On the Exponential Number of Connected Components for the Feasible Set of Optimal Decentralized Control Problems

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Abstract— **The optimal decentralized control (ODC) problem is known to be NP-hard and many sufficient tractability conditions have been derived in the literature for its convex reformulations or approximations. We delve into the computational complexity of the problem to better understand the root cause of the non-existence of efficient methods for solving ODC. We consider the design of a static decentralized controller and show that there is no polynomial upper bound on the number of connected components of the set of stabilizing decentralized controllers. In particular, we present a subclass of problems for which the number of connected components is exponential in the order of the system and, in particular, any point in each of these components is the unique solution of the ODC problem for some quadratic objective functional. The results of this paper have two implications. First, the recent effort in machine learning advocating the use of local search algorithms for nonconvex problems, which has also been successful for the optimal centralized control problem, fails to work for ODC since it needs an exponential number of initializations. Second, a reformulation of the problem through a smooth change of variables does not reduce the complexity since it maintains the number of connected components. On the positive side, we show in special cases that ODC may not be complex for structured systems, such as highly damped systems.**

I. Introduction

Classical state-space solutions to optimal centralized control problems do not scale well as the dimension increases [1]. Real-world controllers also have structural constraints, such as locality and delay. The optimal decentralized control problem (ODC) has been proposed in the literature to bridge this gap. On the one hand, ODC can have nonlinear optimal solutions even for linear systems and is NP-hard in the worst case [2], [3]. On the other hand, the existence of dynamic structured feedback laws is completely captured by the notion of fixed modes [4]. Furthermore, several works have discovered structural conditions on the system and/or the controller under which the ODC problem admits tractable solutions. The conditions include spatially invariance [5], partially nestedness [6], positiveness [7], and quadratic invariance [8]. More recently, the System Level Approach [9] has convexified structural constraints at the expense of working with many impulse response matrices. Promising approximation [10]–[12] and convex relaxation techniques [13]–[16] also exist in the literature.

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A recent line of research, initiated in the machine learning community, suggests using nonlinear programming methods based on local search for the optimal control problems [17]. Local search methods are well-studied for convex problems, and they normally come with optimality guarantees [14]. However, when the problem is non-convex, these methods may converge to a saddle point or a local minimum [18]. Local search algorithms are effective: (i) when they are initialized at a point close enough to the optimal solution, or (ii) when there is no spurious local optimum and it is possible to escape saddle points [19]–[22]. They have also been applied to instances of ODC to obtain approximate solutions [23], [24], but a question arises as to whether local search is effective for ODC.

In this paper, we prove that the chances of success for the global convergence of local search methods applied to a general ODC problem are theoretically slim. Specifically, we prove that the feasible set of the ODC problem in the static case, which includes all structured static controllers that stabilize the system, can be not only non-convex but also disconnected where the number of connected components grows exponentially in the order of the system. Since any point in the feasible set is the unique globally optimal solution of ODC for some quadratic objective functional, this result implies that there is no reformulation of the problem with a smooth change of variables that could convexify the problem. Therefore, one would need to resort to computationally expensive convex hull approaches. Moreover, if one seeks to solve a hard instance of the ODC problem through local search, the algorithm needs to be initialized an exponential number of times unless some prior information about the location of the solution is available in order to start in the correct connected component. This result contrasts with the recent findings in [17], arguing that local search could be useful for optimal control problems because it is guaranteed to work for the optimal centralized control problem. Although the number of connected components is shown to be exponential in this work, we also demonstrate that one single connected component is possible for favorably structured systems.

This work is related to several papers in the literature. The set of stabilizing controllers has been studied from many angles. The work [25] parametrizes the set of stable state-feedback controllers under no structural constraints. The paper [26] studies the connectivity of stable linear systems and concludes that single-input single-output systems of order n have at most $n+1$ connected components, while stable multi-input multi-output systems have only one connected component. The work [27] investigates what types of sparse patterns can sustain stable dynamics, using graph theory. To the authors' best knowledge, the connectivity of decentralized stabilizing controllers has not been studied before.

The remainder of this paper is organized as follows. Notations and problem formulations are given in Section II. We derive elementary connectivity properties of the set of stabilizing controllers and bound the number of connected components for scalar stabilizing controllers in Section III. Section IV examines a subclass of decentralized control problems where the number of connected components is exponential. We also show that highly damped systems admit a connected set of decentralized controllers. Concluding remarks are drawn in Section V.

