A Scalable Method for Designing Distributed Controllers for Systems with Unknown Initial States

Salar Fattahi*, Javad Lavaei*, and Murat Arcak+

*Industrial Engineering and Operations Research, University of California, Berkeley
+Electrical Engineering and Computer Sciences, University of California, Berkeley

***Abstract*— The optimal distributed control (ODC) problem for linear discrete-time systems is studied in this paper. The goal is to design a static stabilizing controller that offers some optimality guarantee for the closed-loop system and yet respects an imposed communication structure. Unlike the traditional centralized control problem, the ODC problem is hard to solve in general. Recently, we have introduced an efficient and scalable algorithm to design a distributed controller whose performance is close to that of a given centralized controller, provided that the initial state of the system is known. In this work, we generalize the proposed method to systems with uncertain initial states. The objective is to make the performance of the designed distributed controller be as closely as possible to that of the optimal centralized counterpart for every initial state in the uncertainty region. It is shown that the developed method requires solving a convex problem. Strong theoretical lower bounds are provided on the optimality guarantee of the synthesized distributed controller. To illustrate the effectiveness of the proposed method, case studies on aircraft formation and frequency control of power systems are offered.**

I. Introduction

The problem of designing an optimal distributed controller is of significant importance for complex large-scale systems that are subject to communication and/or computation restrictions. The main objective is to design a controller with a prescribed sparsity pattern, as opposed to a conventional centralized controller that does not have any restrictions on the controller's structure. In a complex system that is comprised of many subsystems, it is often computationally prohibitive to process and design control policies based on the state of the whole system. Instead, these subsystems are equipped with subcontrollers that interact with each other based on a pre-defined structure. The imposed control structure significantly increases the complexity of the design procedure. This difficulty arises from the fact that well-established methods for the design of traditional centralized controllers often fail as soon as sparsity control constraints are imposed on the controller.

It is formally proven in [1] that the optimal distributed control (ODC) problem is NP-hard in its worst case. Although the problem of designing a distributed controller with an arbitrary structure is a daunting task, several methods have been developed to find an optimal or near-optimal controller for special structures, such as spatially distributed systems [2], [3], localizable systems [4], [5], strongly connected systems [6], optimal static distributed systems [7], decentralized systems over graphs [8], [9], and quadratically-invariant systems [10]. The problem of designing a distributed controller whose performance is close to that of its centralized counterpart is cast as a rank-constrained optimization problem in [11] and [12]. Moreover, it is shown in [13] that the design of an optimal linear-quadratic distributed controller with an arbitrary structure is essentially a convex problem as long as the input weighting or measurement covariance matrices are large enough.

The difficulty of finding the optimal distributed controller is due to the fact that it belongs to a more general class of hard problems, namely *polynomial optimization problems*. Because of the NP-hardness of these problems, several convex relaxation and reformulation techniques have been proposed to find their global or near-global solutions [14], [15]. A convex relaxation of general polynomial optimization problems is studied in [16] and [17], where the underlying structure of the problem is exploited via a weighted graph. Furthermore, it is shown that the exactness of the proposed relaxation depends on the specifications of this graph. By building on this result, it is proved in [18] and [19] that the semidefinite programming relaxation of the ODC problem possesses a low-rank solution.

Recently, we have proposed an easy-to-implement and scalable method to design a distributed controller with a pre-specified sparsity pattern via a transformation of the optimal centralized controller, as long as the exact value of the initial state of the system is known [20]. Later on, this method has been extended to stochastic systems that are subject to measurement and input disturbance noise [21]. One important drawback of this method for deterministic systems is a lack of robustness to the initial state in the sense that the designed distributed controller depends on the exact value of the initial state. However, in almost all real-world problems, the system operators do not know the exact value of the initial state and can only estimate it. This implies that the method introduced in [20] is not directly applicable to the control of systems whose initial state is

Email: fattahi@berkeley.edu, lavaei@berkeley.edu, and arcak@berkeley.edu

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not known a priori. On the other hand, unlike the optimal centralized control problem, the parameters of the optimal distributed controller may not be independent from the initial state. The aim of this paper is to extend the recent work [20] to the case where the initial state is unknown, but belongs to an uncertainty region. In particular, the objective is to design a convex optimization problem whose solution gives rise to a distributed controller that is nearly optimal for every initial state in a given uncertainty region. Furthermore, it is desirable to theoretically certify the performance of the designed distributed controller by deriving strong lower bounds on its optimality guarantee.

To exhibit the performance of the designed distributed controller, two case studies are considered in this paper. The first one is an aircraft formation problem in which each aircraft should make decision solely based on its relative distance from the neighboring agents [22], [23]. The second case study is the frequency control problem of generators in power systems with different topological restrictions on the controller [18]. It will be shown that the synthesized distributed controller can stabilize the closed-loop system while maximizing its performance in both of these case studies.

Notations: The set of real numbers is denoted by $\mathbb{R}$. The symbol $\mathrm{trace}\{W\}$ denotes the trace of a matrix W. The notation I denotes the identity matrix of appropriate dimension. The symbol $(\cdot)^T$ denotes the transpose operator. The symbols $\|W\|_2$ and $\|W\|_F$ are used to denote the 2-norm and Frobenius norm of W, respectively. $W(i,j)$ is the $(i,j)^{th}$ entry of a matrix W. The notations $w(t)$ and $w[t]$ are used for vectors correspondence to time t in continuous and discrete domains, respectively. The symbol $\lambda_W^{\max}$ refers to the maximum eigenvalue of a symmetric matrix W. The maximum absolute value of the eigenvalues of W is denoted by $\rho(W)$ and called the spectral radius of matrix W. The notation $W \succeq 0$ means that the symmetric matrix W is positive semidefinite. For a real number y, the notation $(y)_+$ denotes the maximum of 0 and y.

