Analysis of Semidefinite Programming Relaxation of Optimal Power Flow for Cyclic Networks

Hamid Mahboubi and Javad Lavaei

Abstract— **The optimal power flow (OPF) problem determines an optimal operating point of the power network that minimizes a certain objective function subject to physical and network constraints. There has been a great deal of attention in convex relaxation of the OPF problem in recent years, and it has been shown that a semidefinite programming (SDP) relaxation can solve different classes of the nonconvex OPF problem to global or near-global optimality. Although it is known that the SDP relaxation is exact for radial networks under various conditions, solving the OPF problem for cyclic networks needs further research. In this paper, we propose sufficient conditions under which the SDP relaxation is exact for special but important cyclic networks. More precisely, when the objective function is a linear function of active powers, we show that the SDP relaxation is exact for odd cycles under certain conditions. Also, the exactness of the SDP relaxation for simple cycles of size 3 and 4 is proved under different technical conditions. The existence of rank-1 or -2 SDP solutions for weakly-cyclic networks is also proved in both lossy and lossless cases. In addition, when the objective function is an increasing function of reactive powers, we prove that the SDP relaxation is exact under certain conditions. This result justifies why the sum of reactive powers acts as a low-rank-promoting term for OPF. The findings of this paper provide intuition into the behavior of SDPs for different building blocks of cyclic networks.**

I. Introduction

Optimal power flow is a fundamental problem in power engineering that has attracted considerable research interest for many years, due to its broad range of applications [1], [2], [3], [4], [5]. Examples of such applications include state estimation and energy management, charging of plug-in electric vehicles, unit commitment, voltage stability and voltage regulation in power distribution systems, to name only a few [6], [7], [8], [9], [10]. OPF aims to minimize an objective function (that is generally increasing and convex in active power) subject to network and physical constraints. Since the interrelation among voltage magnitude, active and reactive powers in OPF is nonlinear accompanied by many different types of constraints, solving the optimal power flow problem is very challenging [11], [12].

Convex relaxations of the OPF problem have attracted a great deal of attention in the past decade, and solving OPF through conic relaxation as second-order cone program (SOCP) and semidefinite program (SDP) has been investigated by many researchers in recent years [13], [14], [15], [16]. The authors in [17] use an SDP relaxation for solving OPF. They prove that the SDP relaxation is exact if and only if the duality gap is zero. They also made an important observation that OPF has a zero duality gap in IEEE benchmark networks. It is shown in [18] and [19] that this is due to the physics of power systems. Further analysis of zero relaxation gap is provided in [20] to guarantee the recovery of a global solution of OPF. The authors in [21] propose a sufficient condition to check the global optimality of a candidate OPF solution generated by local search OPF solvers. The paper [22] shows that the SDP relaxation may not be exact, but it has a low-rank solution due to the small treewidth of real-world systems. A sufficient condition for the exactness of an SOCP convex relaxation of OPF in radial distribution networks is proposed in [23]. In this condition, the reverse power flow can be only active, reactive, or none. An online algorithm is proposed in [24] to solve the OPF problem for radial networks. Moreover, the sufficient conditions provided in [24] guarantee convergence to a global optimum. It is known that an SDP relaxation of the OPF problem over radial networks is exact when there is no generation lower bound. The impact of generation lower bounds on the performance of SDP relaxations of the OPF problem is investigated in [25]. A distributed algorithm based on the alternating direction method of multiplier (ADMM) is proposed in [26], which solves the OPF problem for balanced radial networks. A sequential convex optimization method is provided in [27] that can solve large classes of OPF problems over radial networks.

Despite recent success in developing efficient techniques for solving the OPF problem especially for radial networks, numerous issues remain open. For instance, solving the OPF problem for cyclic networks is one of the areas that requires further research. In this work, the exactness of the SDP relaxation of OPF for very special cyclic networks is investigated. When the objective function is a linear function of active powers, we show that the SDP relaxation is exact for odd cycles provided that the optimal lagrange multiplier matrix fits the graph representing the network. We also prove that the SDP relaxation is exact for cycles of size 3 and 4 under certain conditions. In addition, we provide sufficient conditions for the existence of rank-1 or -2 SDP solutions in a weakly-cyclic network for both lossless and lossy cases. Furthermore, sufficient conditions for the exactness of the SDP relaxation are provided when the objective function is an increasing function of reactive powers. The sum of reactive powers has been recently used as a regularization term for a classic OPF minimizing a function of active powers to promote a rank-1 SDP solution [12], [28]. Our

H. Mahboubi and J. Lavaei are with the Department of Industrial Engineering and Operations Research, University of California at Berkeley, Berkeley, CA. 94720, USA, {mahboubi,lavaei}@berkeley.edu

This work was supported by the ARO grant W911NF-17-1-0555, NSF CAREER Award, and NSF EPCN Award.

result explains why this regularization term enforces to obtain a low-rank solution for cyclic networks. This paper targets the building blocks of cyclic networks and provides intuitions into the performance of SDP relaxations.

The organization of the remainder of this paper is as follows. The optimal power flow formulation along with necessary notions and definitions is provided in Section II. The convex relaxation of OPF is proposed in Section III. Section IV presents the main results of the paper, where the exactness of OPF for cyclic networks under different conditions is investigated. Concluding remarks are summarized in Section V.