II. Problem Formulation

Consider the linear time-invariant system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t),\\ y(t) &= Cx(t),\end{aligned}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$ are real matrices of compatible sizes. The vector $x(t)$ is the state of the system and $y(t)$ is the output. We focus on the static case, where the control input $u(t)$ is to be determined via a static output-feedback law $u(t) = Ky(t)$ with the gain $K \in \mathbb{R}^{m\times p}$ such that some performance is optimized. Since the analysis to be conducted next is on the feasible set, the initial state (being deterministic or stochastic) and the objective function (being quadratic or another function of the system's signals) are not important. With no loss of generality, consider the case with the given initial state $x(0) = x_0$ and the quadratic performance measure

$$J(K, x_0) = \int_0^\infty \left[x^T(t)Qx(t) + 2x^T(t)Du(t) + u^T(t)Ru(t)\right] dt \tag{1}$$

where the matrix $L = \begin{bmatrix} Q & D \\ D^T & R\end{bmatrix}$ is positive semi-definite and R is positive definite. We use $L \succeq 0$ and $R \succ 0$ to denote positive semi-definiteness and positive definiteness, respectively. The closed-loop system is

$$\dot{x}(t) = (A + BKC)x(t).$$

The objective is to study the set of structured stabilizing controllers

$$\mathcal{K} = \{K : A + BKC \text{ is stable/Hurwitz}, K \in \mathcal{S}\},$$

where $\mathcal{S} \subseteq \mathbb{R}^{m\times p}$ is a linear subspace of matrices, often specified by fixing certain entries of the matrix to zero. Decentralized and distributed controllers could be specified by the set $\mathcal{S}$ with a prescribed sparsity pattern. The connectivity properties of $\mathcal{K}$ will be studied under Euclidean topology.

III. Connectivity Properties in Special Cases

In this section, we prove global geometric properties of the stabilizing set $\mathcal{K}$ for certain choices of B, C and $\mathcal{S}$ using elementary arguments. The main result of this paper in its full generality will be stated in the next section.

Recall that one can characterize the stability of matrices in different ways. Lyapunov's characterization [28, §4.1] states that a matrix M is stable if and only if there is a solution $P \succ 0$ to the equation $MP + PM^T + I = 0$. Another approach is based on the Routh-Hurwitz criterion [29, §11.17], which states that a matrix is stable if the coefficients of its characteristic polynomial satisfy a set of polynomial inequalities. Using these basic techniques, we first study the case with no structural constraints and full state measurements.

Lemma 1: Assume that $\mathcal{S} = \mathbb{R}^{m\times p}$ and $C = I$. The set $\mathcal{K}$ is connected, but generally non-convex.

Proof: Observe that $\mathcal{K}$ is the continuous image of the set

$$\mathcal{H} = \{(R, P) : AP + BR + PA^T + R^T B^T = -I, P \succ 0\}$$

through the map $(R, P) \to RP^{-1}$. Moreover, $\mathcal{H}$ is connected since it is the intersection of a linear space and a convex cone. The map is well-defined as P is positive definite; it is also surjective from the Lyapunov's characterization: whenever $A+BK$ is stable, there is a $P \succ 0$ such that $(A+BK)P + P(A+BK)^T = -I$ and the tuple (R, P) can be mapped to the desired K under the formula $KP = R$.

To show that $\mathcal{K}$ is not always convex, consider the second-order system

$$A = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1\end{bmatrix}, B = \begin{bmatrix} 0 & b_0 \\ 1 & b_1\end{bmatrix}, K = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22}\end{bmatrix}$$

where A and the first column of B are in the canonical form to ensure controllability. The closed-loop matrix is equal to

$$A + BK = \begin{bmatrix} b_0 k_{21} & 1 + b_0 k_{22} \\ -a_0 + k_{11} + b_1 k_{21} & -a_1 + k_{12} + b_1 k_{22}\end{bmatrix}.$$

To analyze the stability, we use the Routh-Hurwitz criterion and write

$$\mathcal{K} = \{K : \mathrm{tr}(A + BK) < 0, \det(A + BK) > 0\}. \tag{2}$$

Notice that $\mathcal{K}$ is not convex in general since its intersection with the lower dimensional subspace $k_{21} = 0$ is given by

$$\left\{K = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22}\end{bmatrix} : \begin{matrix} -a_1 + k_{12} + b_1 k_{22} < 0, \\ (1 + b_0 k_{22})(-a_0 + k_{11}) < 0\end{matrix}\right\},$$

which turns out to be the union of two disjoint polyhedrons if $b_0 \neq 0$ (due to the product in the second condition). ■

An implication of Lemma 1 is that the feasible set of the linear-quadratic optimal centralized control problem is connected, which justifies the success of the local search algorithm proven in [17] for centralized controllers. Another insightful, but impractical, scenario is the case with $B = C = I$ and a mostly arbitrary $\mathcal{S}$. This will be studied below.

Lemma 2: Assume that $B = C = I$ and that $\mathcal{S}$ contains $-I$. Then, the set $\mathcal{K}$ is connected.