II. Problem Formulation

In this paper, we consider the optimal distributed control (ODC) problem with an uncertain initial state. The goal is to develop a fast and scalable algorithm for the design of a near-globally optimal distributed controller with a pre-defined structure for linear time-invariant (LTI) systems whose initial state is unknown, but belongs to an uncertainty region.

Definition 1: The set $\mathcal{K} \in \mathbb{R}^{m\times n}$ is defined as a linear subspace with some pre-specified sparsity structure. This structure captures the communication topology of the distributed controller (enforced zeros in certain entries). A static feedback gain that belongs to $\mathcal{K}$ is called a **distributed controller** with the sparsity pattern captured by $\mathcal{K}$. If there is no structural constraint on the controller, then $\mathcal{K} = \mathbb{R}^{m\times n}$ and the static feedback gain is called a **centralized controller**. The notations K_c, K_d, K are used for the optimal centralized controller, the designed distributed controller, and a variable controller in optimization problems, respectively.

Consider the discrete-time system

$$x[\tau+1] = Ax[\tau] + Bu[\tau], \qquad \tau = 0, 1, ..., \infty \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are some known matrices that depend on the structure of the LTI system. Assume that the initial state $x[0]$ is unknown, but it belongs to a compact and bounded set $\mathcal{E}$ that is referred to as the *uncertainty region* of the initial state. For the sake of simplicity of notations, the initial state is denoted as x henceforth. The goal is to design a static gain $K_d \in \mathcal{K}$ such that the system (1) under the distributed controller $u[\tau] = K_d x[\tau]$ satisfies some performance and stability criteria. Associated with the system (1), define the cost function

$$J(K) = \sum_{\tau=0}^{\infty} \left(x[\tau]^T Q x[\tau] + u[\tau]^T R u[\tau] \right) \tag{2}$$

where $Q \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times m}$ are constant positive-definite matrices. Assume that the pair (A, B) is stabilizable. The optimization problem of

$$\min_{K \in \mathbb{R}^{m\times n}} J(K) \tag{3}$$

subject to (1), $u[\tau] = Kx[\tau]$, and the closed-loop stability condition is called the *optimal centralized control problem*, where the optimal controller gain can be efficiently obtained from the Riccati equation. While finding the optimal centralized controller is a tractable problem, the design of an optimal distributed controller subject to the constraint $K \in \mathcal{K}$ is computationally hard in general.

Definition 2: Consider the system (1) with the cost function (2). Given $K_d \in \mathcal{K}$ and a number $\mu \in [0, 1]$, it is said that the distributed controller has the optimality guarantee μ if it satisfies the inequality

$$\frac{J(K_c)}{J(K_d)} \geq \mu \tag{4}$$

We interchangeably denote the optimality guarantee as a number μ between 0 and 1 or in percentage as $100 \times \mu\%$. For example, if $\mu = 0.95$, then the inequality (4) implies that the underlying distributed controller K_d is at most 5% worse than the optimal centralized controller K_c. This means that if there exists a better distributed controller, it would outperform K_d by at most 5%. The goal of this work is twofold:

- Given the sparsity structure $\mathcal{K}$, it is desirable to develop a cheap and scalable method for designing a distributed controller $K_d \in \mathcal{K}$ that maximizes the performance for every initial state x in a given uncertainty region $\mathcal{E}$.
- The second objective is to analyze the proposed design method and provide optimality guarantees on the obtained distributed controller.

III. Distributed Controller Design

Assume for now that the uncertainty region $\mathcal{E}$ is the singleton a. In this case, the initial state of the system is known a priori. Define P_a as the unique positive semidefinite solution of the Lyapunov equation

$$(A + BK_c)P(A + BK_c)^\top - P + aa^\top = 0 \tag{5}$$

Consider the optimization problem

$$\min\ C(K) \qquad \text{s.t.} \qquad K \in \mathcal{K} \tag{6}$$

where

$$\begin{aligned} C(K) = \omega \times & \underbrace{\text{trace}\left\{(K_c - K)P_a(K_c - K)^T\right\}}_{C_1(P_a,K)} \\ +(1-\omega)\times & \underbrace{\text{trace}\left\{(K_c-K)^T B^T B(K_c-K)\right\}}_{C_2(K)} \end{aligned} \tag{7}$$

for a regularization coefficient ω between 0 and 1. In what follows, it will be explained that the optimality guarantee of a distributed controller K_d can be lower bounded in terms of $C_1(P_a, K_d)$. Therefore, the minimization of $C_1(P_a, K_d)$ would increase the optimality guarantee of the designed controller K_d. Denote $\kappa(V)$ as the condition number of the eigenvector matrix V of $A + BK_d$ in 2-norm.

