Identifying the Connectivity of Feasible Regions for Optimal Decentralized Control Problems

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***Abstract*—The optimal decentralized control (ODC) is an NP-hard problem with many applications in real-world systems. There is a recent trend of using local search algorithms for solving optimal control problems. However, the effectiveness of these methods depends on the connectivity property of the feasible region of the underlying optimization problem. In this paper, we develop the notion of stable expandability and use it to obtain a novel criterion for certifying the connectivity of the feasible region for a class of ODC problems. This criterion can be checked via an efficient algorithm. Based on the developed mathematical technique, we prove that the feasible region is guaranteed to be connected in presence of only a small number of communication constraints. We also show that among the exponential number of possible communication networks (named patterns), a square root of them lead to connected feasible regions. A by-product of this result is that a high-complexity ODC problem may be approximated with a simpler one by replacing its pattern with a favorable pattern that makes the feasible region connected.**

I. INTRODUCTION

The field of optimal decentralized control (ODC) emerges as communication constraints among agents are prevalent in many real-world systems, including power grids [1], computer networks [2] and robotics [3]. Being a nonconvex optimization problem, the general ODC problem has been proved to be computationally intractable [4], [5]. Many techniques have been proposed in the literature to convexify or solve special cases of the ODC problem [6]–[12].

Inspired by the learning algorithms in the field of machine learning, the recent work [13] suggests using local search methods to solve the optimal control problems. Local search methods have several advantages, such as having low computational and memory complexities and the ability of being implemented without explicitly establishing the underlying model. The main issue with these methods is that they are not guaranteed to find the global optimal solution if the problem does not have a convex structure. However, for the classical (centralized) LQR optimal control problem, [13] proves that the gradient descent method will converge to the global optimal solution despite the nonconvexity of the problem. Given this surprising result, it is natural to ask whether local search methods are also effective for ODC problems.

The effectiveness of local search methods depends on the connectivity properties of the feasible region. If the feasible region is connected, a local search method only needs to take feasible directions. As being successful in many machine learning problems, approaches such as stochastic gradient method are able to find near-globally optimal solutions even in the presence of spurious local minima [14]. However, if the feasible region is disconnected, then the algorithm needs to either initiate in each of the connected components or take infeasible directions during the iteration, which significantly increases the computational burden and is the underlying reason for the NP-hardness of many problems.

The recent work [15] has found a class of ODC problems with n state variables whose feasible regions have $O(2^n)$ connected components. This negative result shows that local search methods are not effective for general ODC problems, since there could be an exponential number of local minima that are far away from each other. The later paper [16] characterizes the connectivity property for single-input-single-output systems. However, for general multiple-input-multiple-output systems, there are only a few cases in which the connectivity of the feasible region has been determined.

In this paper, for ODC problems of systems with direct state-feedback, we develop a new criterion for the connectivity of the feasible region, which can be verified by an efficient algorithm. Furthermore, based on the new tool, the following two results can be obtained: (i) The feasible region is connected for most dense communication networks; (ii) There are an exponential number of communication networks with connected feasible regions. These networks constitute a set of easier problems that can be used as approximations for other ODC problems, which is similar to the common approach of using convex functions to approximate nonconvex functions.

This paper is organized as follows. The notation and problem formulation are introduced in Section II. In Section III, we first develop a powerful connectivity criterion. Then, we design an efficient algorithm to check the connectivity based on this criterion. In Section IV, we construct and prove the connectivity for certain classes of ODC problems. Concluding remarks are given in Section V, followed by some proofs in the Appendix.

II. NOTATION AND PROBLEM FORMULATION

Before proceeding with the problem formulation, we summarize the common notations used in the paper below:

- I_n is the $n \times n$ identity matrix.
- $\mathcal{S}_n$ is the set of $n \times n$ symmetric matrices.

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- $\mathcal{P}_n$ is the set of $n \times n$ positive definite matrices.
- $\mathcal{K}_n$ is the set of $n \times n$ stable matrices, i.e., matrices whose eigenvalues have a negative real part.

Consider the continuous-time linear system:

$$\dot{x}(t) = Ax(t) + Bu(t),$$

where $x(t) \in \mathbb{R}^n$ is the system state and $u(t) \in \mathbb{R}^m$ is the control input. The optimal static state-feedback control problem is to design a feedback controller $u(t) = -Kx(t)$ with $K \in \mathbb{R}^{m\times n}$ while minimizing certain cost functional. For example, in the classical infinite-horizon LQR problem, the objective is to minimize

$$J = \int_0^{+\infty} (x^T(t)Qx(t) + u^T(t)Ru(t) + 2x^T(t)Nu(t))\mathrm{d}t$$

subject to the constraint that the control law K must stabilize the closed-loop system

$$\dot{x}(t) = (A - BK)x(t).$$

Consider the ODC problem of designing an optimal static state-feedback controller minimizing an arbitrary cost functional (not necessarily a quadratic one), where there are some communication constraints enforcing certain entries of K to be zero. Let

$$\mathscr{P} \subseteq \{(i, j)|1 \le i \le m,\ 1 \le j \le n\}$$

be the set of indices for all free variables K_{ij} that are not restricted by the communication constraints. The set $\mathscr{P}$ will be referred to as a *pattern* in this paper. After substituting

$$x(t) = \mathrm{e}^{(A-BK)t}x(0), \quad u(t) = -K\mathrm{e}^{(A-BK)t}x(0)$$

into the cost functional, the ODC problem can be formulated as the minimization of a cost function with the only variable K over the feasible region

$$\mathcal{D} = \{K \in \mathcal{L}(\mathscr{P})|A - BK \in \mathcal{K}_n\},$$

where the linear subspace $\mathcal{L}(\mathscr{P})$ is given by

$$\mathcal{L}(\mathscr{P}) = \{K \in \mathbb{R}^{m\times n}|K_{ij} = 0,\ \forall (i, j) \notin \mathscr{P}\}.$$

The performance of local search methods for solving ODC through this formulation (or others) is directly related to the geometric properties of the feasible region $\mathcal{D}$. The main goal of this work is to study the connectivity of $\mathcal{D}$ under the usual Euclidean topology.