Proof: Since $\mathcal{S}$ is a linear subspace, we have $-\lambda I \in \mathcal{S}$ for every $\lambda \in \mathbb{R}$. Given two arbitrary matrices $K_1, K_2 \in \mathcal{K}$, consider the following connected path from $A + K_1$ to $A + K_2$:

$$\begin{aligned} A + K_1 &\overset{\text{increase } \lambda}{\rightarrow} A + K_1 - \lambda I \\ &\overset{K_1 \to K_2}{\rightarrow} A + K_2 - \lambda I \\ &\overset{\text{decrease } \lambda}{\rightarrow} A + K_2, \end{aligned}$$

where

- $\lambda \geq 0$ is first increased to a large value;
- we move from $A + K_1 - \lambda I$ to $A + K_2 - \lambda I$ via an arbitrary continuous path between K_1 and K_2 in $\mathcal{S}$;
- λ is decreased eventually.

The parameter λ can always be chosen so large that all matrices on the path from $A + K_1 - \lambda I$ to $A + K_2 - \lambda I$ could be regarded as a small (on the order of $K_2 - K_1$) perturbation of the large matrix $A + K_1 - \lambda I$ so that their stability condition is the same as that for $A + K_1 - \lambda I$. The proof is completed by noting that the designed path for the closed-loop matrix is associated with a path between K_1 and K_2 such that the path is continuous, involves only controllers in $\mathcal{S}$, and passes through only stabilizing matrices. ■

If the measurement matrix C is not the identity matrix, the set may become disconnected even if $K = k \in \mathbb{R}$ is a scalar. This will be shown in an example below. For clarity, we will use the vector notations $B = b$ and $C = c^T$ in this case, where $b, c \in \mathbb{R}^n$.

Example 1: Suppose that $A \in \mathbb{R}^{3\times 3}$, (A, b) is controllable, and $c \neq 0$. Then, the set $\mathcal{K}$ can have at most two connected components. To prove this statement, with no loss of generality assume that the system is in the controllable canonical form, i.e.,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}, \ b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \ c^T = \begin{bmatrix} c_0 & c_1 & c_2 \end{bmatrix}.$$

Using the Routh-Hurwitz method, the stability condition reduces to the set of inequalities

$$\begin{aligned} a_0 - kc_0 &> 0, \\ a_1 - kc_1 &> 0, \\ a_2 - kc_2 &> 0, \\ (a_0 - kc_0) &< (a_2 - kc_2)(a_1 - kc_1). \end{aligned}$$

Consider the quadratic function $f(k) = (a_2 - kc_2)(a_1 - kc_1)$. This function can have at most two branches that lie above the line $a_0 - kc_0$. The intersection of these branches with the interval defined by the first three linear inequalities leads to at most 2 connected components. An example with exactly two components can be produced by the parameters

$$(a_0, a_1, a_2) = (-5, -1, 1), \quad (c_0, c_1, c_2) = (0.85, 0.2, 0.2).$$

Figure 1 verifies the above result by plotting the maximum real part of the closed-loop eigenvalues versus k.

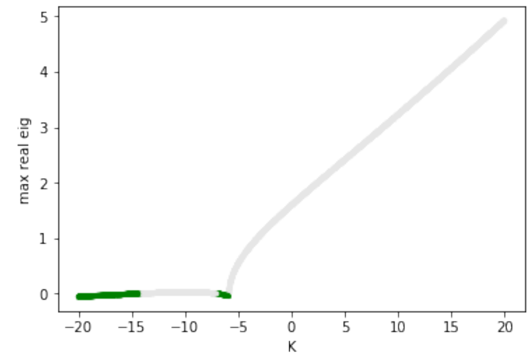

Fig. 1. As discussed in Example 1, the set of stabilizing controllers can have two connected components in a third-order system. Observe that there are two intervals for k that produce eigenvalues in the left-half complex plane.

It can be inferred from Example 1 that the coordinates of the set of stabilizing controllers are "one-sided". This is not surprising since when $A + BKC$ is stable, we have $\text{tr}(A + BKC) < 0$. We elaborate on this result below.

Lemma 3: Consider the case $m = p = 1$. Suppose that (A, b) is controllable and $c \neq 0$. Then, the scalar set $\mathcal{K}$ cannot extend to infinity on both sides.

Proof: As before, with no loss of generality consider the canonical form

$$A = \begin{bmatrix} 0 & & I \\ -a_0 & \dots & -a_{n-1} \end{bmatrix}, b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, c^T = [c_0, \dots, c_{n-1}].$$

The matrix $A + bkc^T$ has the characteristic polynomial

$$(a_0 - c_0 k) + (a_1 - c_1 k)x + \dots, + (a_{n-1} - c_{n-1} k)x^{n-1} + x^n = 0.$$

It follows from the Routh-Hurwitz criterion that the coefficients of this polynomial must be positive. Since $c \neq 0$, there is some entry $c_{i_0} \neq 0$ and, as a result, k is prevented from extending to infinity on one side by the inequality $a_{i_0} - c_{i_0} k > 0$. ■

In what follows, we will bound the number of connected components for scalar controllers.