Theorem 1 ([21]): Assume that $Q = I_n$ and $R = I_m$. Consider the set $\mathcal{E} = \{a\}$, the optimal centralized gain K_c and an arbitrary stabilizing gain $K_d \in \mathcal{K}$ for which $A + BK_d$ is diagonalizable. The controller $u[\tau] = K_d x[\tau]$ has the optimality guarantee μ, where

$$\mu = \frac{1}{\left(1 + \zeta\sqrt{C_1(P_a, K_d)}\right)^2} \tag{8}$$

and

$$\zeta = \max\left\{\frac{\kappa(V)\|B\|_2}{(1-\rho(A+BK_d))\sqrt{\sum_{\tau=0}^{\infty}\|x_c[\tau]\|_2^2}}\ ,\ \frac{1-\rho(A+BK_d)+\kappa(V)\|K_d\|_2\|B\|_2}{(1-\rho(A+BK_d))\sqrt{\sum_{\tau=0}^{\infty}\|u_c[\tau]\|_2^2}}\right\} \tag{9}$$

Remark 1: As delineated in [21], the value of $C_1(P_a, K_d)$ can be interpreted as a measure of closeness between the state (or input) trajectories of the optimal centralized control system and their counterparts in the designed distributed control systems in the case where the initial state is equal to a. If $C_1(P_a, K_d)$ is equal to zero for a designed distributed controller K_d, the optimality guarantee of the distributed controller is 100%. The incorporation of $C_2(K)$ via the regularization coefficient ω in the objective function (7) indirectly enforces the stability of the closed-loop system. It is worthwhile to mention that a small value for the function $C_1(P_a, K_d)$ does not necessarily guarantee the closed-loop stability. Instead, such small value only ensures that the system started from the initial state a resides in the stable manifold of $x[\tau+1] = (A + BK_d)x[\tau]$.

Note that although Theorem 1 holds for $Q = I_n$ and $R = I_m$, a similar bound can be derived for the general case through a simple transformation of states and inputs in the ODC problem (see [21]). Note also that (6) has an explicit solution that can be derived by solving a system of linear equations. This closed-form solution can be found in [20].

As described in Section I, it is often the case that the initial state at which the system starts to operate is not known precisely *a priori*. For these types of systems, the equation (5) cannot be used because it depends on the unknown initial state. In this section, the objective is to modify (5) and (6) to account for an unknown initial state. In particular, the goal is to design a distributed controller that provides a high optimality guarantee for every initial point that belongs to the uncertainty region $\mathcal{E}$. Throughout the rest of this paper, assume that

$$\mathcal{E} = \{a + Mu : u \in \mathbb{R}^{n\times n}, \|u\|_2 \le 1\} \tag{10}$$

for a vector $a \in \mathbb{R}^{n\times 1}$ and a symmetric matrix $M \in \mathbb{R}^{n\times n}$. Indeed, if $\mathcal{E}$ does not have the above ellipsoidal expression, one may use its outer ellipsoidal approximation as the uncertainty region of the initial state at the expense of designing more conservative controllers (see [15] for more details). Define

$$\mathcal{L}(P, x) = (A + BK_c)P(A + BK_c)^\top - P + xx^\top \tag{11}$$

Based on the definition of $\mathcal{E}$, it may not be possible to find a matrix P that satisfies $\mathcal{L}(P, x) = 0$ for every $x \in \mathcal{E}$. Instead, we introduce a new optimization problem to design a matrix P such that $\mathcal{L}(P, x)$ is maintained close to zero for every $x \in \mathcal{E}$. This problem is given below:

$$\min_{\alpha, P} \quad \alpha \tag{12a}$$

$$\text{s.t.} \quad -\alpha I \preceq \mathcal{L}(P, x) \preceq \alpha I, \qquad \forall x \in \mathcal{E} \tag{12b}$$

$$P \succeq 0 \tag{12c}$$

Notice that (12) is a semidefinite programming (SDP) with an infinite number of constraints. One may speculate that using the optimal solution of (12) as a surrogate for P_a in (6) would not lead to a high performance distributed controller (since the optimality guarantee introduced in Theorem 1 no longer holds for the designed distributed controller). Furthermore, (12) is an infinite-dimensional optimization problem and cannot be solved efficiently using the available solvers. In the sequel, we will remedy both of the above-mentioned problems. First, we will introduce explicit solutions that are nearly optimal for (12). Second, we will derive bounds similar to (8) on the performance of the designed distributed controller under all initial states belonging to the uncertainty region.

The next theorem studies the solution of the optimization problem (12). For the sake of simplicity of notations, define

$$s(\mathcal{E}) = \max_{||y||_2=1}\left\{|a^\top y| \times \|My\|_2\right\} \tag{13}$$

Theorem 2: Suppose $\mathcal{E} = \{a + Mu \mid u \in \mathbb{R}^{n\times n}\}$ is the uncertainty region of the initial state and α^* is the optimal objective value of (12). Furthermore, define P^* as the unique solution of the Lyapunov equation

$$(A + BK_c)P(A + BK_c)^\top - P + aa^\top + M^2 = 0 \tag{14}$$

Then, the following statements hold:

1. (β, P^*) is a feasible solution for (12), where

$$\beta = 2\max\left\{s(\mathcal{E}), (\lambda_M^{\max})^2\right\} \tag{15}$$

2. $0 \leq \beta - \alpha^* \leq \left((\lambda_M^{\max})^2 - s(\mathcal{E})\right)_+$