In this paper, we will focus on the important special case where the system has direct state-feedback, i.e., $m = n$ and B is identity. In this case, similar to the notations used in [17], the pattern $\mathscr{P}$ can be represented by both a matrix and a directed graph with possible self-loops. For instance, the pattern

$$\{(1, 1), (1, 2), (2, 3), (3, 1)\} \tag{1}$$

can be described in the matrix form

$$\begin{bmatrix} * & * & 0 \\ 0 & 0 & * \\ * & 0 & 0 \end{bmatrix}$$

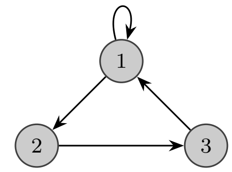

Fig. 1. Graph representation of the pattern (1).

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or equivalently in the graph form given by Fig. 1. In addition, for a pattern $\mathscr{P}$ viewed as a graph, we denote its complement graph as $\mathscr{P}^c$ and the number of edges in $\mathscr{P}^c$, i.e., the number of "0"s in $\mathscr{P}$, as $|\mathscr{P}^c|$.

III. Main Results

A. The Connectivity Criterion

In this section, we will first develop a powerful criterion for ensuring the connectivity of the feasible region corresponding to a pattern $\mathscr{P}$, and then devise an accompanying algorithm to check the connectivity based on the criterion. To enable the mathematical analysis of $\mathcal{D}$, it is beneficial to introduce a notion that is stronger that connectivity, as stated below.

Definition 1: A pattern $\mathscr{P}$ is said to be *stably expandable* if the following property holds: for every two stable matrices $K^0 \in \mathcal{K}_n$ and $K^1 \in \mathcal{K}_n$, together with arbitrary continuous functions $\gamma_{ij}(\tau) : [0, 1] \to \mathbb{R}$ defined for all $(i, j) \notin \mathscr{P}$ with the endpoint conditions

$$K^0_{ij} = \gamma_{ij}(0), \quad K^1_{ij} = \gamma_{ij}(1),$$

there exists a (continuous) path $K(\tau) : [0, 1] \to \mathcal{K}_n$ with $K(0) = K^0$, $K(1) = K^1$ expanding the functions $\gamma_{ij}(\tau)$, i.e.,

$$K_{ij}(\tau) = \gamma_{ij}(\tau), \quad \forall\tau \in [0, 1],\ \forall (i, j) \notin \mathscr{P}.$$

In the special case where the functions $\gamma_{ij}(\tau)$ are selected to be constant, we immediately arrive at the following result.

Proposition 1: If a pattern $\mathscr{P}$ is stably expandable, then the corresponding feasible region $\mathcal{D}$ is connected for all $A \in \mathbb{R}^{n\times n}$. ∎

Some basic properties of the stable expandability are:

1) If the pattern $\mathscr{P}$ is stably expandable and $\mathscr{P} \subseteq \mathscr{Q}$, then $\mathscr{Q}$ is also stably expandable.
2) If the pattern $\mathscr{P}$ is stably expandable, its transpose $\mathscr{P}^T$ is also stably expandable, since any matrix is similar to its transpose.
3) If the pattern $\mathscr{P}$ is stably expandable and $\mathscr{Q}$ is isomorphic (as a graph) to $\mathscr{P}$, then $\mathscr{Q}$ is also stably expandable, since any matrix is similar to the matrix obtained by simultaneously applying the same permutation on its rows and columns.

It is shown in [15] that the feasible region $\mathcal{D}$ is connected if the diagonal elements of K are free. The following proposition shows that such patterns also satisfy the stronger property of stable expandability.

Proposition 2: A pattern $\mathscr{P}$ is stably expandable if its complement graph $\mathscr{P}^c$ does not have self-loops.

Proof: Given stable matrices K^0, K^1 and functions $\gamma_{ij}(\tau)$, consider an arbitrary path $K'(\tau)$ satisfying all of the requirements in Definition 1 except the stability of $K'(\tau)$ for all $\tau \in [0,1]$. Define the function $\sigma(\tau)$ to be the largest real part of the eigenvalues of $K'(\tau)$. Then, $\sigma(\tau)$ is a continuous function and the path

$$K(\tau) = K'(\tau) - (\max\{\sigma(\tau), 0\} + \tau(1-\tau))I_n,$$

is guaranteed to be contained in $\mathcal{K}_n$. Since $K(0) = K'(0)$, $K(1) = K'(1)$ and the two paths have the same off-diagonal elements, $K(\tau)$ is a path satisfying all of the requirements in Definition 1. ■

It is desirable to show that the stable expendability of a pattern may be checked by analyzing smaller subpatterns.

Theorem 3: The $n \times n$ pattern

$$\begin{array}{r} n-m\left\{\vphantom{\begin{matrix}a\\a\end{matrix}}\right. \\ m\left\{\vphantom{\begin{matrix}a\\a\end{matrix}}\right. \end{array}\!\!\underbrace{\left[\begin{array}{c:c} \mathscr{A} & \mathscr{B} \\ \hdashline \mathscr{C} & \mathscr{D} \end{array}\right]}_{n-m \quad m}$$

is stably expandable if

1) The pattern $\mathscr{A}$ is stably expandable.
2) The number of rows in $\mathscr{B}$ without "0"s is at least $m+1$.
3) The pattern $\mathscr{C}$ does not contain "0"s.