Theorem 1: Consider the case $m = p = 1$. The scalar set $\mathcal{K}$ can have at most n connected components.

Proof: If there is no stabilizing controller in $\mathcal{S}$, then $\mathcal{K} = \emptyset$; otherwise one can first stabilize A with some controller k_0 and then analyze the set of shifted controllers $k - k_0$. As a result, without loss of generality one can assume that A is stable. We call a controller k *critical* when it is on the boundary of the closure of the set stabilizing controllers, meaning that it produces some closed-loop eigenvalues on the imaginary axis. We need to solve the equation

$$\begin{aligned} 0 &= \det(\mathbf{j}wI - A - kbc^T) \\ &= \det(\mathbf{j}wI - A) \det(1 - kc^T(\mathbf{j}wI - A)^{-1}b) \end{aligned} \quad (3)$$

(the symbol $\mathbf{j}$ denotes the imaginary unit). Since A is stable, the first term in the second line of (3) is not zero and therefore the second term must be zero. Taking its real and imaginary part yields that

$$1-k\times \mathrm{Re}\{c^T(\mathbf{j}wI-A)^{-1}b\}=0, \qquad (4)$$
$$\mathrm{Im}\{c^T(\mathbf{j}wI-A)^{-1}b\}=0. \qquad (5)$$

Equation (5) is of the form $\mathrm{Im}\left\{\frac{f(\mathbf{j}w)}{g(\mathbf{j}w)}\right\}=0$ with $g(\mathbf{j}w)\neq 0$; equivalently, one can write $\mathrm{Im}\{f(\mathbf{j}w)\overline{g(\mathbf{j}w)}\}=0$ where $f(\mathbf{j}w)$ is a polynomial of degree $n-1$ and $g(\mathbf{j}w)=\det(\mathbf{j}wI-A)$ is a polynomial of degree n . A polynomial of degree $2n-1$ in w can have at most $2n-1$ real solutions. For each solution, there is at most one critical scalar k that solves (4). Those $2n-1$ critical scalars divide the real line into at most $2n$ intervals of interlacing stable-unstable regions, leading to at most n stable components. ■

Theorem 1 states that the number of connected components would grow with the dimension of the system even in the special case $m=p=1$. The intuition behind the proof of Theorem 1 is that one can calculate how many times the Nyquist plot crosses the real line. For an n-th order system, Argument Principle suggests that the Nyquist Plot can wind at most n cycles and intersect with the real line at most $2n$ times.

IV. Exponential Subclass

One of the main results of this paper will be stated below.

Theorem 2: There is no polynomial function with respect to the order of the system that can serve as an upper-bound on the number of connected components of the set of decentralized stabilizing controllers.

To prove the above theorem, it suffices to discover a subclass of decentralized control problems whose set of stabilizing controllers has an exponential number of connected components. Our proof is based on a lemma that characterizes the stability of tri-diagonal matrices whose diagonal elements are mostly purely imaginary complex numbers. Define the inertia $\mathrm{In}(G)$ of an $n\times n$ matrix G as the triplet $\mathrm{In}(G)=(\pi(G),\nu(G),\delta(G))$, where $\pi(G)$, $\nu(G)$, and $\delta(G)$ are the number of eigenvalues of G with positive, negative, and zero real parts, respectively.

Lemma 4 (From [30]): Given the tri-diagonal matrix

$$G=\begin{bmatrix} f_1+\mathbf{j}g_1 & f_2 & 0 & \cdots & \cdots & 0\\ -h_2 & \mathbf{j}g_2 & f_3 & \ddots & & \vdots\\ 0 & -h_3 & \mathbf{j}g_3 & f_4 & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & \ddots & 0\\ \vdots & & \ddots & -h_{n-1} & \mathbf{j}g_{n-1} & f_n\\ 0 & \cdots & \cdots & 0 & -h_n & \mathbf{j}g_n \end{bmatrix},$$

where f_i, g_i and h_i are real for $i=1,...,n$, $f_1\neq 0$, and $f_ih_i\neq 0$ for $i=2,\dots,n$. Then,

$$\mathrm{In}(G)=\mathrm{In}(D),$$

where

$$D=\mathrm{diag}(f_1,f_1f_2h_2,f_1f_2f_3h_2h_3,\dots,f_1\cdots f_nh_2\cdots h_n).$$

A corollary of the above lemma for the stability of real tri-diagonal matrices is given below.