II. Optimal Power Flow

A. Notations and Definitions

Let $\mathbb{C}, \mathbb{R}, \mathbb{R}^+$ and $\mathbb{R}^-$ denote the sets of complex numbers, real numbers, positive real numbers and negative real numbers, respectively. We denote the real and imaginary parts of complex number $x \in \mathbb{C}$ by Re(x) and Im(x), respectively. The scalars and vector variables are in small letters, generally. However, certain power system quantities, namely V_k, P_{G_k}, P_{D_k}, P_k^{min}, Q_k^{min}, are in capital letters. Matrices are also in bold capital letters. For $x \in \mathbb{C}$, $\measuredangle x$ denotes the angle of x. We denote conjugate transpose by $(\cdot)^*$. The (l,m) entry of matrix $\mathbf{X}$ is denoted by X_{lm}. The rank and trace of $\mathbf{X}$ are denoted by rank$(\mathbf{X})$ and trace$(\mathbf{X})$, respectively. We denote a globally optimal value of a parameter by $(\cdot)^{opt}$. The transpose and Hermitian (complex conjugate) transpose of $\mathbf{X}$ are denoted by $\mathbf{X}^T$ and $\mathbf{X}^H$, respectively. The sets of all $n \times n$ Hermitian and Hermitian positive semidefinite matrices are denoted by $\mathbf{H}^n$ and $\mathbf{H}^n_+$, respectively. The trace inner product of two $m \times n$ matrices $\mathbf{X}$ and $\mathbf{Y}$ is denoted by $\mathbf{X} \bullet \mathbf{Y}$, which is equal to

$$\mathbf{X} \bullet \mathbf{Y} = \text{trace}(\mathbf{X}^T\mathbf{Y}) = \sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}Y_{ij}$$

Definition 1. Consider a graph G with n vertices and a matrix $\mathbf{A}$ of size $n \times n$. The matrix $\mathbf{A}$ is said to fit the graph G if for any $i \neq j$, $A_{ij} = 0$ if and only if (i,j) is not an edge of G.

Definition 2. A network is called *weakly-cyclic* if every edge of its representative graph belongs to at most one cycle of the graph. In other words, the intersection of any two arbitrary cycles of a weakly-cyclic network is empty or a single node.

Definition 3. A network is called *lossless* if it cannot absorb power. In a lossless network, the real part of the admittance of each line is zero. A network is called *lossy* if it is not lossless.

B. Optimal Power Flow

Consider a power network modeled by a connected undirected graph $G(\mathcal{N}, \mathcal{L})$, where $\mathcal{N} := \{1, 2, \ldots, n\}$ and $\mathcal{L} \subseteq N \times \mathcal{N}$. In this network, $\mathcal{N}$, $\mathcal{L}$ and $\mathcal{G} \subseteq \mathcal{N}$ represent the set of buses, the set of flow lines and the set of generator buses, respectively. For each line $(l,m) \in \mathcal{L}$, let y_{lm} denote the complex admittance of the line. The known constant power load of each bus $k \in \mathcal{N}$ is denoted by $P_{D_k} + Q_{D_k}\mathbf{i}$. Also, the unknown complex output of each generator bus $k \in \mathcal{G}$ is denoted by $P_{G_k} + Q_{G_k}\mathbf{i}$. To simplify the optimal power follow formulation, with no loss of generality we assume that $\mathcal{G} = \mathcal{N}$. Let $V = [V_1, V_2, \ldots, V_n]^T \in \mathbb{C}^n$ and $I = [I_1, I_2, \ldots, I_n]^T \in \mathbb{C}^n$ be the vectors of bus voltages and currents, respectively, where I_k is the total current flowing out of bus k to the remainder of the network. According to Kirchhoff's law, the complex power injected at bus k is $S_k = P_k + Q_k = V_k I_k^*$, where P_k and Q_k are real power and reactive power, respectively. Note that $P_k = P_{G_k} - P_{D_k}$ and $Q_k = Q_{G_k} - Q_{D_k}$. Let P_{lm} denote the active power transferred from bus l to the rest of the network through the line (l,m). Define $\mathrm{P}_G = [P_{G_1}, \ldots, P_{G_n}]^T$, $\mathrm{Q}_G = [Q_{G_1}, \ldots, Q_{G_n}]^T$, $\mathrm{P}_D = [P_{D_1}, \ldots, P_{D_n}]^T$ and $\mathrm{Q}_D = [Q_{D_1}, \ldots, Q_{D_n}]^T$.

A general OPF problem is often formulated as:

$$\min_{\mathrm{P}_G, \mathrm{Q}_G, \mathrm{V}} \sum_{k \in \mathcal{N}} f_k(P_{G_k}, Q_{G_k}) \tag{1a}$$

subject to

i) the following constraints for each bus $k \in \mathcal{N}$:

$$P_{G_k} - P_{D_k} = \sum_{l \in N(k)} \text{Re}\{V_k(V_k^* - V_l^*)y_{kl}^*\} \tag{1b}$$

$$Q_{G_k} - Q_{D_k} = \sum_{l \in N(k)} \text{Im}\{V_k(V_k^* - V_l^*)y_{kl}^*\} \tag{1c}$$

$$P_k^{min} \leq P_{G_k} \leq P_k^{max} \tag{1d}$$

$$Q_k^{min} \leq Q_{G_k} \leq Q_k^{max} \tag{1e}$$

$$V_k^{min} \leq |V_k| \leq V_k^{max} \tag{1f}$$

ii) a capacity constraint for each line $(l,m) \in \mathcal{L}$:

$$|V_l - V_m| \leq \Delta V_{lm}^{max} \tag{1g}$$

$$|P_{lm}| = |\text{Re}\{V_l(V_l^* - V_m^*)y_{lm}^*\}| \leq P_{lm}^{max} \tag{1h}$$

$$|S_{lm}| = |V_l(V_l^* - V_m^*)y_{lm}^*| \leq S_{lm}^{max} \tag{1i}$$

$$|\theta_{lm}| = |\measuredangle V_l - \measuredangle V_m| \leq \theta_{lm}^{max} \tag{1j}$$

where $f_k(P_{G_k}, Q_{G_k})$ is an increasing convex function and $N(k)$ denotes the neighboring nodes of bus k [12].