Proof: See the Appendix. ■

To check the connectivity of the feasible region $\mathcal{D}$ associated with a pattern $\mathscr{P}$, one can partition the vertices of $\mathscr{P}$ appropriately and then apply Theorem 3 multiple times. This will be formalized below.

Corollary 4: A pattern $\mathscr{P}$ is stably expandable if there exists a partition $\{S_1, S_2, \ldots, S_m\}$ of the vertices such that

1) For every $1 \le k < l \le m$, there is no edge from S_l to S_k in the complement graph $\mathscr{P}^c$.
2) The subpattern with the vertex set S_1 is stably expandable.
3) For each $k > 1$, if d_k denotes the number of vertices i with the property

$$i \notin S_k \text{ and } \exists j \in S_k \text{ s.t. } (i,j) \in \mathscr{P}^c$$

and r_k denotes the number of vertices in S_k, then

$$\sum_{l=1}^{k-1} r_l > d_k + r_k.$$

Proof: Let $\mathscr{P}_k$ be the subpattern of $\mathscr{P}$ with the vertex set $\cup_{l=1}^{k} S_l$. We prove by induction that $\mathscr{P}_k$ is stably expandable. The base step $k = 1$ is obviously true. Now, assume that $\mathscr{P}_{k-1}$ is stably expandable. After ordering the vertices, one can write the subpattern $\mathscr{P}_k$ in matrix form as follows:

$$\begin{bmatrix} \mathscr{P}_{k-1} & \mathscr{B} \\ \mathscr{C} & \mathscr{D} \end{bmatrix}.$$

In the above pattern, $\mathscr{P}_{k-1}$ is a stably expandable pattern of size $\sum_{l=1}^{k-1} r_l$ and $\mathscr{D}$ is the subpattern of $\mathscr{P}$ with the vertex set S_k. By Condition 1, $\mathscr{C}$ does not have "0"s. Moreover, the number of rows in $\mathscr{B}$ containing "0"s is exactly d_k. Therefore, in light of Condition 3, the number of rows in $\mathscr{B}$ without "0"s can be computed as

$$\sum_{l=1}^{k-1} r_l - d_k > r_k.$$

By Theorem 3, the subpattern $\mathscr{P}_k$ is stably expandable. This completes the proof. ■

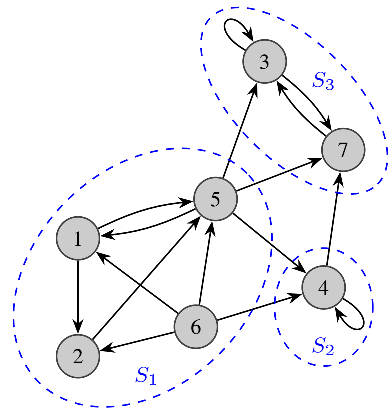


Fig. 2. The complement graph of the pattern studied in Example 1.

Example 1: To illustrate the application of Corollary 4, consider the 7×7 pattern $\mathscr{P}$ whose complement graph $\mathscr{P}^c$ is given by Fig. 2. One can partition the pattern into three parts S_1, S_2 and S_3 as shown in Fig. 2, where

$$r_1 = 4, \quad r_2 = 1, \quad r_3 = 2, \quad d_2 = 2, \quad d_3 = 2.$$

For this partition, the subpattern corresponding to S_1 is stably expandable due to Proposition 2, and the other conditions in Corollary 4 can be directly verified. As a result, the pattern $\mathscr{P}$ is stably expandable and the feasible region of the ODC problem is connected no matter what the matrix A is.

B. The Connectivity Detection Algorithm

The objective of this part is to develop an algorithm that finds a suitable partition for an arbitrary pattern $\mathscr{P}$ to reason about its stable expandability based on Corollary 4. To design the algorithm, the first step is to choose S_1. Since the only stably expandable patterns known initially are the ones satisfying Proposition 2, S_1 should only contain vertices without self-loops in $\mathscr{P}^c$. On the other hand, the set S_1 should be as large as possible to increase the probability that Condition 3 in Corollary 4 is satisfied. Based on these guidelines, we select S_1 to be the set of all vertices not reachable in the complement graph $\mathscr{P}^c$ from any vertex with a self-loop in $\mathscr{P}^c$.

Next, since the partition in Corollary 4 has an acyclic structure, it is natural to consider the strongly connected components of the complement graph $\mathscr{P}^c$. By the definition of S_1, there is no edge from the vertices in $\{1, \ldots, n\} - S_1$ to the vertices in S_1. Now, we further divide the prior set into strongly connected components $S_2, \ldots, S_m$ of $\mathscr{P}^c$. The remaining task is to find a ordering for the sets $S_2, \ldots, S_m$ such that Conditions 1 and 3 in Corollary 4 are satisfied.

The above ordering problem is analogous to the task scheduling problem studied in [18]. If each set S_k is regarded as a task that requires r_k time to complete, then the goal is to find an ordering of all tasks satisfying the precedence constraints in such a way that S_1 becomes the first task and the starting time for each remaining task S_k becomes strictly later than $d_k + r_k$.

We propose Algorithm 1 based on the above ideas. If the algorithm returns “succeeded”, then the feasible region $\mathcal{D}$ associated with the given pattern has been proved to be connected. However, the algorithm returning “failed” means that it cannot determine whether or not the feasible region is connected.