Corollary 1: Given the tri-diagonal real matrix A of the form

$$A=\begin{bmatrix} f_1 & f_2 & 0 & \cdots & \cdots & 0\\ -h_2 & 0 & f_3 & 0 & & \vdots\\ 0 & -h_3 & 0 & f_4 & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & \ddots & 0\\ \vdots & & \ddots & -h_{n-1} & 0 & f_n\\ 0 & \cdots & \cdots & 0 & -h_n & 0 \end{bmatrix}, \qquad (6)$$

it holds that

- If $f_1<0$ and $f_ih_i>0$ for $i=2,\dots,n$, then A is stable.
- If $f_ih_i<0$ for some $i=2,\dots,n$, then A is unstable.

Remark 1: The sparse stable matrices theory [27] states that the graph associated with the sparsity pattern of the matrix in (6) is a chain and has nested Hamiltonian sub-graphs that are sufficient to sustain stable dynamics. Moreover, the space is also minimally stable because: (i) if f_1 is set to zero, then the trace of the matrix becomes zero and therefore at least one eigenvalue should be unstable, (ii) if any non-diagonal element is set to zero, then the matrix decomposes into a block triangular form where the lower diagonal block has a zero trace, leading to instability.

Due to the above remark, Corollary 1 gives necessary and sufficient conditions for the stability of a class of matrices, which can be used to analyze both connected components and separating hypersurfaces. In what follows, we will first show the possibility of 2^{n-1} connected components in the case with a non-identify C and then develop a similar result for $C=I$.

Theorem 3: Let $A\in\mathbb{R}^{n\times n}$ be in the form of (6), and set $B\in\mathbb{R}^{n\times(2n-2)}$, $C\in\mathbb{R}^{(2n-2)\times n}$ and $K\in\mathbb{R}^{(2n-2)\times(2n-2)}$ to

$$B=\left[\begin{array}{cccc|cccc} 0 & \cdots & \cdots & 0 & +1 & 0 & \cdots & 0\\ -1 & \ddots & & \vdots & 0 & \ddots & \ddots & \vdots\\ 0 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 0\\ \vdots & \ddots & \ddots & 0 & \vdots & & \ddots & +1\\ 0 & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 \end{array}\right],$$

$$C=\left[\begin{array}{ccccc} 1 & 0 & \cdots & \cdots & 0\\ 0 & \ddots & \ddots & & \vdots\\ \vdots & \ddots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & 1 & 0\\ \hline 0 & 1 & 0 & \cdots & 0\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ \vdots & & \ddots & \ddots & 0\\ 0 & \cdots & \cdots & 0 & 1 \end{array}\right],$$

$$K=\mathrm{diag}(k_2,\dots,k_n,k_2,\dots,k_n).$$

Suppose $f_1 < 0$ and $f_i \neq h_i$ for $i = 2, \ldots, n$. Then, the set $\mathcal{K}$ has at least 2^{n-1} connected components.

Proof: The closed-loop matrix $A + BKC$ can be expressed as

$$\begin{bmatrix} f_1 & f_2+k_2 & 0 & \cdots & \cdots & 0 \\ -h_2-k_2 & 0 & f_3+k_3 & \ddots & & \vdots \\ 0 & -h_3-k_3 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 0 & f_n+k_n \\ 0 & \cdots & \cdots & 0 & -h_n-k_n & 0 \end{bmatrix}.$$

It results from Corollary 1 and Remark 1 that the closed-loop stability is equivalent to the conditions $(h_i+k_i)(f_i+k_i) > 0$ for $i = 2, \ldots, n$, which reduce to $k_i < \min(-h_i, -f_i)$ or $k_i > \max(-h_i, -f_i)$. In particular, if any k_i takes the value $-(f_i + h_i)/2$, the matrix becomes unstable. Therefore, the region of stabilizing K, parametrize in $(k_2, \ldots, k_n) \in \mathbb{R}^{n-1}$, is separated by $n-1$ hyperplanes $k_i = -(f_i + h_i)/2$ for $i = 2, \ldots, n$, and there are stable regions on both sides of each of those hyperplanes. The overall number of connected components becomes at least 2^{n-1}. ■

The conclusion of Theorem 3 is demonstrated in the top plot in Figure 2 for $n = 3$. Note that the "one-sided" result of Lemma 3 does not hold here since K is not scalar.