III. Convex Relaxation

The OPF problem (1) is nonconvex due to the presence of nonlinear terms in its formulation, and is an NP-hard problem in the worse case. A convex relaxation of OFP is studied in [17], which is based on the fact that all constraints of OPF can be expressed as linear functions of the entries of the quadratic matrix VV^*. By replacing the matrix VV^* with a new matrix variable $\mathbf{W} \in \mathbf{H}^n$, all constraints of OPF become convex in $\mathbf{W}$. To preserve the equivalence of the two OPF problem formulations, we should add the constraints rank$(\mathbf{W}) = 1$ and $\mathbf{W} \succeq 0$. Note that the condition rank$(\mathbf{W}) = 1$ is the only nonconvex constraint of the reformulated OPF problem [12], [13]. Hence, an SDP relaxation of OPF can be defined as follows:

$$\min_{P_G, Q_G, \mathbf{W}\in H_+^n} \sum_{k\in\mathcal{N}} f_k(P_{G_k}, Q_{G_k}) \tag{2a}$$

subject to

i) the following constraints for each bus $k \in \mathcal{N}$:

$$P_{G_k} - P_{D_k} = \sum_{l\in N(k)} \mathrm{Re}\{(W_{kk} - W_{kl})y_{kl}^*\} \tag{2b}$$

$$Q_{G_k} - Q_{D_k} = \sum_{l\in N(k)} \mathrm{Im}\{(W_{kk} - W_{kl})y_{kl}^*\} \tag{2c}$$

$$P_k^{min} \le P_{G_k} \le P_k^{max} \tag{2d}$$

$$Q_k^{min} \le Q_{G_k} \le Q_k^{max} \tag{2e}$$

$$(V_k^{min})^2 \le W_{kk} \le (V_k^{max})^2 \tag{2f}$$

ii) a convexified capacity constraint for each line :

$$W_{ll} + W_{mm} - W_{lm} - W_{ml} \le (\Delta V_{lm}^{max})^2 \tag{2g}$$

$$\mathrm{Re}\{(W_{ll} - W_{lm})y_{lm}^*\} \le P_{lm}^{max} \tag{2h}$$

$$|(W_{ll} - W_{lm})y_{lm}^*| \le S_{lm}^{max} \tag{2i}$$

$$\mathrm{Im}\{W_{lm}\} \le \mathrm{Re}\{W_{lm}\}\tan(\theta_{lm}^{max}) \tag{2j}$$

The existence of a rank-1 matrix solution $\mathbf{W}^{opt}$ for the SDP relaxation given in (2) guarantees the equivalence of the nonconvex OPF problem (1) and its associated convex SDP relaxation (2), and hence enables the recovery of a globally optimal solution of (1). In this case, it is said that the relaxation is exact.

The authors in [12] prove that the capacity constraints (1g)-(1j) are all mathematically related. We consider the constraint (1g) in all parts of this paper except for Theorem 6 in which the constraint (1h) is considered. Equations (2b), (2c) and inequalities (2d)-(2g) can be written in the following matrix forms:

$$P_k^{min} - P_{D_k} \le \mathbf{C}_k \bullet \mathbf{W} \le P_k^{max} - P_{D_k} \quad \forall k \in \mathcal{N} \tag{3a}$$

$$Q_k^{min} - Q_{D_k} \le \mathbf{E}_k \bullet \mathbf{W} \le Q_k^{max} - Q_{D_k} \quad \forall k \in \mathcal{N} \tag{3b}$$

$$(V_k^{min})^2 \le \mathbf{M}_k \bullet \mathbf{W} \le (V_k^{max})^2 \quad \forall k \in \mathcal{N} \tag{3c}$$

$$\mathbf{Z}_{lm} \bullet \mathbf{W} \le (\Delta V_{lm}^{max})^2 \quad \forall (l,m) \in \mathcal{L} \tag{3d}$$

where $\mathbf{C}_k$, $\mathbf{E}_k$, $\mathbf{M}_k$ and $\mathbf{Z}_{lm}$ are given by the following equations for a cycle of size n:

$$[\mathbf{C}_k]_{ij} = \begin{cases} \mathrm{Re}(y_{k(k-1)}^*) + \mathrm{Re}(y_{k(k+1)}^*) & ij = kk \\ -\frac{1}{2}\mathrm{Re}(y_{kl}^*) - \frac{\mathbf{i}}{2}\mathrm{Im}(y_{kl}^*) & i = k,\ |i-j| \in \{1, n-1\} \\ -\frac{1}{2}\mathrm{Re}(y_{kl}^*) + \frac{\mathbf{i}}{2}\mathrm{Im}(y_{kl}^*) & j = k,\ |i-j| \in \{1, n-1\} \\ 0 & \text{otherwise} \end{cases}$$