Algorithm 1 (Checking Connectivity for Pattern $\mathscr{P}$):

Compute S_1 through a breadth-first search.
if $S_1 = \emptyset$ **then**
    return failed
end if
Divide $\{1, \ldots, n\} - S_1$ into strongly connected components $S_2, \ldots, S_m$ of $\mathscr{P}^c$.
Compute d_k and r_k for each S_k.
Remove S_1 from the graph.
$T \leftarrow r_1$
while $T < n$ **do**
    Find an unprocessed set S_k with no incoming edges in $\mathscr{P}^c$ and $T > d_k + r_k$.
    if not found **then**
        return failed
    end if
    Remove S_k from the graph.
    $T \leftarrow T + r_k$
end while
return succeeded

Remark 1: When executing Algorithm 1, there could be multiple feasible sets S_k being found at the beginning of each iteration. One may ask whether the choice of S_k would affect the result of the algorithm. To investigate this issue, assume that there are two available sets S_k and $S_{k'}$ at some step, and that the algorithm selects S_k and finally returns succeeded with a desired ordering

$$S_1, S_{l_1}, \ldots, S_{l_s}, S_k, S_{l_{s+1}}, \ldots, S_{l_t}, S_{k'}, S_{l_{t+1}}, \ldots$$

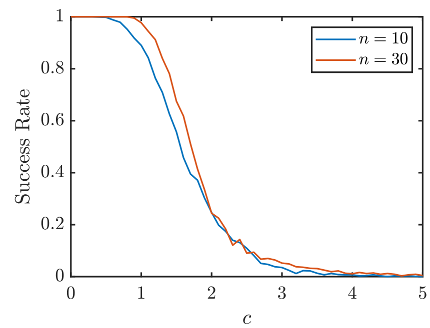
However, if $S_{k'}$ was selected instead of S_k, one could still find the ordering

$$S_1, S_{l_1}, \ldots, S_{l_s}, S_{k'}, S_k, S_{l_{s+1}}, \ldots, S_{l_t}, S_{l_{t+1}}, \ldots$$

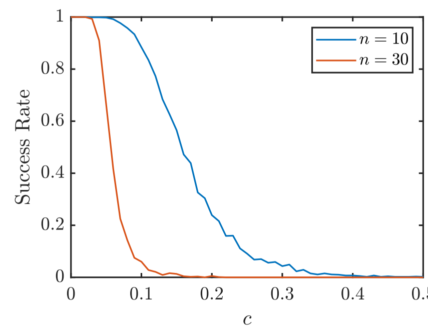
which also satisfies all of the conditions in Corollary 4. Therefore, having multiple possibilities for the sets in each iteration of Algorithm 1 will not change the outcome of the algorithm.

C. Numerical Examples

To demonstrate the performance of Algorithm 1, it is desirable to provide some numerical examples with randomly



(a) The dense case



(b) The sparse case

Fig. 3. The success rate of Algorithm 1 on random examples with different values for n and c.

generated patterns. Let each entry in the pattern be chosen as “0” independently with a fixed probability. Two types of random instances are considered: (i) the *dense* case where

$$\Pr(\mathscr{P}_{ij} = 0) = \frac{c}{n} \tag{2}$$

with some parameter c, (ii) and the *sparse* case where

$$\Pr(\mathscr{P}_{ij} = 0) = c.$$

For each of the two cases, we generate 1000 random samples according to the above probability distribution, run Algorithm 1 on these samples and calculate the empirical probability that the algorithm finds patterns with connected feasible regions.

The result for the dense case is given in Fig. 3(a). It can be observed that most patterns generated by $c < 1$, i.e., patterns whose number of “0” is approximately less than n, are associated with connected feasible regions. This observation will be mathematically supported in Section IV.

The result for the sparse case is given in Fig. 3(b). Despite the fact that the success rate goes to 0 quickly when c increases, it will be also shown in Section IV that it is still possible to construct an exponential number of patterns, with approximately up to half of the entries being “0”, which can all be certified via Theorem 3.

IV. Applications of the Connectivity Criterion

The proposed connectivity criterion in Section III has two important usages. First, one can apply Theorem 3 to certify the connectivity for certain classes of patterns. ODC problems with these patterns are well suited for local search methods due to the connectivity of the feasible region. Second, one can construct patterns making the feasible region $\mathcal{D}$ connected by exploiting the conditions in Theorem 3. The designed patterns can be used to relax a general ODC problem with an unfavorable pattern $\mathscr{P}$ by replacing $\mathscr{P}$ with a favorable pattern $\mathscr{Q}$ such that $\mathscr{P} \subseteq \mathscr{Q}$. Then, the relaxed problem may be solved by local search methods and its solution could be rounded into an approximate solution to the original ODC problem.

A. Proving Connectivity

As an application of Theorem 3, it is desirable to prove that most dense patterns with a small number of '0"s lead to a connected feasible region $\mathcal{D}$.

Theorem 5: Let r denote the number of vertices in the largest strongly connected component of the complement graph $\mathscr{P}^c$. If

$$|\mathscr{P}^c| \leq n - \max\{r, 2\},$$

then $\mathscr{P}$ is stably expandable.

Proof: Following the argument used in Algorithm 1, let S_1 be the set of all vertices not reachable in $\mathscr{P}^c$ from any vertex with a self-loop in $\mathscr{P}^c$. Observe that each vertex not in S_1 has either a self-loop or an incoming edge in $\mathscr{P}^c$. Since there are at most $n-2$ edges in $\mathscr{P}^c$, the set S_1 cannot be empty. We partition the remaining vertices $\{1, \ldots, n\} - S_1$ into strongly connected components of $\mathscr{P}^c$ and perform a topological sorting over these components. The result is a list of strongly connected components $\{S_2, \ldots, S_m\}$ for which there is no edge from S_l to S_k in $\mathscr{P}^c$ for $k < l$.