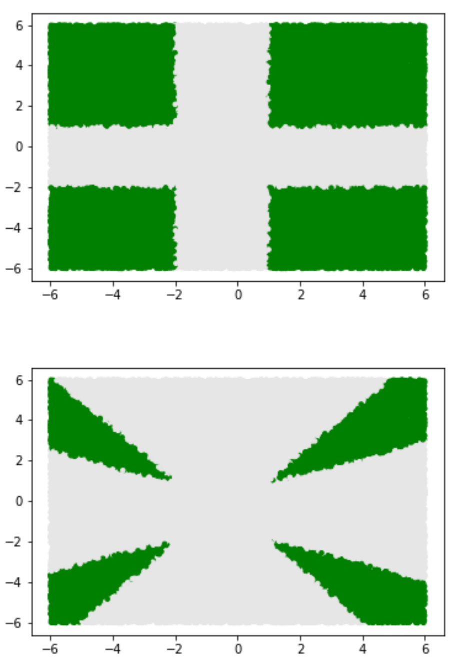

Fig. 2. We randomly sample K and check the closed-loop stability for an instance of the system in Theorem 3. The controller is parametrized in terms of (k_2, k_3) where $n = 3$, with $f_i = -1$ and $h_i = 2$ for $i = 1, 2, 3$. The top figure shows that there are $2^{n-1} = 4$ connected components, where each coordinate takes values in $(-\infty, -2)$ or $(1, \infty)$ to be stable. The bottom figure shows the connected components when the number 0.2 is added to each diagonal entry of A (the projection of the set $\mathcal{K}$ onto the 2-dimensional space corresponding to (k_2, k_3) is shown in green).

Remark 2: Note that eigenvalues are continuous functions of the entries of a matrix and that the connected components studied in the proof of Theorem 3 are separated by a positive margin. Therefore, one may speculate that a small perturbation to A will not change the number of connected components. This is not the case in general since the eigenvalues of $A + BKC$ can become arbitrarily close to the imaginary axis when $\|K\|$ is large, as illustrated in Figure 3. However, some parts from each connected component are resistant to perturbations. For example, the set $\{K : (A + \epsilon I) + BKC \text{ stable }\} \subseteq \{K : A + BKC \text{ stable }\}$ with $\epsilon > 0$ contains only those controllers that make the closed-loop eigenvalues at least ϵ away from the imaginary axis, and ϵ can be set so small that at least one point from each component remains stable. In other words, a new A obtained by adding ϵ to the diagonal of the matrix in (6) gives an exponential number of connected components where the number cannot change with a very small perturbation of its elements. This is illustrated in the bottom plot in Figure 2.

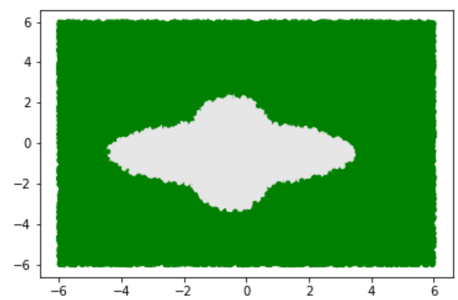

Fig. 3. This figure shows that if the diagonal of A are reduced by 0.2, then the set $\mathcal{K}$ becomes connected (the projection of the set $\mathcal{K}$ onto the 2-dimensional space corresponding to (k_2, k_3) is shown in green).

The subclass of problems studied in Theorem 3 may be unsatisfactory as it requires that the free elements of K repeat themselves and that $C \neq I$. The next theorem addresses these issues.

Theorem 4: Let A be in the form

$$A = \begin{bmatrix} f_1+\epsilon & f_2 & 0 & \cdots & \cdots & 0 \\ -h_2 & \epsilon & f_3 & \ddots & & \vdots \\ 0 & -h_3 & \epsilon & f_4 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & -h_{n-1} & \epsilon & f_n \\ 0 & \cdots & \cdots & 0 & -h_n & \epsilon \end{bmatrix}, \tag{7}$$

where $\epsilon > 0$, $f_1 < 0$, and $(-1)^i(f_i - h_{i+1}) > 0$ for $i = 2, \ldots, n$. Consider $B \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{n\times n}$ and $K \in \mathbb{R}^{n\times n}$

to be

$$B=\begin{bmatrix} 0 & 1 & & & \\ -1 & \ddots & \ddots & & \\ & & \ddots & 0 & 1 \\ & & & -1 & 0 \end{bmatrix}, \quad C=I,$$
$$K=\mathrm{diag}(k_1,k_2,\dots,k_n).$$

For a small enough ϵ, the set $\mathcal{K}$ has at least F_n connected components, where $F_0=1, F_1=1, F_{i+2}=F_{i+1}+F_i$ for $i=0,1,...$ is the Fibonacci sequence, which grows in the order of $\left(\frac{1+\sqrt{5}}{2}\right)^n$.