$$[\mathbf{E}_k]_{ij} = \begin{cases} \mathrm{Im}(y_{k(k-1)}^*) + \mathrm{Im}(y_{k(k+1)}^*) & ij = kk \\ -\frac{1}{2}\mathrm{Im}(y_{kl}^*) + \frac{\mathbf{i}}{2}\mathrm{Re}(y_{kl}^*) & i = k,\ |i-j| \in \{1, n-1\} \\ -\frac{1}{2}\mathrm{Im}(y_{kl}^*) - \frac{\mathbf{i}}{2}\mathrm{Re}(y_{kl}^*) & j = k,\ |i-j| \in \{1, n-1\} \\ 0 & \text{otherwise} \end{cases}$$

$$[\mathbf{M}_k]_{ij} = \begin{cases} 1 & ij = kk \\ 0 & \text{otherwise} \end{cases}$$

$$[\mathbf{Z}_{lm}]_{ij} = \begin{cases} 1 & ij = ll \ \text{ or } \ ij = mm \\ -1 & ij = lm \ \text{ or } \ ij = ml \\ 0 & \text{otherwise} \end{cases}$$

IV. Convex Relaxation for Cyclic Networks

In this section, we investigate the exactness of the convex relaxation for certain networks. Since the resistance and reactance of real-world lines are positive, assume that the real and imaginary parts of y_{lm}^* are nonnegative for all $(l,m) \in \mathcal{L}$, i.e.,

$$\mathrm{Re}(y_{lm}^*) \ge 0 \tag{4a}$$

$$\mathrm{Im}(y_{lm}^*) > 0 \tag{4b}$$

In this section, we first consider the case where the objective function (1a) is equal to the reactive power $\sum_{k\in\mathcal{N}} Q_{G_k}$. The objective function has been recently used as a regularization term for a classic OPF minimizing a function of active powers to promote a rank-1 SDP solution [12], [28]. In this part, we analyze this regularization term, namely the sum of reactive powers, to understand why it enforces to obtain a low-rank solution. This objective function can be written in the following matrix form:

$$\sum_{k\in\mathcal{N}} Q_{G_k} = \mathbf{F} \bullet \mathbf{W} + \sum_{k\in\mathcal{N}} Q_{D_k} \tag{5}$$

where $\mathbf{F}$ is given by the following equation for a cycle of size n:

$$[\mathbf{F}]_{ij} = \begin{cases} \mathrm{Im}(y_{i(i-1)}^*) + \mathrm{Im}(y_{i(i+1)}^*) & i = j \\ -\mathrm{Im}(y_{ij}^*) & |i-j| \in \{1, n-1\} \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

Let $\nu_k^a, \lambda_k^a, \mu_k^a, \psi_{lm}$ and $\nu_k^b, \lambda_k^b, \mu_k^b$ denote the lagrange multipliers corresponding to the upper bounding constraints (3a)-(3d) and the lower bounding constraints (3a)-(3c), respectively. The lagrange multiplier corresponding to $\mathbf{W} \succeq 0$ is also denoted by $\mathbf{A} \in \mathbf{H}_+^n$.

By obtaining the lagrangian of optimization (2) with the objective function (5), one can arrive at:

$$\begin{aligned} A_{lm} =& [-\psi_{lm} - \psi_{ml} - \frac{1}{2}\mathrm{Im}(y_{lm}^*)(\lambda_l^a + \lambda_m^a - \lambda_l^b - \lambda_m^b) \\ & - \mathrm{Im}(y_{lm}^*) - \frac{1}{2}\mathrm{Re}(y_{lm}^*)(\nu_l^a + \nu_m^a - \nu_l^b - \nu_m^b)] \\ & + [\frac{1}{2}\mathrm{Re}(y_{lm}^*)(\lambda_l^a - \lambda_m^a - \lambda_l^b + \lambda_m^b) \\ & - \frac{1}{2}\mathrm{Im}(y_{lm}^*)(\nu_l^a - \nu_m^a - \nu_l^b + \nu_m^b)]\ \mathbf{i} \quad \forall (l,m) \in \mathcal{L} \end{aligned} \tag{7}$$

and

$$A_{lm} = 0 \quad \forall (l,m) \notin \mathcal{L} \tag{8}$$

The matrix $\mathbf{A}$ is in the form given in (9) (see the top of next page). In addition, the complementary slackness condition

$$
\mathbf{A} = \begin{bmatrix}
A_{11} & A_{12} & 0 & \dots & \dots & \dots & \dots & 0 & A_{1n} \\
A_{21} & A_{22} & A_{23} & \dots & \dots & \dots & \dots & 0 & 0 \\
0 & A_{32} & A_{33} & A_{34} & \dots & \dots & \dots & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & \dots & \dots & \dots & \dots & A_{(n-2)(n-3)} & A_{(n-2)(n-2)} & A_{(n-2)(n-1)} & 0 \\
0 & 0 & \dots & \dots & \dots & \dots & A_{(n-1)(n-2)} & A_{(n-1)(n-1)} & A_{(n-1)n} \\
A_{n1} & 0 & \dots & \dots & \dots & \dots & 0 & A_{n(n-1)} & A_{nn}
\end{bmatrix} \tag{9}
$$

yields that $\mathbf{W} \bullet \mathbf{A} = 0$ at optimality. From $\mathbf{W} \bullet \mathbf{A} = 0$ and $\mathbf{W}, \mathbf{A} \in \mathbf{H}_+^n$, the following inequality can be concluded at optimality:

$$\text{rank}(\mathbf{W}) + \text{rank}(\mathbf{A}) \leq n \tag{10}$$

Lemma 1. *Consider arbitrary complex numbers c_1 and h_1. If $d_{11} = c_1 c_1^* + h_1 h_1^*$, then any $n \times n$ Hermitian positive semidefinite matrix*

$$
\boldsymbol{A} = \begin{bmatrix}
d_{11} & d_{12} & \dots & d_{1n} \\
d_{21} & d_{22} & \dots & d_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
d_{n1} & d_{n2} & \dots & d_{nn}
\end{bmatrix}
$$

with rank($\boldsymbol{A}$) ≤ 2 and $n \geq 2$ can be written as $\boldsymbol{A} = \underline{c}\ \underline{c}^* + \underline{h}\ \underline{h}^*$, where $\underline{c} = [c_1\ c_2\ \dots\ c_n]^T$ and $\underline{h} = [h_1\ h_2\ \dots\ h_n]^T$.