Proof: The optimization problem (12) can be written as

$$\min \quad \alpha \tag{16a}$$
$$\text{s.t.} \quad \mathcal{L}(P,x) \preceq \alpha I, \qquad \forall x \in \mathcal{E} \tag{16b}$$
$$-\alpha I \preceq \mathcal{L}(P,x), \qquad \forall x \in \mathcal{E} \tag{16c}$$
$$P \succeq 0 \tag{16d}$$

Notice that (16b) is equivalent to $y^\top \mathcal{L}(P,x) y \leq \alpha$ for every $x \in \mathcal{E}$ and y such that $\|y\|_2 = 1$. This implies that (16b) is equivalent to

$$y^\top \mathcal{L}(P,0) y + \max_{\substack{x=a+Mu \\ \|u\|_2=1}} (y^\top x)^2 \leq \alpha \tag{17}$$

for every y such that $\|y\|_2 = 1$. Now, consider

$$\max_{\substack{x=a+Mu \\ \|u\|_2\leq 1}} \{(y^\top x)^2\} \tag{18}$$

Using S-procedure, it can be easily verified that (18) is equal to $(|a^\top y| + \|My\|_2)^2$. Therefore, (17) can be reduced to

$$y^\top \mathcal{L}(P,0) y + (|a^\top y| + \|My\|_2)^2 \leq \alpha \tag{19}$$

Now, consider (16c). Similar to the previous case, this constraint is equivalent to

$$-\alpha \leq y^\top \mathcal{L}(P,0) y + \min_{\substack{x=a+Mu \\ \|u\|_2\leq 1}} (y^\top x)^2 \tag{20}$$

for every y such that $\|y\|_2 = 1$. One can use strong duality to show that

$$\min_{\substack{x=a+Mu \\ \|u\|_2\leq 1}} \{(y^\top x)^2\} = \left((|a^\top y| - \|My\|_2)_+\right)^2 \tag{21}$$

Therefore, (20) is equivalent to

$$-\alpha \leq y^\top \mathcal{L}(P,0) y + \left((|a^\top y| - \|My\|_2)_+\right)^2 \tag{22}$$

Now, it follows from (19) and (22) that the inequalities

$$0 \leq \alpha + y^\top \mathcal{L}(P,0) y + \left(|a^\top y| - \|My\|_2\right)^2 \tag{23a}$$
$$0 \leq \alpha - y^\top \mathcal{L}(P,0) y - (|a^\top y| + \|My\|_2)^2 \tag{23b}$$

should be satisfied for every y such that $\|y\|_2 = 1$ (note that we did not use the "+" operator in the above equations). Combining (23a) and (23b), one can verify that the inequality

$$2(|a^\top y|)\|My\|_2 \leq \alpha \tag{24}$$

is satisfied for every y such that $\|y\|_2 = 1$. This implies that

$$s(\mathcal{E}) \leq \alpha \tag{25}$$

Next, it will be proved that the defined pair of (β, P^*) is indeed feasible for (16). First, notice that since P^* satisfies (14) and $aa^T + M^2 \succeq 0$, it yields that $P^* \succeq 0$. Now, the goal is to show the feasibility of (16b) and (16c) via their equivalence to (19) and (22), respectively. Combining the definitions of β and P^* with (19) leads to the inequality

$$0 \leq \beta + (a^\top y)^2 + \|My\|_2^2 - (|a^\top y| + \|My\|_2)^2 \tag{26}$$

This is equivalent to

$$2|a^\top y|\|My\|_2 \leq \beta \tag{27}$$

which holds for every y such that $\|y\|_2 = 1$, due to the definition of β. This implies that (β, P^*) satisfies (16b). Similarly, one can substitute β and P^* in (22) to derive the inequality

$$0 \leq \beta - (a^\top y)^2 - \|My\|_2^2 + ((|a^\top y| - \|My\|_2)_+)^2 \tag{28}$$

If $|a^\top y| \geq \|My\|_2$, the above inequality holds due to (27). Now, assume that $|a^\top y| < \|My\|_2$. It is useful to show that

$$(a^\top y)^2 + \|My\|_2^2 \leq \beta \tag{29}$$

for every y such that $\|y\|_2 = 1$. Observe that

$$(a^\top y)^2 + \|My\|_2^2 \leq 2\|My\|_2^2 \leq 2(\lambda_M^{\max})^2 \leq \beta \tag{30}$$

which certifies that (29) holds for every feasible y. This implies that (β, P^*) satisfies (16c) and, hence, it is feasible for (16). The second part of the theorem follows from the definition of β and the fact that $s(\mathcal{E}) \leq \alpha^*$ (due to (25)). ■

Remark 2: Notice that if $s(\mathcal{E}) \geq (\lambda_M^{\max})^2$, the pair (β, P^*) is an optimal solution of (16). One sufficient condition for the satisfaction of $s(\mathcal{E}) \geq (\lambda_M^{\max})^2$ is the inequality $\|Ma\|_2 \geq (\lambda_M^{\max})^2$. Roughly speaking, this condition holds for those ellipsoids with the properties that the ratio of their largest and smallest diameters is not very large and that the center of the ellipsoid is sufficiently far from the origin. For example, it holds whenever the feasible region is equal to a 2-norm ball that does not include the origin.