Proof: First, assume that $\epsilon=0$ and consider the closed-loop matrix $A+BKC$:

$$\begin{bmatrix} f_1 & f_2+k_2 & 0 & \cdots & \cdots & 0 \\ -h_2-k_1 & 0 & f_3+k_3 & \ddots & & \vdots \\ 0 & -h_3-k_2 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 0 & f_n+k_n \\ 0 & \cdots & \cdots & 0 & -h_n-k_{n-1} & 0 \end{bmatrix}.$$

In light of Corollary 1 and Remark 1, the necessary and sufficient conditions for the closed-loop stability are $(h_i+k_{i-1})(f_i+k_i)>0$ for $i=2,...,n$. As a result, if $h_2+k_1>0$, then it holds that $f_2+k_2>0$. Now, because $h_3<f_2$, the term h_3+k_2 can be positive or negative. If it is positive, then f_3+k_3 must be positive, and we can move on to study the sign of h_4+k_3. As we proceed, it should be noted that not all sign assignments for h_i+k_{i-1} and f_i+k_i are possible due to the assumptions on f_i and h_i. The enumeration procedure is illustrated in Figure 4. Any path from the root to the bottom level leaf passes through a set of linear inequalities that together enclose an open polyhedron of stable regions. These stable regions are separated by the hyperplanes $h_{i+1}+k_i=0$ for $i=1,2,\dots,n-1$ and $f_i+k_i=0$ for $i=2,3,\dots,n$.

Next, we count the number of branches. If $h_{i+1}+k_i>0$ (or equivalently $f_{i+1}+k_{i+1}>0$) appears m_i times and $h_{i+1}+k_i<0$ (or equivalently $f_{i+1}+k_{i+1}<0$) appears n_i times, assuming $m_i\geq n_i$, the next level will have at most $(m_i+n_i)+\max(m_i,n_i)=2m_i+n_i$ branches. This number is achievable if $f_{i+1}<h_{i+2}$, which means keeping all the children of the inequalities $f_{i+1}+k_{i+1}>0$ and pruning one child from each of the inequality $f_{i+1}+k_{i+1}<0$. Then, $m_{i+1}=m_i$, $n_{i+1}=m_i+n_i$, and $n_{i+1}\geq m_{i+1}$, reversing the order of m_i and n_i. It can be verified that the total number of connected regions m_i+n_i satisfies the iteration of the Fibonacci sequence.

The above connected regions are separated by the hyperplanes $k_i=-f_i$ or $k_i=-h_{i+1}$ with no margin. However, when $\epsilon>0$, the connected components will be separated. More precisely, whenever $k_i=-f_i$ or $k_i=-h_{i+1}$, the matrix $A+BKC$ decomposes into a block triangular form where the lower diagonal block has a positive trace, which means that the matrix cannot be stable. When ϵ is small enough, the original connected regions described by linear inequalities do not shrink abruptly — in fact, at least one point from every polyhedron remains stable. As a result, these stable regions are the true connected components of the stabilizing controller set. ■

To illustrate Theorem 4, consider the matrix

$$A=\begin{bmatrix} -1+\epsilon & 2 & 0 & & & & \\ -2 & \epsilon & 1 & 0 & & & \\ 0 & -1 & \epsilon & 2 & 0 & & \\ & 0 & -2 & \epsilon & 1 & 0 & \\ & & 0 & -1 & \epsilon & 2 & 0 \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}. \quad (8)$$

The corresponding set $\mathcal{K}$ obtained by sampling random matrices K and checking the closed-loop stability is provided in Figure 5 for $n=3$.

The previous results all suggest that the diagonal entries of A being positive contribute to the complexity of the feasible set $\mathcal{K}$. We will next show that the diagonal of A being negative is a desirable structure in the sense that if A is highly dampened, then the feasible set is connected.

Theorem 5: Given arbitrary matrices A, B and C of compatible dimensions and a linear subspace of matrices $\mathcal{S}$, the set

$$\mathcal{K}_\lambda=\{K: A-\lambda I+BKC \text{ is stable }, K\in\mathcal{S}\}$$

is connected when $\lambda>0$ is large enough.

Proof: Consider a number μ and let λ be a parameter that increases from μ toward ∞. Since $\lambda\geq\mu$, we have $\mathcal{K}_\lambda\supseteq\mathcal{K}_\mu$, and therefore $\mathcal{K}_\lambda$ contains all components of K_μ but could possibly connect them or add new components. The addition of new components with the increase of λ could occur only a finite number of times. The reason is that, due to the Routh-Hurwitz criterion, K_λ can be described by polynomial inequalities in the entries of $A-\lambda I+BKC$, and hence the boundary of K_λ described as the solutions of some polynomial equalities has a maximum number of connected components, which is finite given the order of the system. Therefore, we first increase λ until no new connected component appears, then select a controller from each connected component, and cover all those controllers (points) with a ball $B\subseteq\mathcal{S}$. By making λ so large that all controllers in B become stable, we glue all of the connected components. ■

The interpretation of the result of Theorem 5 is that if the open-loop matrix of the system can be written as $A-\lambda I$ for a large λ, then the feasible set of ODC is connected. This corresponds to highly damped systems.