Proof: The proof is based on standard linear algebra and is omitted for brevity. ■

Theorem 1. *Consider the OPF problem for a weakly-cyclic network. The following statements hold:*
a) The SDP relaxation has a solution with the property rank($\boldsymbol{W}^{opt}$) ≤ 2 in the lossless case, provided $Q_k^{min} = -\infty$ for every $k \in \mathcal{N}$.
b) The SDP relaxation has a solution with the property rank($\boldsymbol{W}^{opt}$) ≤ 2 in the lossy case, provided $P_k^{min} = Q_k^{min} = -\infty$ for every $k \in \mathcal{N}$.

Proof: First, assume that the network is a single cycle of size n. In part (a) since $Q_k^{min} = -\infty$, the lagrange multipliers λ_k^b do not appear in (7). Also, since $\text{Re}(y_{lm}^*) = 0$ in a lossless network, we have

$$
\begin{aligned}
A_{lm} =& [-\psi_{lm} - \psi_{ml} - \text{Im}(y_{lm}^*)(1 + \frac{\lambda_l^a}{2} + \frac{\lambda_m^a}{2})] \\
& - [\frac{1}{2}\text{Im}(y_{lm}^*)(\nu_l^a - \nu_m^a - \nu_l^b + \nu_m^b)]\mathbf{i} \quad \forall (l,m) \in \mathcal{L}
\end{aligned} \tag{11}
$$

In part (b) since $P_k^{min} = Q_k^{min} = -\infty$, the lagrange multipliers λ_k^b and ν_k^b do not appear in (7). As a result,

$$
\begin{aligned}
A_{lm} =& [-\psi_{lm} - \psi_{ml} - \text{Im}(y_{lm}^*)(1 + \frac{\lambda_l^a}{2} + \frac{\lambda_m^a}{2}) \\
& - \frac{1}{2}\text{Re}(y_{lm}^*)(\nu_l^a + \nu_m^a)] \\
& + [\frac{1}{2}\text{Re}(y_{lm}^*)(\lambda_l^a - \lambda_m^a) - \frac{1}{2}\text{Im}(y_{lm}^*)(\nu_l^a - \nu_m^a)]\mathbf{i}
\end{aligned} \tag{12}
$$

for all $(l,m) \in \mathcal{L}$. Since $\psi_{lm}, \psi_{ml}, \nu_l^a, \nu_m^a, \lambda_l^a, \lambda_m^a$ are all nonnegative, it can be concluded from (11), (12) and (4) that in both cases the real part of A_{lm} is negative. i.e.,

$$\text{Re}(A_{lm}) < 0 \quad \forall (l,m) \in \mathcal{L} \tag{13}$$

Due to strict feasibility, the solutions $\mathbf{W}^{opt}$ and $\mathbf{A}^{opt}$ are attainable. Now, according to (10), in order to prove rank($\mathbf{W}^{opt}$) ≤ 2, it suffices to show that rank($\mathbf{A}^{opt}$) $\geq n-2$. With a light abuse of notation, we use the shorthand notation $\mathbf{A}$ for $\mathbf{A}^{opt}$. We prove the inequality rank($\mathbf{A}$) $\geq n-2$ by contradiction.
Assume that rank($\mathbf{A}$) $\leq n-3$. The determinant of every $(n-2)\times(n-2)$ submatrix of $\mathbf{A}$ must be zero. In particular, consider the following submatrix $\hat{\mathbf{A}}$:

$$
\hat{\mathbf{A}} = \begin{bmatrix}
0 & 0 & \dots & \dots & 0 & A_{1n} \\
A_{23} & \dots & \dots & \dots & 0 & 0 \\
A_{33} & A_{34} & \dots & \dots & \dots & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots \\
0 & \dots & A_{(n-2)(n-3)} & A_{(n-2)(n-2)} & A_{(n-2)(n-1)} & 0
\end{bmatrix} \tag{14}
$$

Since $det(\hat{\mathbf{A}}) = 0$, one can conclude

$$(-1)^{n-1} A_{1n} A_{23} A_{34} \dots A_{(n-2)(n-1)} = 0 \tag{15}$$

which contradicts (13). This completes the proof for a single cycle of size n.