Theorem 2 does not offer closed-form formulas for α^* and β. However, notice that only the matrix P^* is required for the design of the distributed controller, and this matrix can be efficiently found by solving the Lyapunov equation (14). Based on the above analysis, the proposed method for finding a high performance distributed controller for LTI systems with an uncertain initial state can be summarized as follows:

1. Solve the Lyapunov equation (14) in order to find P^*.
2. Solve the optimization problem (6) after replacing P_a with P^* to obtain a distributed controller.

IV. Lower Bounds on Optimality Guarantee

So far, we have combined two convex optimization problems to design a distributed controller with the prescribed sparsity structure. It is also argued that this distributed controller would have a similar performance to the optimal centralized one. In this section, this statement will be formalized by showing that the proposed method finds a distributed controller that maximizes the optimality guarantee. In particular, Theorem 1 will be generalized to derive a lower bound on the optimality guarantee of the designed distributed controller for a system with an uncertain initial state.

Definition 3 ([24]): For a stable matrix X, define the radius of stability as $r(X) = \inf_{0\leq\theta\leq 2\pi} \|(e^{\mathbf{i}\theta} - X)^{-1}\|^{-1}$. It can be verified that $r(X) > 0$ and $r(X) + \rho(X) \leq 1$.

Lemma 1 ([24], [25]): Assume that R satisfies the Lyapunov equation $XRX^T - R + Y = 0$ for a stable matrix X. Then, the inequality

$$\|R\|_2 \leq \frac{\|Y\|_2}{r(X)^2} \tag{31}$$

holds.

For every $x \in \mathcal{E}$, define P_x as the unique positive semidefinite solution of

$$\mathcal{L}(P, x) = 0 \tag{32}$$

Lemma 2: The relation

$$\|P - P_x\|_2 \leq \frac{\alpha}{r(A + BK_c)^2} \tag{33}$$

holds for every $x \in \mathcal{E}$ and every feasible solution (α, P) of the optimization problem (12).

Proof: One can write

$$-\alpha I \preceq \mathcal{L}(P, x) \preceq \alpha I \tag{34}$$

for every $x \in \mathcal{E}$. Subtracting $\mathcal{L}(P_x, x) = 0$ from (34) yields

$$-\alpha I \preceq \mathcal{L}(P - P_x, 0) \preceq \alpha I \tag{35}$$

According to (35) and Lemma 1, one can write

$$\|P - P_x\|_2 \leq \frac{\|\mathcal{L}(P - P_x, 0)\|_2}{r(A + BK_c)^2} \leq \frac{\alpha}{r(A + BK_c)^2} \tag{36}$$

This completes the proof. ■

The following theorem is an extension of Theorem 1 for systems with uncertain initial states.

Theorem 3: Assume that $Q = I_n$ and $R = I_m$. Consider the optimal centralized gain K_c, an arbitrary stabilizing gain $K_d \in \mathcal{K}$, and a feasible solution (α, P) for the optimization problem (12). The controller $u[\tau] = K_d x[\tau]$ has the optimality guarantee μ for every initial state $x \in \mathcal{E}$, where

$$\mu = \frac{1}{\left(1 + \zeta\sqrt{C_1(P, K_d) + \gamma\alpha}\right)^2} \tag{37}$$

and

$$\zeta = \max\left\{\frac{\kappa(V)\|B\|_2}{(1 - \rho(A + BK_d))\sqrt{\sum_{\tau=0}^{\infty}\|x_c[\tau]\|_2^2}}, \right.$$
$$\left.\frac{1 - \rho(A + BK_d) + \kappa(V)\|K_d\|_2\|B\|_2}{(1 - \rho(A + BK_d))\sqrt{\sum_{\tau=0}^{\infty}\|u_c[\tau]\|_2^2}}\right\} \tag{38a}$$

$$\gamma = \frac{\|K_c - K_d\|_F^2}{r(A + BK_c)^2} \tag{38b}$$

Proof: It yields that

$$\begin{aligned}
&\text{trace}\{(K_c - K_d)P_x(K_c - K_d)^T\}\\
&=\text{trace}\{(K_c - K_d)P(K_c - K_d)^T\}\\
&\quad + \text{trace}\{(K_c - K_d)(P_x - P)(K_c - K_d)^T\}
\end{aligned} \tag{39}$$

On the other hand, according to Lemma 2, one can verify that

$$\text{trace}\{(K_c - K_d)(P_x - P)(K_c - K_d)^T\} \tag{40}$$
$$= \text{trace}\{(K_c - K_d)^T(K_c - K_d)(P_x - P)\} \tag{41}$$
$$\leq \lambda_{\max}(P_x - P)\text{trace}\{(K_c - K_d)^T(K_c - K_d)\} \tag{42}$$
$$\leq \|P - P_x\|_2\|K_c - K_d\|_F^2 \tag{43}$$
$$\leq \frac{\|K_c - K_d\|_F^2}{r(A + BK_c)^2}\alpha \tag{44}$$

Hence, the relation

$$C_1(P_x, K_d) \leq C_1(P, K_d) + \frac{\|K_c - K_d\|_F^2}{r(A + BK_c)^2}\alpha \tag{45}$$

holds for every $x \in \mathcal{E}$. The proof is completed using (45) and Theorem 1. ■

Remark 3: Theorem 3 states that minimizing both α and $C_1(P, K)$ increases the optimality guarantee of the designed distributed controller. More specifically, if the uncertainty region $\mathcal{E}$ is a singleton ($M = 0$), the derived lower bound on the optimality guarantee is identical to the one introduced in Theorem 1.