Remark 3: It is noted in [31] that if we consider the discounted cost

$$\int_0^\infty e^{-2\lambda t}(x^TQx+2u^TDx+u^TRu)dt,$$

or equivalently make a change of variables $\hat{x}(t)=e^{-\lambda t}x(t)$ and $\hat{u}(t)=e^{-\lambda t}u(t)$, then the closed-loop dynamics become equal to $\dot{\hat{x}}(t)=(A-\lambda I+BKC)\hat{x}(t)$. Therefore, it follows

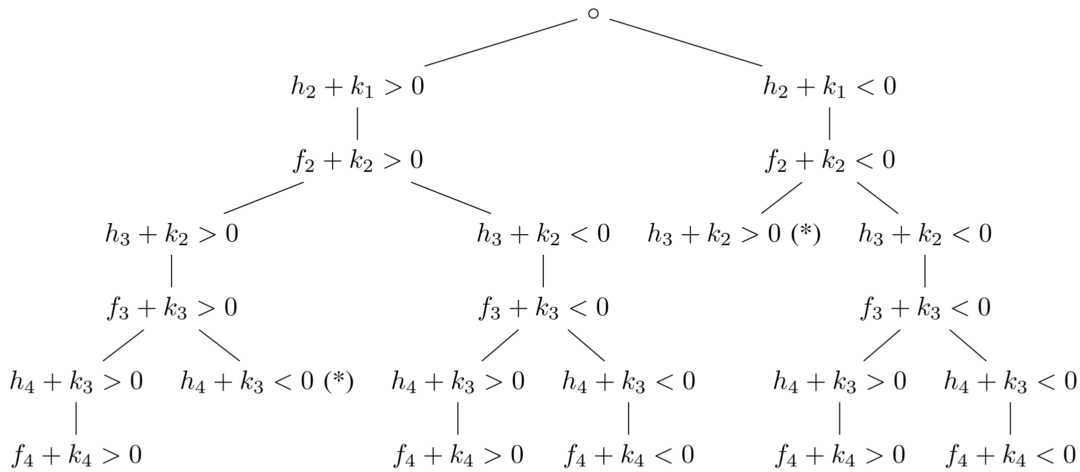

Fig. 4. This tree shows the enumerating signs of the closed-loop matrix entries for $n = 4$. The branch marked with $(*)$ has contradictory inequalities.

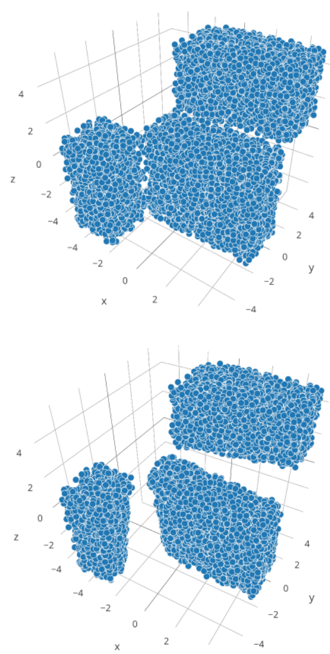

Fig. 5. We randomly sample K and check the closed-loop stability for an instance of the system in Theorem 4 with $n = 3$, the matrix A given in (8), and $K = \text{diag}(k_1, k_2, k_3)$. The projection of the set $\mathcal{K}$ onto the 3-dimensional space corresponding to (k_1, k_2, k_3) is shown in blue in the top picture for $\epsilon = 0$ and in the bottom picture for $\epsilon = 0.02$.

from Theorem 5 that the feasible set of the ODC problem is connected for discounted costs with a large forgetting factor.

Remark 4: It is known in the context of inverse optimal control [31] that any static state-feedback gain K is the unique minimizer of some quadratic performance measure (1) for all initial states. One such measure is

$$\int_0^\infty (u(t) - Kx(t))^T R\,(u(t) - Kx(t))\,dt.$$

where R is a positive definite matrix, and Q and D can be computed accordingly. As a result, each point in any connected component is an optimal solution to some ODC problem. Since there are an exponential number of connected components in certain cases, it is unlikely for a random initialization to successfully locate the optimal component unless some prior information is available or the system is favorably structured. Local search algorithms, therefore, fail for the general ODC problem.

V. Conclusion

In this paper, we studied global geometric properties of the set of static stabilizing decentralized controllers. We demonstrated through a subclass of problems that the NP-hardness of optimal decentralized control could be attributed to a large number of connected components. In particular, we proved that the number of connected components for chain subsystems would follow a Fibonacci sequence. We bound the number of connected components in the scalar case. We also showed that connectivity would not be an issue for highly damped systems. Our results qualified the application of local search algorithms to optimal decentralized control problems and emphasized structural considerations. Future work includes the analysis of the connectivity properties of dynamic controllers and the identification of system/control structural properties that guarantee the connectivity of the feasible set.

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