For a general network with multiple cycles, let O_2 and O_n ($n \geq 3$) denote the set of all bridge edges and all n-bus cyclic subgraphs of the power network, respectively. By adapting the proof delineated above for single cycles of size n, it can be shown that the SDP relaxation has a solution $\mathbf{W}$ with the following property:

$$\text{rank}\big(\mathbf{W}(S)\big) \leq 2, \quad \forall S \in \bigcup_{i=2}^{n} O_i \tag{16}$$

where $\mathbf{W}(S)$ is a sub-matrix of $\mathbf{W}$ obtained by selecting every row and column of $\mathbf{W}$ whose index corresponds to a vertex of the subgraph S. According to (16), the matrix $\mathbf{W}(S)$ can be written as follows:

$$\mathbf{W}(S) = \hat{V}\ \hat{V}^* + \hat{Z}\ \hat{Z}^*, \quad \forall S \in \bigcup_{i=2}^{n} O_i \tag{17}$$

Moreover, according to the definition of weakly cyclic networks, two arbitrary bridge edges or cycles S_1 and $S_2 \in \bigcup_{i=2}^{n} O_i$ have at most one common vertex. Therefore, $\mathbf{W}(S_1)$ and $\mathbf{W}(S_2)$ have at most one common element, which is located in their diagonals. Therefore, according to Lemma 1, one can find vectors V and Z such that $V\ V^* + Z\ Z^*$ is a solution of the SDP relaxation. ■

Theorem 2. *Consider the OPF problem for a lossy cyclic network of size 3. The SDP relaxation of OPF for this network is exact if $Re(y_{lm}^*)/Im(y_{lm}^*)$ is equal for all lines of the network and $P_k^{min} = Q_k^{min} = -\infty$.*

Proof: Since $P_k^{min} = Q_k^{min} = -\infty$, the lagrange multipliers λ_k^b and ν_k^b do not appear in (7). As a result,

$$\begin{aligned} A_{lm} =& [-\psi_{lm} - \psi_{ml} - \frac{1}{2}\mathrm{Re}(y_{lm}^*)(\nu_l^a + \nu_m^a) \\ & - \mathrm{Im}(y_{lm}^*) - \frac{1}{2}\mathrm{Im}(y_{lm}^*)(\lambda_l^a + \lambda_m^a)] \\ & + [\frac{1}{2}\mathrm{Re}(y_{lm}^*)(\lambda_l^a - \lambda_m^a) - \frac{1}{2}\mathrm{Im}(y_{lm}^*)(\nu_l^a - \nu_m^a)]\mathbf{i} \end{aligned} \tag{18}$$

for all $(l, m) \in \mathcal{L}$. Since $\psi_{lm}, \psi_{ml}, \nu_l^a, \nu_m^a, \lambda_l^a, \lambda_m^a$ are all nonnegative, it follows from (18) and (4) that the real part of A_{lm} is negative for all $(l, m) \in \mathcal{L}$. For $n = 3$, the matrix $\mathbf{A}$ given in (9) is in the following form:

$$\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \tag{19}$$

Since all principal minors of a Hermitian positive semidefinite matrix are nonnegative, and also the real part of each A_{lm} is negative, it can be shown that none of A_{11}, A_{22} and A_{33} are zero. In other words:

$$A_{11}, A_{22}, A_{33} \in \mathbb{R}^+ \tag{20}$$

According to (10), in order to prove that $\mathrm{rank}(\mathbf{W}^{opt}) = 1$, it suffices to show that $\mathrm{rank}(\mathbf{A}) = 2$. We prove the inequality $\mathrm{rank}(\mathbf{A}) = 2$ by contradiction. Assume that $\mathrm{rank}(\mathbf{A}) = 1$. Therefore, the determinant of every 2×2 submatrix of $\mathbf{A}$ must be zero. In particular,

$$\det \begin{bmatrix} A_{12} & A_{13} \\ A_{22} & A_{23} \end{bmatrix} = A_{12}A_{23} - A_{13}A_{22} = 0 \tag{21}$$

Since $A_{22} \in \mathbb{R}^+$ and $\measuredangle A_{13} = -\measuredangle A_{31}$, one can conclude from (21) that:

$$\measuredangle A_{12} + \measuredangle A_{23} + \measuredangle A_{31} = 0 \tag{22}$$

Let $[\mathrm{Re}(y_{lm}^*)/\mathrm{Im}(y_{lm}^*)] = \alpha$ for all $(l, m) \in \mathcal{L}$. Then:

$$\begin{aligned} & \frac{\mathrm{Im}(A_{12})}{\mathrm{Im}(y_{12}^*)} + \frac{\mathrm{Im}(A_{23})}{\mathrm{Im}(y_{23}^*)} + \frac{\mathrm{Im}(A_{31})}{\mathrm{Im}(y_{31}^*)} = \\ & \quad \frac{1}{2}\alpha[(\lambda_1^a - \lambda_2^a) + (\lambda_2^a - \lambda_3^a) + (\lambda_3^a - \lambda_1^a)] \\ & \quad - \frac{1}{2}[(\nu_1^a - \nu_2^a) + (\nu_2^a - \nu_3^a) + (\nu_3^a - \nu_1^a)] = 0 \end{aligned} \tag{23}$$

According to (4) and (23), it can be concluded that $\mathrm{Im}(A_{12})$, $\mathrm{Im}(A_{23})$ and $\mathrm{Im}(A_{31})$ are not positive or negative simultaneously. If two of them are nonpositive and the third one is nonnegative, then since their real parts are negative, one can conclude that

$$\frac{5\pi}{2} < \measuredangle A_{12} + \measuredangle A_{23} + \measuredangle A_{31} < 4\pi \tag{24}$$

and if two of them are nonnegative and the third one is nonpositive, then because of their negative real parts, one can arrive at:

$$2\pi < \measuredangle A_{12} + \measuredangle A_{23} + \measuredangle A_{31} < \frac{7\pi}{2} \tag{25}$$

Both (24) and (25) contradict (22). This completes the proof. ■

According to the results given in [12], $Q_k^{max} = +\infty$ is a sufficient condition for the exactness of the SDP relaxation in a lossy cyclic network of size 3. However, Theorem 2 demonstrates that this condition is not necessary for the exactness when $\mathrm{Re}(y_{lm}^*)/\mathrm{Im}(y_{lm}^*)$ is equal for all lines of the network.