In this paper, we have not considered stochastic systems that are subject to measurement and input disturbance noise. For those systems, either the time average or the asymptotic behavior of the states and inputs is often considered as a measure of their performance, as opposed to the cost function (2) used for deterministic systems. It can be observed that if the system is subject to noise, the objective function defined in (2) would be unbounded due to the summation of non-vanishing noise covariances. Therefore, the choice of the objective function in stochastic systems implies that the optimal distributed controller is independent of the value of the initial state. For such systems, the method introduced in [21] can be used to design distributed controllers.

V. Case Studies

In this section, the developed design procedure will be applied to two case studies on multi-agent and power systems to demonstrate its efficacy in different real-world problems.

A. Multi-Agent Systems

Consider N identical LTI systems, named *agents*, with the following state-space representation in the continuous domain:

$$\dot{x}^i(t) = Ax^i(t) + Bu^i(t) \tag{46a}$$
$$y^i(t) = Cx^i(t) \tag{46b}$$

where $x^i(t)$ and $u^i(t)$ denote the state and the input of agent i at time t and $y^i(t)$ is the output that the controller can use to design the feedback law. The agents are coupled through a global objective, which is to maintain their relative positions in the system. The communication among these agents is captured via an undirected graph. If there exists an edge between agents i and j, it means that both agents have access to the difference of their outputs $y^i(t) - y^j(t)$. Let

$N(i)$ denote the set of neighbors of agent i, and consider the control law

$$u^i(t) = \sum_{j \in N(i)} k_{ij} \left(y^j(t) - y^i(t) \right) \tag{47}$$

There is no assumption on the symmetry of the control gain and, therefore, it may occur that $k_{ij} \neq k_{ji}$. The structure of the distributed controller is defined as

$$K_d(i,j) = \begin{cases} k_{ij} & \text{if} \quad j \in N(i) \\ 0 & \text{if} \quad j \notin N(i) \end{cases} \tag{48}$$

for every pair of agents i and j.

The goal is to design a distributed controller K_d that minimizes the objective function

$$J_c(K_d) = \int_0^\infty \left(y(t)^\top Q y(t) + u(t)^\top R u(t) \right) dt \tag{49}$$

while ensuring the stability of the closed-loop system. Here, $y(t)$ and $u(t)$ are the concatenation of the output and input vectors of all agents, respectively. To illustrate the performance of the proposed method on multi-agent systems, consider the planar vertical takeoff and landing (PVTOL) of a set of aircraft, where the model of each aircraft is given as [26]:

$$\ddot{X}^i(t) = v(t), \quad \ddot{\theta}^i(t) = \frac{1}{\delta} \left(\sin\theta^i(t) + v^i(t)\cos\theta^i(t) \right) \tag{50}$$

Note that X and θ are the horizontal position and angle of aircraft i, respectively, and δ depends on the coupling between the rolling moment and lateral acceleration of the aircraft. Assuming that all agents are stabilized vertically, we only study their horizontal position in this problem. Consider the feedback rule

$$v^i(t) = \alpha \dot{X}^i(t) + \beta\theta^i(t) + \gamma\dot{\theta}^i(t) + u^i(t) \tag{51}$$

for the control of aircraft i. The first three terms in (51) are used for the internal stability of the horizontal speed and angle of each aircraft. The last term needs to be designed using a controller with the structure delineated in (47) and (48) such that the agents maintain their relative positions.

As an example, consider a system consisting of 4 aircraft whose communication structure is in the form of a path graph. This communication structure also defines the sparsity of the to-be-designed distributed controller. Assume that the desired distance between adjacent agents is equal to d. By defining the state of aircraft $i \in \{1,2,3,4\}$ as

$$x^i(t) = [\dot{X}^i(t),\ \theta^i(t),\ \dot{\theta}^i(t)]^\top \tag{52}$$

the linearized model of this system can be described as (46) where

$$A = \begin{bmatrix} \alpha & \beta & \gamma \\ 0 & 0 & 1 \\ \frac{\alpha}{\delta} & \frac{\beta+1}{\delta} & \frac{\gamma}{\delta} \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ \frac{1}{\delta} \end{bmatrix} \tag{53}$$

and $C = I$ [27]. Note that the parameters α, β, and γ are used to guarantee the internal stability of each aircraft. As explained in [27], for $\delta = 0.1$, the values $\alpha = 90.62$, $\beta = -42.15$ and $\gamma = -13.22$ in the state feedback controller of each agent ensure its internal stability. Since the agents have access to their relative distance from their neighbors, define

$$z^i(t) = [X^i(t) - X^{i+1}(t) - d,\ x^i(t)^\top]^\top \tag{54}$$

for $i = 1,2,3$ and $z^4(t) = x^4(t)$. Based on the above definition, the state-space model of the entire system can be described as

$$\dot{z}(t) = \underbrace{\begin{bmatrix} \tilde{A} & H_4 & 0 & 0 \\ 0 & \tilde{A} & H_4 & 0 \\ 0 & 0 & \tilde{A} & H_3 \\ 0 & 0 & 0 & A \end{bmatrix}}_{\tilde{\mathcal{A}}} z(t) + \underbrace{\begin{bmatrix} \tilde{B} & 0 & 0 & 0 \\ 0 & \tilde{B} & 0 & 0 \\ 0 & 0 & \tilde{B} & 0 \\ 0 & 0 & 0 & B \end{bmatrix}}_{\tilde{\mathcal{B}}} u(t) \tag{55}$$