For any subset S of vertices, let $E(S)$ denote the number of edges (including self-loops) in $\mathscr{P}^c$ whose destinations belong to S. Using this notation, it can be concluded that

$$E(S_l) \geq r_l, \quad \forall l = 2, \ldots, m. \tag{3}$$

This obviously holds true for a non-singleton S_l since S_l is strongly connected in $\mathscr{P}^c$. On the other hand, if $S_l = \{i\}$ is a singleton, then $i \notin S_1$. The observation made at the beginning of the proof implies that $E(S_l) \geq 1 = r_l$.

We claim that the partition $\{S_1, \ldots, S_m\}$ satisfies all of the conditions in Corollary 4. If not, then there exists some $k > 1$ such that

$$d_k + r_k \geq \sum_{l=1}^{k-1} r_l. \tag{4}$$

In this case, there exists at least one vertex j in $\cup_{l=2}^k S_l$ with a self-loop in $\mathscr{P}^c$; otherwise, by the definition all these vertices in $\cup_{l=2}^k S_l$ should belong to S_1, which is not possible.

Now, we investigate two scenarios. If the strongly connected component S_k is a singleton, then

$$\sum_{l=2}^{k} E(S_l) \geq d_k + 1,$$

where the additional "1" above counts for the self-loop of vertex j. By (3), (4) and the above inequality, one can write

$$\begin{aligned} |\mathscr{P}^c| &\geq \sum_{l=2}^{m} E(S_l) \geq d_k + 1 + \sum_{l=k+1}^{m} r_l \\ &\geq \sum_{l=1}^{k-1} r_l + \sum_{l=k+1}^{m} r_l = n - 1 \\ &> n - \max\{r, 2\}, \end{aligned}$$

which is a contradiction.

For the scenario in which S_k is not a singleton, since S_k is strongly connected, it holds that

$$\sum_{l=2}^{k} E(S_l) \geq d_k + r_k + 1.$$

Similar to the previous scenario, one can write

$$\begin{aligned} |\mathscr{P}^c| &\geq \sum_{l=2}^{m} E(S_l) \geq d_k + r_k + 1 + \sum_{l=k+1}^{m} r_l \\ &\geq \sum_{l=1}^{k-1} r_l + 1 + \sum_{l=k+1}^{m} r_l = n - r_k + 1 \\ &> n - \max\{r, 2\}, \end{aligned}$$

which is also a contradiction. ∎

Remark 2: For each pattern $\mathscr{P}$ whose complement graph $\mathscr{P}^c$ is acyclic[1], Theorem 5 implies that $\mathscr{P}$ is stably expandable as long as $|\mathscr{P}^c| \leq n-2$. This bound is tight. The $n \times n$ pattern

$$\mathscr{P} = \begin{bmatrix} * & \ldots & * & * & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ * & \ldots & * & * & 0 \\ * & \ldots & * & * & * \\ * & \ldots & * & * & 0 \end{bmatrix}$$

has an acyclic $\mathscr{P}^c$ with $|\mathscr{P}^c| = n-1$, but the corresponding feasible region $\mathcal{D}$ for the case $A = 0$ is not connected. The reason is that the stable matrices

$$\begin{bmatrix} -1 & & & & \\ & \ddots & & & \\ & & -1 & & \\ & & & -1 & -1 \\ & & & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} -1 & & & & \\ & \ddots & & & \\ & & -1 & & \\ & & & -1 & 1 \\ & & & -1 & 0 \end{bmatrix}$$

conform with the pattern $\mathscr{P}$, while any path within $\mathscr{P}$ between these matrices must pass through an unstable matrix whose last column is all zero.

Remark 3: For random patterns subject to the distribution given by (2) with a parameter $c < 1$, [19] has proven that with high probability the largest strongly connected component of $\mathscr{P}^c$ is of size $O(\log n)$. Therefore, as long as n is sufficiently large such that

$$|\mathscr{P}^c| \approx cn < n - O(\log n),$$

[1]Here, we allow acyclic graphs to contain self-loops.

$$\begin{bmatrix} \ast & 0 & \ast & 0 & \ast & \ast & \ast \\ 0 & \ast & \ast & \ast & 0 & \ast & \ast \\ \ast & \ast & 0 & \ast & 0 & 0 & \ast \\ \ast & \ast & \ast & 0 & \ast & 0 & \ast \\ \ast & \ast & \ast & \ast & 0 & 0 & \ast \\ \ast & \ast & \ast & \ast & \ast & 0 & \ast \\ 0 & 0 & \ast & \ast & 0 & 0 & 0 \end{bmatrix}$$

Fig. 4. A 7×7 pattern with seventeen "0"s generated by Algorithm 2.

Theorem 5 implies that the feasible region $\mathcal{D}$ associated with the pattern is connected. This explains why most patterns in Fig. 3(a) with $c < 1$ are certified.

B. Constructing Patterns with Connected Feasible Regions

According to the numerical result in Fig. 3(b), Algorithm 1 cannot certify the connectivity for many sparse patterns. This is not surprising since the complement graphs of most of these patterns are strongly connected and cannot be decomposed as required by Corollary 4. However, using Theorem 3, one can still construct an exponential class of desirable patterns with up to approximately half of entries being "0". The construction procedure is provided by Algorithm 2 below. Fig. 4 shows some connected sparse pattern generated by Algorithm 2 achieving the maximum possible number of "0"s.