Theorem 3. *Consider the OPF problem for a lossless cycle of size 4. The SDP relaxation of OPF for this network is exact if $Q_k^{min} = -\infty$ and $Q_k^{max} = +\infty$ for all $k \in \mathcal{N}$ and after relaxing the line capacity constraints.*

Proof: Since $Q_k^{min} = -\infty$, $Q_k^{max} = +\infty$ and the capacity constraints are relaxed, the lagrange multipliers λ_k^a, λ_k^b, ψ_{lm} and ψ_{ml} do not appear in (7). Also, in the lossless network, we have $\mathrm{Re}(y_{lm}^*) = 0$. Therefore,

$$A_{lm} = -\mathrm{Im}(y_{lm}^*) - \frac{1}{2}\mathrm{Im}(y_{lm}^*)(\nu_l^a - \nu_m^a - \nu_l^b + \nu_m^b)\mathbf{i} \tag{26}$$

For $n = 4$, the matrix $\mathbf{A}$ given in (9) is in the following form:

$$\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} & 0 & A_{14} \\ A_{21} & A_{22} & A_{23} & 0 \\ 0 & A_{32} & A_{33} & A_{34} \\ A_{41} & 0 & A_{43} & A_{44} \end{bmatrix} \tag{27}$$

According to (10), in order to prove that $\mathrm{rank}(\mathbf{W}^{opt}) = 1$, it suffices to show that $\mathrm{rank}(\mathbf{A}) = 3$. We prove the equality $\mathrm{rank}(\mathbf{A}) = 3$ by contradiction. Assume that $\mathrm{rank}(\mathbf{A}) \leq 2$. Therefore, the determinant of every 3×3 submatrix of $\mathbf{A}$ must be zero. In particular,

$$\begin{aligned} \det \begin{bmatrix} A_{12} & 0 & A_{14} \\ A_{22} & A_{23} & 0 \\ A_{32} & A_{33} & A_{34} \end{bmatrix} =& A_{12}A_{23}A_{34} + \\ & A_{14}(A_{22}A_{33} - A_{23}A_{32}) = 0 \end{aligned} \tag{28}$$

Since all principal minors of a Hermitian positive semidefinite matrix are nonnegative, we have

$$A_{22}A_{33} - A_{23}A_{32} \geq 0 \tag{29}$$

If $A_{22}A_{33} - A_{23}A_{32} = 0$, then it results from (28) that $A_{12}A_{23}A_{34} = 0$, which contradicts the fact that the real part of every A_{lm} is a negative number.

If $A_{22}A_{33} - A_{23}A_{32} > 0$, then according to (28), we have

$$\measuredangle A_{12} + \measuredangle A_{23} + \measuredangle A_{34} + \measuredangle A_{41} = \pi \tag{30}$$

Also, it can be concluded from (26) that

$$\tan(\measuredangle A_{lm}) = \frac{1}{2}(\nu_l^a - \nu_m^a - \nu_l^b + \nu_m^b) \tag{31}$$

In light of (31), one can arrive at the following equation:

$$\tan(\measuredangle A_{12}) + \tan(\measuredangle A_{23}) + \tan(\measuredangle A_{34}) + \tan(\measuredangle A_{41}) = 0 \tag{32}$$

On the other hand,

$$\tan(\measuredangle A_{12}) + \tan(\measuredangle A_{23}) = \tan(\measuredangle A_{12} + \measuredangle A_{23})\times \big(1 - \tan(\measuredangle A_{12})\tan(\measuredangle A_{23})\big) \tag{33a}$$

$$\tan(\measuredangle A_{34}) + \tan(\measuredangle A_{41}) = \tan(\measuredangle A_{34} + \measuredangle A_{41})\times \big(1 - \tan(\measuredangle A_{34})\tan(\measuredangle A_{41})\big) \tag{33b}$$

Since $\tan(\pi - \alpha) = -\tan(\alpha)$, it can be deduced from (30)-(33) that

$$\tan(\measuredangle A_{12} + \measuredangle A_{23})\big(\tan(\measuredangle A_{34})\tan(\measuredangle A_{41}) - \tan(\measuredangle A_{12})\tan(\measuredangle A_{23})\big) = 0 \tag{34}$$

Also, since the real part of A_{lm} is negative, one can write

$$\frac{\pi}{2} < \measuredangle A_{lm} \leq \pi \qquad \text{if} \quad \mathrm{Im}(A_{lm}) \geq 0 \tag{35a}$$

$$\pi < \measuredangle A_{lm} < \frac{3\pi}{2} \qquad \text{if} \quad \mathrm{Im}(A_{lm}) < 0 \tag{35b}$$