Note that the vectors $z(t)$ and $u(t)$ are the concatenation of $z^i(t)$ and $u^i(t)$ for all agents, respectively. The matrix H_4 (or H_3) is a 4×4 (or 3×3) matrix whose $(i,j)^{\text{th}}$ entry is equal to -1 if $(i,j) = (1,2)$ (or $(i,j) = (1,1)$) and is equal to 0, otherwise. Finally, $\tilde{A}$ and $\tilde{B}$ are defined as

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \alpha & \beta & \gamma \\ 0 & 0 & 0 & 1 \\ 0 & \frac{\alpha}{\delta} & \frac{\beta+1}{\delta} & \frac{\gamma}{\delta} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \frac{1}{\delta} \end{bmatrix} \tag{56}$$

The structure of the distributed controller for the system described in (55) can be viewed as

$$K_d = \left[\begin{array}{cccc|cccc|cccc|ccc} \star & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \star & 0 & 0 & 0 & \star & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \star & 0 & 0 & 0 & \star & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \star & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \tag{57}$$

where each "$\star$" corresponds to a to-be-designed free element of the distributed controller. Since the goal is to bring each aircraft to its pre-specified relative location as quickly as possible with the least amount of effort, we define the weighting matrix Q to be a diagonal matrix with $Q(k,k)$ equal to 100 if the k^{th} element of $x(t)$ corresponds to the relative positions of neighboring agents and 1 otherwise. Furthermore, we choose R to be an identity matrix with appropriate dimension. Let the estimate of the initial state of the whole system be equal to a vector a whose k^{th} element is uniformly drawn from the interval $[-2\ 2]$. Moreover, we consider a maximum amount of $0.2 \times |a|$ for the estimation error, where $|\cdot|$ is the entry-wise absolute value operator. This means that the initial state of the system can reside anywhere between $a - 0.2 \times |a|$ and $a + 0.2 \times |a|$. It is easy to observe that the smallest-volume outer ellipsoidal approximation of this uncertainty region can be described as $\mathcal{E} = \{a + Mu : u \in \mathbb{R}^{15\times 1}, \|u\|_2 \leq 1\}$, where M is a diagonal matrix with the k^{th} diagonal entry equal to $0.2 \times |a_k| \times \sqrt{15}$.

We discretize the system using the zero-order hold method with the sampling time equal to 0.01 and then find the distributed controller via the method presented in this paper. The free entries of the designed distributed controller are

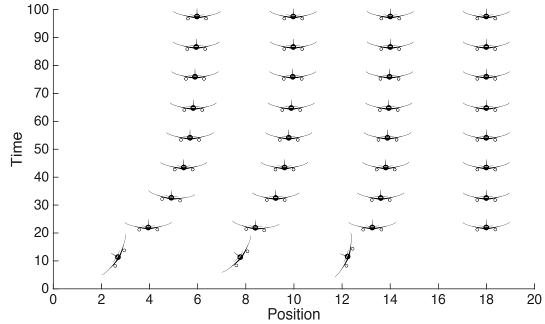


Fig. 1: Formation of 4 aircraft over the horizontal axis with $d = 4$.

obtained as:

$$
\begin{aligned}
&K_d(1,1) = -8.84,\ K_d(2,1) = 4.72,\ K_d(2,5) = -7.30\\
&K_d(3,5) = 6.18,\ K_d(3,9) = -4.90,\ K_d(4,9) = 9.67
\end{aligned} \tag{58}
$$

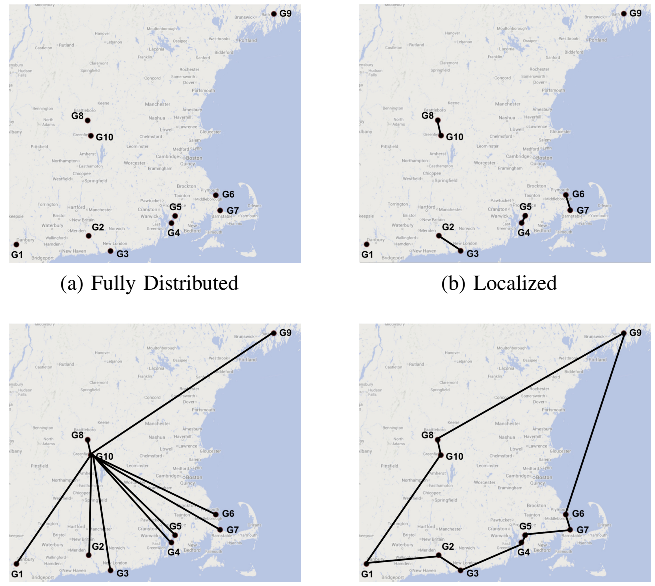
This controller makes the closed-loop system stable. We also find the optimal centralized LQR controller for the continuous system in order to measure the optimality guarantee of the designed distributed controller. For 100 uniformly and independently chosen initial states from the uncertainty region, the average cost function using the optimal centralized LQR controller is 7793.49, whereas the average cost function for the designed distributed controller is 8178.40. Moreover, the average optimality guarantee of these trials is 95.28% with the standard deviation of 0.32. Figure 1 shows a snapshot of the coordination of the four aircraft for one of these trials.