Algorithm 2 (Generating Patterns with Connected Feasible Regions):

Set $\mathscr{P}$ to be a 2×2 stably expandable pattern such as

$$\mathscr{P} \leftarrow \begin{bmatrix} \ast & 0 \\ 0 & \ast \end{bmatrix}.$$

for $i \leftarrow 3$ **to** n **do**

Add one row and column at the bottom and right side of $\mathscr{P}$.

Fill the newly added entries with "$\ast$".

Choose at most $i - 3$ elements from $\{1, \ldots, i-1\}$.

For each chosen element j, set $\mathscr{P}_{ji} \leftarrow 0$.

Optionally set $\mathscr{P}_{ii} \leftarrow 0$.

Optionally set $\mathscr{P} \leftarrow \mathscr{P}^T$.

end for

The next theorem shows that the number of favorable patterns generated by Algorithm 2 is roughly the square root of $O(2^{n\times n})$ or the total number of possible patterns. Although the constructed patterns only account for a small proportion of all patterns, they are abundant enough to be used as approximations for general ODC problems.

Theorem 6: There are at least $O(2^{n(n-1)/2})$ patterns whose corresponding feasible regions $\mathcal{D}$ are connected.

Proof: Let $f(n)$ be the number of different $n \times n$ patterns that can be generated by Algorithm 2. Note that $f(2) = 1$. For $n \geq 3$, the algorithm essentially allows us to arbitrarily choose "0" or "$\ast$" for the entries $\mathscr{P}_{1n}, \ldots, \mathscr{P}_{(n-1)n}$, except the cases that all theses entries are "0" or only one of them is "$\ast$". Based on this observation, on can write

$$f(n) = f(n-1) \times (2^{n-1} - 1 - (n-1)) \times 2 \times 2,$$

where the second last "2" counts for the choice of $\mathscr{P}_{nn}$ and the last "2" counts for the choice of using a transpose. Since

$$2^{n-1} - 1 - (n-1) \geq 2^{n-3}, \quad \forall n \geq 3,$$

it holds that

$$f(n) \geq f(n-1) \times 2^{n-1}, \quad \forall n \geq 3.$$

By induction, $f(n) \geq O(2^{n(n-1)/2})$. ∎

V. Conclusion

In this paper, we studied the connectivity property of the feasible regions of ODC problems. After introducing the notion of stable expandability, we developed a novel criterion together with an efficient algorithm to identify the connectivity for systems with direct state-feedback. Subsequently, we proved that the feasible region of the ODC problem is connected for most dense communication patterns as well as an exponential class of patterns.

Future research directions include generalizing the above results to ODC problems without direct state-feedback. Moreover, for problems with proven connected feasible regions, it is helpful to analyze the convergence behavior of local search methods on these problems.

Appendix
Proof of the Connectivity Criterion

We first recall some fundamental facts about stable matrices. Assume that $A \in \mathcal{K}_n$ is a stable matrix and $Q \in \mathbb{R}^{n\times n}$ is an arbitrary matrix. The standard Lyapunov equation

$$PA + A^T P = -Q \tag{5}$$

is equivalent to

$$(I_n \otimes A^T + A^T \otimes I_n) \operatorname{vec} P = -\operatorname{vec} Q,$$

where $\operatorname{vec} X$ is the vector resulted from stacking all the columns of a matrix X, and $\otimes$ is the Kronecker product. Since A is stable,

$$I_n \otimes A^T + A^T \otimes I_n$$

is invertible (see [20, Sec. 12.11]) and thus the Lyapunov equation (5) has a unique solution denoted by $\Phi(A, Q)$ such that

$$\operatorname{vec} \Phi(A, Q) = -(I_n \otimes A^T + A^T \otimes I_n)^{-1} \operatorname{vec} Q.$$

The above equation implies that $\Phi(A, Q)$ continuously depends on the matrices A and Q. Furthermore, if Q is positive definite, then this unique solution $\Phi(A, Q)$ is also positive definite. Conversely, given a matrix $A \in \mathbb{R}^{n\times n}$, if there exist two positive definite matrices $P, Q \in \mathcal{P}_n$ satisfying the Lyapunov equation (5), then A is stable.

Lemma 7: A block matrix

$$\begin{array}{r} n\{ \\ m\{ \end{array} \left[\begin{array}{c:c} A & B \\ \hdashline C & D \end{array}\right]_{\underbrace{\hphantom{A}}_{n}\underbrace{\hphantom{B}}_{m}}$$

is stable if and only if there exist $Q \in \mathbb{R}^{m\times n}$ and $R \in \mathcal{P}_m$ such that

1) $A' = A - BR^{-1}Q$ is stable.
2) $QB + RD + B^TQ^T + D^TR = -I_m$.
3) $C = \Gamma(A', B, D, Q, R)$, where

$$\Gamma(A', B, D, Q, R) = -R^{-1}(QA' - R^{-1}Q - RDR^{-1}Q + B^T\Phi(A', I_n + Q^T(R^{-1})^2Q)). \quad (6)$$

Proof: The original matrix is stable if and only if there exists a positive definite matrix

$$\begin{bmatrix} P & Q^T \\ Q & R \end{bmatrix},$$

blocked in the same way as the original matrix, satisfying the Lyapunov equation:

$$\begin{bmatrix} P & Q^T \\ Q & R \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix} + \begin{bmatrix} A^T & C^T \\ B^T & D^T \end{bmatrix}\begin{bmatrix} P & Q^T \\ Q & R \end{bmatrix} = -I_{n+m}.$$

