + \psi^T(\tau) = \varphi(\tau),$$
$$\psi(0) = Q^0 B^0, \quad \psi(1) = Q^1 B^1, \tag{13}$$

which suggests finding a path $Q(\tau)$ from Q^0 to Q^1 and a path $B(\tau)$ from B^0 to B^1 satisfying $Q(\tau)B(\tau) = \psi(\tau)$. After that, we have

$$\begin{aligned} &Q(\tau)B(\tau) + R(\tau)D(\tau) + B^T(\tau)Q^T(\tau) + D^T(\tau)R(\tau) \\ &\quad = \psi(\tau) + \psi^T(\tau) - I_q - \varphi(\tau) = -I_q. \end{aligned} \tag{14}$$

Denote B^0 as

$$B^0 = \begin{bmatrix} X^0 \\ Y^0 \end{bmatrix}$$

and Q^0 as

$$Q^0 = \begin{bmatrix} Z^0 & W^0 \end{bmatrix},$$

where X^0 is the submatrix of the first $q+1$ rows of B^0 and Z^0 is the submatrix of the first $q+1$ columns of Q^0. Similar notations can also be introduced for the corresponding submatrices of B^1 and Q^1. Now, we choose a path $Y(\tau)$ from Y^0 to Y^1 based on the given functions $\gamma_{ij}(\tau)$, and we choose $W(\tau)$ to be an arbitrary path from W^0 to W^1. Then, the question of finding the paths $Q(\tau)$ and $B(\tau)$ satisfying $Q(\tau)B(\tau) = \psi(\tau)$ is equivalent to finding a path $X(\tau)$ from X^0 to X^1 and another path $Z(\tau)$ from Z^0 to Z^1 satisfying

$$Z(\tau)X(\tau) = \psi(\tau) - W(\tau)Y(\tau). \tag{15}$$

We first consider the case in which both X^0 and X^1 are of full column rank. In this case, there exist invertible $(q+1)\times(q+1)$ matrices $\tilde{X}^0$ and $\tilde{X}^1$ with positive determinants whose columns except the last one are the same as X^0 and X^1, respectively. Since the set of matrices with positive determinants is path-connected, one can find a path $\tilde{X}(\tau)$ of invertible matrices from $\tilde{X}^0$ to $\tilde{X}^1$. Let $X(\tau)$ be the first q columns of $\tilde{X}(\tau)$ and $\phi(\tau) \in \mathbb{R}^{q\times(q+1)}$ be the path of the form

$$\phi(\tau) = \begin{bmatrix} \psi(\tau) - W(\tau)Y(\tau) & a(\tau) \end{bmatrix},$$

where $a(\tau) \in \mathbb{R}^{q\times 1}$ is an arbitrary path with $a(0)$ being the last column of $Z^0\tilde{X}^0$ and $a(1)$ being the last column of $Z^1\tilde{X}^1$. Now, we choose $Z(\tau) = \phi(\tau)\tilde{X}^{-1}(\tau)$. Then, $Z(\tau)$ is continuous and by (13) it holds that

$$\begin{aligned} &Z(0)\tilde{X}^0 = \phi(0) = \begin{bmatrix} \psi(0) - W^0Y^0 & a(0) \end{bmatrix} \\ &= \begin{bmatrix} Q^0B^0 - W^0Y^0 & a(0) \end{bmatrix} = \begin{bmatrix} Z^0X^0 & a(0) \end{bmatrix} = Z^0\tilde{X}^0. \end{aligned}$$

Therefore, $Z(0) = Z^0$ and similarly $Z(1) = Z^1$. Furthermore, since $Z(\tau)X(\tau)$ is the first q columns of $\phi(\tau)$, the equation (15) is satisfied. Combining the paths $X(\tau)$, $Y(\tau)$ and $Z(\tau)$, $W(\tau)$ leads to the desired paths $Q(\tau)$ and $B(\tau)$.

Let

$$A'^0 = A^0 - B^0(R^0)^{-1}Q^0, \quad A'^1 = A^1 - B^1(R^1)^{-1}Q^1.$$

For each $(i,j) \notin \mathscr{P}$ with $1 \le i,j \le n-q$, we also set

$$\gamma'_{ij}(\tau) = \gamma_{ij}(\tau) - b_{ij}(\tau),$$

where $b_{ij}(\tau)$ is the (i,j) component of $B(\tau)R^{-1}(\tau)Q(\tau)$. By the assumption that the pattern $\mathscr{A}$ is stably expandable, one can find a path $A'(\tau)$ of stable matrices from A'^0 to A'^1 whose components are coincided with the functions $\gamma'_{ij}(\tau)$. Then, we can choose

$$A(\tau) = A'(\tau) + B(\tau)R^{-1}(\tau)Q(\tau) \tag{16}$$

with $A'(\tau)$ guaranteed to be stable.

Finally, let

$$C(\tau) = \Gamma(A'(\tau), B(\tau), D(\tau), Q(\tau), R(\tau)), \tag{17}$$

where the function $\Gamma(\cdot)$ is defined in (11). Then, $C(\tau)$ is continuous and thus a path from C^0 to C^1. Combining the paths $A(\tau)$, $B(\tau)$, $C(\tau)$, $D(\tau)$, one can obtain the desired path $K(\tau)$ from K^0 to K^1. By (14), (16), (17) and Lemma 15, $K(\tau)$ must be stable.

In the above argument, we still need to handle the cases in which either X^0 or X^1 is not of full column rank. Assume that X^0 is not of full column rank. Since the set $\mathcal{K}_n$ of stable matrices is open, there exists $\epsilon > 0$ such that $\mathcal{B}(K^0, \epsilon) \subseteq \mathcal{K}_n$, where

$$\mathcal{B}(K^0, \epsilon) = \{P \in \mathbb{R}^{n\times n} | |P_{ij} - K^0_{ij}| < \epsilon, \, \forall 1 \le i,j \le n\}.$$

On the other hand, since the set of matrices of full column rank is a dense subset of $\mathbb{R}^{(q+1)\times q}$, there exists a matrix $X' \in \mathbb{R}^{(q+1)\times q}$ of full column rank whose components satisfy

$$|X'_{ij} - X^0_{ij}| < \epsilon, \quad \forall 1 \le i \le q+1, \, \forall 1 \le j \le q.$$

Next, we find $0 < s < 1$ such that

$$|\gamma_{ij}(\tau) - K^0_{ij}| < \epsilon, \quad \forall (i,j) \notin \mathscr{P}, \, \forall \tau \in [0,s].$$

Then, one can design a path $\bar{K}(\tau) : [0,s] \to \mathcal{B}(K^0, \epsilon)$ of stable matrices starting from K^0 such that its components are prescribed by the functions $\gamma_{ij}(\tau)$ and X' is the corresponding submatrix of $\bar{K}(s)$. Now, we replace K^0 by $\bar{K}(s)$ in the above path construction procedure. After obtaining the constructed path with the starting matrix $\bar{K}(s)$, we can concatenate it with the path $\bar{K}(\tau)$ here to obtain a path starting from K^0. The case in which X^1 is not of full column rank can be dealt with similarly. ■