B. Power Systems

In this case study, we consider the frequency control problem for power systems. The aim is to control the frequency of the power system with a distributed controller that respects a certain sparsity structure. This sparsity structure determines which generators can share their rotor angle and frequency with each other. We consider the IEEE 39-bus New England Power System. The relationship between the rotor angle and frequency of each generator can be described by the per-unit swing equation

$$M_i\ddot{\theta}_i + D_i\dot{\theta}_i = P_{Mi} - P_{Ei} \tag{59}$$

where θ_i is the voltage (or rotor) angle at a generator bus i (in rad), P_{M_i} denotes the mechanical power input to the generator at bus i (in per unit), P_{E_i} shows the electrical active power injection at bus i (in per unit), M_i is the inertia coefficient of the generator at bus i (in pu-sec^2/rad), and D_i is the damping coefficient of the generator at bus i (in pu-sec/rad) [28]. The relationship between the electrical active power injection P_{E_i} and the voltage angles can be described by a set of nonlinear equations, known as AC power flow equations. In order to simplify these equations and to linearize the representation of the system, a widely-used method is to utilize the following DC power flow equations as an approximation of the nonlinear relationship



(a) Fully Distributed

(b) Localized

(c) Star (G10 in center)

(d) Ring

Fig. 2: Communication structures studied for the IEEE 39-bus test system (borrowed from [18]).

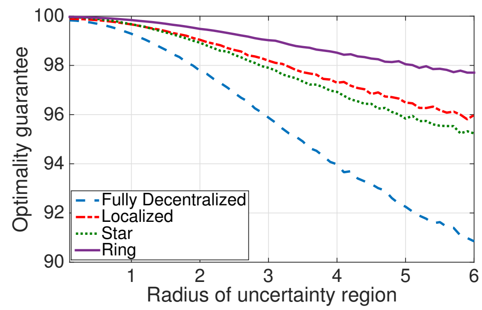


Fig. 3: Optimality gap of the designed distributed controller for different topologies.

between the active power injection and voltage angles:

$$P_{Ei} = \sum_{j=1}^{n} B_{ij}(\theta_i - \theta_j) \tag{60}$$

where n is the number of buses in the system and B_{ij} is the susceptance of the line (i, j). Writing (60) in a matrix form gives rise to the following state-space representation of the frequency control problem:

$$\begin{bmatrix}\dot{\theta}(t)\\ \dot{w}(t)\end{bmatrix} = \begin{bmatrix}0_{n\times n} & I_n\\ -M^{-1}L & -M^{-1}D\end{bmatrix}\begin{bmatrix}\theta(t)\\ w(t)\end{bmatrix} + \begin{bmatrix}0_{n\times n}\\ M^{-1}\end{bmatrix}P_M(t) \tag{61}$$

where $\theta(t) = [\theta_1(t), \ldots, \theta_n(t)]^T$ and $w(t) = [w_1(t), \ldots, w_n(t)]^T$ represent the state of the rotor angles and the frequency of generators at time t, respectively. Furthermore, L is the Laplacian matrix, $M = \text{diag}(M_1, \ldots, M_n)$ and $D = \text{diag}(D_1, \ldots, D_n)$. The IEEE 39-bus system has 10 generators. We consider four different topologies for the distributed controller: Fully decentralized, Localized, Star, and Ring. Figure 2 visualizes these structures on the map. The state and input weighting matrices are chosen as I and $0.1 \times I$, respectively.

We discretize the system using the zero-order hold method with the sampling rate of 0.2. Consider an uncertainty region in the form of a sphere that is centered at $[1, 1, ..., 1]^T$ and has a radius ψ to be specified later. In order to show the performance of the proposed method, we analyze the optimality guarantee of the designed distributed controller for different topologies with respect to the radius of the uncertainty region varied from 0.1 to 6. For each radius and topology, we consider 1000 independent trials with initial states uniformly chosen from the spherical uncertainty region. The results are summarized in Figure 3. It can be observed that the ring topology has the best performance for different radii. The maximum and minimum optimality guarantees for this structure are equal to 99.97% and 97.70% (corresponding to the radii 0.1 and 6), respectively. Moreover, the worst performance corresponds to the fully decentralized controller with the maximum and minimum optimality guarantees equal to 99.83% and 90.85%, respectively. Finally, it can be observed that the star and localized structures have relatively similar performances with respect to the radius of the uncertainty region.

VI. Conclusion

This paper studies the optimal distributed control (ODC) problem for deterministic discrete-time linear systems. The goal is to design a static controller that obeys a user-defined sparsity pattern for the system. This distributed controller should maximize the performance of the controlled system while stabilizing the closed-loop system. Recently, we have developed an efficient method to design a distributed controller whose state and input trajectories resemble those of the optimal centralized controller, provided that the initial state is known precisely. In this paper, we extend this method to the case where the initial state belongs to an uncertainty region and, therefore, a robust and high-performance controller is sought. It is proved that the designed distributed controller yields a high optimality guarantee for every initial state in the prescribed uncertainty region. Extensive simulations are performed on two real-world problems, namely aircraft formation as a multi-agent system and frequency control of power systems, to demonstrate the efficacy of the developed method.

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