Using the property of the Schur complement, the above condition is equivalent to the existence of $P \in \mathcal{S}_n$, $Q \in \mathbb{R}^{m\times n}$ and $R \in \mathcal{P}_m$ satisfying

$$PA + Q^TC + A^TP + C^TQ = -I_n, \quad (7a)$$
$$QB + RD + B^TQ^T + D^TR = -I_m, \quad (7b)$$
$$QA + RC + B^TP + D^TQ = 0, \quad (7c)$$
$$P - Q^TR^{-1}Q \in \mathcal{P}_n. \quad (7d)$$

The equation (7b) is the same as Condition 2 in the statement of the lemma. Let $A' = A - BR^{-1}Q$ and $P' = P - Q^TR^{-1}Q$. Then,

$$\begin{aligned} P'A' + (A')^TP' &= PA + A^TP \\ &+ Q^TC + C^TQ - Q^T(R^{-1})^2Q \\ &- Q^TR^{-1}(QA + RC + B^TP + D^TQ) \\ &- (A^TQ^T + C^TR + PB + Q^TD)R^{-1}Q \\ &+ Q^TR^{-1}(I_m + QB + RD + B^TQ^T + D^TR)R^{-1}Q. \end{aligned}$$

Therefore, given (7b) and (7c), equation (7a) is equivalent to

$$P'A' + (A')^TP' = -I_n - Q^T(R^{-1})^2Q.$$

The above equation and (7d) are further equivalent to the stability of A' and

$$P' = \Phi(A', I_n + Q^T(R^{-1})^2Q).$$

In addition, equation (7c) can be rewritten as

$$\begin{aligned} C &= -R^{-1}(QA + B^TP + D^TQ) \\ &= -R^{-1}(QA' + QBR^{-1}Q + B^TP' \\ &\quad + B^TQ^TR^{-1}Q + D^TQ) \\ &= -R^{-1}(QA' - R^{-1}Q - RDR^{-1}Q + B^TP') \\ &= -R^{-1}(QA' - R^{-1}Q - RDR^{-1}Q \\ &\quad + B^T\Phi(A', I_n + Q^T(R^{-1})^2Q)), \end{aligned}$$

which leads to Condition 3 in the statement. ■

Proof of Theorem 3: Without loss of generality, assume that the first $m+1$ rows of $\mathscr{B}$ do not contain "0"s while the remaining rows of $\mathscr{B}$ and the pattern $\mathscr{D}$ are all "0"s.

Given stable matrices

$$K^0 = \begin{bmatrix} A^0 & B^0 \\ C^0 & D^0 \end{bmatrix}, \quad K^1 = \begin{bmatrix} A^1 & B^1 \\ C^1 & D^1 \end{bmatrix}$$

and functions $\gamma_{ij}(\tau)$ as in Definition 1, there exist $Q^0, Q^1 \in \mathbb{R}^{m\times(n-m)}$ and $R^0, R^1 \in \mathcal{P}_m$ satisfying the corresponding conditions in Lemma 7. Our main idea is to expand $\gamma_{ij}(\tau)$ into a path $K(\tau)$ from K^0 to K^1 and construct additional paths $Q(\tau)$ and $R(\tau)$ with given endpoints at the same time such that the conditions in Lemma 7 are satisfied for $K(\tau)$, $Q(\tau)$ and $R(\tau)$ at every $\tau \in [0, 1]$.

Let $D(\tau)$ be the path from D^0 to D^1 whose components are completely determined by the functions $\gamma_{ij}(\tau)$ above, and let $R(\tau)$ be a path of positive definite matrices from R^0 to R^1 whose existence is guaranteed by the convexity of $\mathcal{P}_m$. Define

$$\varphi(\tau) = -I_m - R(\tau)D(\tau) - D^T(\tau)R(\tau).$$

Then, $\varphi(\tau)$ is a path of symmetric matrices with

$$\varphi(0) = Q^0B^0 + (B^0)^T(Q^0)^T,$$
$$\varphi(1) = Q^1B^1 + (B^1)^T(Q^1)^T.$$

Let

$$\psi(\tau) = \frac{1}{2}(\varphi(\tau) + (1-\tau)Q^0B^0 - (1-\tau)(B^0)^T(Q^0)^T + \tau Q^1B^1 - \tau(B^1)^T(Q^1)^T).$$

The above path $\psi(\tau)$ satisfies

$$\psi(\tau) + \psi^T(\tau) = \varphi(\tau),$$
$$\psi(0) = Q^0B^0, \quad \psi(1) = Q^1B^1, \quad (8)$$

which suggests we find a path $Q(\tau)$ from Q^0 to Q^1 and a path $B(\tau)$ from B^0 to B^1 satisfying $Q(\tau)B(\tau) = \psi(\tau)$. After that, we have

$$Q(\tau)B(\tau) + R(\tau)D(\tau) + B^T(\tau)Q^T(\tau) + D^T(\tau)R(\tau) = \psi(\tau) + \psi^T(\tau) - I_m - \varphi(\tau) = -I_m. \quad (9)$$

Denote B^0 as

$$B^0 = \begin{bmatrix} X^0 \\ Y^0 \end{bmatrix}$$

and Q^0 as

$$Q^0 = \begin{bmatrix} Z^0 & W^0 \end{bmatrix},$$

where X^0 is the submatrix of the first $m+1$ rows of B^0 and Z^0 is the submatrix of the first $m+1$ columns of Q^0. Similar notations can also be introduced for the corresponding submatrices of B^1 and Q^1. Now, we choose a path $Y(\tau)$ from Y^0 to Y^1 based on the given functions $\gamma_{ij}(\tau)$, and we choose $W(\tau)$ to be an arbitrary path from W^0 to W^1. Then, the question of finding the paths $Q(\tau)$ and $B(\tau)$ satisfying $Q(\tau)B(\tau) = \psi(\tau)$ is equivalent to finding a path $X(\tau)$ from X^0 to X^1 and another path $Z(\tau)$ from Z^0 to Z^1 satisfying

$$Z(\tau)X(\tau) = \psi(\tau) - W(\tau)Y(\tau). \quad (10)$$

We first consider the case in which both X^0 and X^1 are of full column rank. In this case, there exist invertible $(m+1)\times(m+1)$ matrices $\tilde{X}^0$ and $\tilde{X}^1$ with positive determinants whose columns except the last one are the same as X^0 and X^1, respectively. Since the set of matrices with positive determinants is path-connected, one can find a path $\tilde{X}(\tau)$ of invertible matrices from $\tilde{X}^0$ to $\tilde{X}^1$. Let $X(\tau)$ be the first m columns of $\tilde{X}(\tau)$ and $\phi(\tau)\in\mathbb{R}^{m\times(m+1)}$ be the path of the form

$$\phi(\tau)=\begin{bmatrix}\psi(\tau)-W(\tau)Y(\tau) & a(\tau)\end{bmatrix},$$

where $a(\tau)\in\mathbb{R}^{m\times 1}$ is an arbitrary path with $a(0)$ being the last column of $Z^0\tilde{X}^0$ and $a(1)$ being the last column of $Z^1\tilde{X}^1$. Now, we choose $Z(\tau)=\phi(\tau)\tilde{X}^{-1}(\tau)$. Then, $Z(\tau)$ is continuous and by (8) it holds that

$$\begin{aligned}&Z(0)\tilde{X}^0=\phi(0)=\begin{bmatrix}\psi(0)-W^0Y^0 & a(0)\end{bmatrix}\\ &=\begin{bmatrix}Q^0B^0-W^0Y^0 & a(0)\end{bmatrix}=\begin{bmatrix}Z^0X^0 & a(0)\end{bmatrix}=Z^0\tilde{X}^0.\end{aligned}$$

Therefore, $Z(0)=Z^0$ and similarly $Z(1)=Z^1$. Furthermore, since $Z(\tau)X(\tau)$ is the first m columns of $\phi(\tau)$, the equation (10) is satisfied. Combining the paths $X(\tau)$, $Y(\tau)$ and $Z(\tau)$, $W(\tau)$ leads to the desired paths $Q(\tau)$ and $B(\tau)$.

Let

$$A'^0=A^0-B^0(R^0)^{-1}Q^0,\quad A'^1=A^1-B^1(R^1)^{-1}Q^1.$$

For each $(i,j)\notin\mathscr{P}$ with $1\le i,j\le n-m$, we also set

$$\gamma'_{ij}(\tau)=\gamma_{ij}(\tau)-b_{ij}(\tau),$$

where $b_{ij}(\tau)$ is the (i,j) component of $B(\tau)R^{-1}(\tau)Q(\tau)$. By the assumption that the pattern $\mathscr{A}$ is stably expandable, one can find a path $A'(\tau)$ of stable matrices from A'^0 to A'^1 whose components are coincided with the functions $\gamma'_{ij}(\tau)$. Then, we can choose

$$A(\tau)=A'(\tau)+B(\tau)R^{-1}(\tau)Q(\tau)\tag{11}$$

with $A'(\tau)$ guaranteed to be stable.

Finally, let

$$C(\tau)=\Gamma(A'(\tau),B(\tau),D(\tau),Q(\tau),R(\tau)),\tag{12}$$

where the function Γ is defined in (6). Then, $C(\tau)$ is continuous and thus a path from C^0 to C^1. Combining the paths $A(\tau)$, $B(\tau)$, $C(\tau)$, $D(\tau)$, one can obtain the desired path $K(\tau)$ from K^0 to K^1. By (9), (11), (12) and Lemma 7, $K(\tau)$ must be stable.

In the above argument, we still need to handle the cases in which either X^0 or X^1 is not of full column rank. Assume that X^0 is not of full column rank. Since the set $\mathcal{K}_n$ of stable matrices is open, there exists $\epsilon>0$ such that $\mathcal{B}(K^0,\epsilon)\subseteq\mathcal{K}_n$, where

$$\mathcal{B}(K^0,\epsilon)=\{P\in\mathbb{R}^{n\times n}||P_{ij}-K^0_{ij}|<\epsilon,\ \forall 1\le i,j\le n\}.$$

On the other hand, since the set of matrices of full column rank is a dense subset of $\mathbb{R}^{(m+1)\times m}$, there exists a matrix $X'\in\mathbb{R}^{(m+1)\times m}$ of full column rank whose components satisfy

$$|X'_{ij}-X^0_{ij}|<\epsilon,\quad\forall 1\le i\le m+1,\ \forall 1\le j\le m.$$

Next, we find $0<s<1$ such that

$$|\gamma_{ij}(\tau)-K^0_{ij}|<\epsilon,\quad\forall(i,j)\notin\mathscr{P},\ \forall\tau\in[0,s].$$

Then, one can design a path $\bar{K}(\tau):[0,s]\to\mathcal{B}(K^0,\epsilon)$ of stable matrices starting from K^0 such that its components are prescribed by the functions $\gamma_{ij}(\tau)$ and X' is the corresponding submatrix of $\bar{K}(s)$. Now, we replace K^0 by $\bar{K}(s)$ in the above path construction procedure. After obtaining the constructed path with the starting matrix $\bar{K}(s)$, we can concatenate it with the path $\bar{K}(\tau)$ here to obtain a path starting from K^0. The case in which X^1 is not of full column rank can be dealt with similarly. ■

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