It follows from (34) that $\tan(\measuredangle A_{12} + \measuredangle A_{23}) = 0$ or $\tan(\measuredangle A_{34})\tan(\measuredangle A_{41}) = \tan(\measuredangle A_{12})\tan(\measuredangle A_{23})$. If $\tan(\measuredangle A_{12} + \measuredangle A_{23}) = 0$, then (30) yields that $\measuredangle A_{12} + \measuredangle A_{23} = \pi$ or $\measuredangle A_{34} + \measuredangle A_{41} = \pi$. With no loss of generality, assume that $\measuredangle A_{12} + \measuredangle A_{23} = \pi$. However, based on (35) one can conclude that $\pi < \measuredangle A_{12} + \measuredangle A_{23} < 3\pi$, which contradicts the equation $\measuredangle A_{12} + \measuredangle A_{23} = \pi$.
If $\tan(\measuredangle A_{34})\tan(\measuredangle A_{41}) = \tan(\measuredangle A_{12})\tan(\measuredangle A_{23})$, then according to (32) two elements of the set $\{A_{12}, A_{23}, A_{34}, A_{41}\}$ have nonnegative imaginary parts, and the other two elements have nonpositive imaginary parts. Then, based on (35) one can arrive at

$$3\pi < \measuredangle A_{12} + \measuredangle A_{23} + \measuredangle A_{34} + \measuredangle A_{41} < 5\pi$$

which contradicts (30). This completes the proof. ■

Theorem 4. *Consider the OPF problem for a lossless cyclic network of size n. The SDP relaxation of OPF for this network is exact if $Q_k^{min} = P_k^{min} = -\infty$ and $P_k^{max} = +\infty$ for every $k \in \mathcal{N}$.*

Proof: Since $Q_k^{min} = P_k^{min} = -\infty$ and $P_k^{max} = +\infty$, the lagrange multipliers ν_k^a, ν_k^b and λ_k^b do not appear in (7). Moreover, the equation $\mathrm{Re}(y_{lm}^*) = 0$ holds in the lossless network. As a result,

$$A_{lm} = -\psi_{lm} - \psi_{ml} - \frac{1}{2}\mathrm{Im}(y_{lm}^*)(\lambda_l^a + \lambda_m^a) - \mathrm{Im}(y_{lm}^*) \tag{36}$$

Since $\psi_{lm}, \psi_{ml}, \lambda_l^a$ and λ_m^a are all nonnegative and $\mathrm{Im}(y_{lm}^*) > 0$, it follows from (36) that A_{lm} is a negative real number for all $(l, m) \in \mathcal{L}$, i.e.,

$$A_{lm} \in \mathbb{R}^- \quad \forall (l, m) \in \mathcal{L} \tag{37}$$

Recall that the matrix $\mathbf{A}$ is in the form given in (9), and also inequality (10) holds at optimality.

Let $A_{\{i\},\{j\}}$ be a matrix obtained from $\mathbf{A}$ by removing its i-th row and j-th column. Likewise, define $A_{\{i,k\},\{j,l\}}$ as a matrix obtained from $\mathbf{A}$ by removing its i-th and k-th rows and j-th and l-th columns. All principal minors of a Hermitian positive semidefinite matrix are nonnegative. Particularly,

$$det\big(A_{\{1,n\},\{1,n\}}\big) \geq 0 \tag{38}$$

According to (10), in order to prove that $\mathbf{W}^{opt}$ has rank 1, it suffices to show that rank$(\mathbf{A}) = n - 1$. We prove this by contradiction. Assume that rank$(\mathbf{A}) < n - 1$. Therefore, the determinant of each $(n-1) \times (n-1)$ submatrix $\mathbf{A}$ must be zero. In particular,

$$\begin{aligned} det\big(A_{\{n\},\{1\}}\big) = & A_{12}A_{23}\ldots A_{(n-1)n} \\ & + (-1)^n A_{1n} det\big(A_{\{1,n\},\{1,n\}}\big) = 0 \end{aligned} \tag{39}$$

According to (37) and (38) for $n = 2k$, we have

$$A_{12}A_{23}\ldots A_{(n-1)n} < 0 \tag{40}$$

$$(-1)^n A_{1n} det\big(A_{\{1,n\},\{1,n\}}\big) \leq 0 \tag{41}$$

From (40) and (41), one can arrive at $A_{12}A_{23}\ldots A_{(n-1)n} + (-1)^n A_{1n} det\big(A_{\{1,n\},\{1,n\}}\big) < 0$, which contradicts (39). Similarly for $n = 2k + 1$, we have

$$A_{12}A_{23}\ldots A_{(n-1)n} > 0 \tag{42}$$

$$(-1)^n A_{1n} det\big(A_{\{1,n\},\{1,n\}}\big) \geq 0 \tag{43}$$

From (42) and (43), one can conclude that $A_{12}A_{23}\ldots A_{(n-1)n} + (-1)^n A_{1n} det\big(A_{\{1,n\},\{1,n\}}\big) > 0$, which contradicts (39). This completes the proof. ■

Theorem 5. *Consider the OPF problem with the objective function $\sum_{k\in\mathcal{N}} f_k(Q_{G_k})$, where $f_k(Q_{G_k})$ is an increasing function for all $k \in \mathcal{N}$. The SDP relaxation is exact for a lossless network of size n, provided that $Q_k^{min} = -\infty$, $P_k^{min} \leq P_{D_k}$ and $P_k^{max} \geq P_{D_k}$ for every $k \in \mathcal{N}$.*

Proof: Consider an arbitrary solution $(\hat{\mathbf{W}}, P^{opt}, Q^{opt})$ of the SDP relaxation. In the sequel, we show that the following rank-1 Hermitian positive semidefinite matrix $\bar{\mathbf{W}}$ is a solution of the SDP relaxation:

$$\bar{W}_{ij} = \begin{cases} \hat{W}_{ij} & i = j \\ \sqrt{\hat{W}_{ii}\hat{W}_{jj}} & i \neq j \end{cases} \tag{44}$$

Since the diagonal entries of a Hermitian matrix are real numbers, in light of (2b) and (2c) we have

$$\begin{aligned} Q_{G_k} = Q_{D_k} + & \sum_{l\in N(k)} W_{kk}\mathrm{Im}(y_{kl}^*) \\ & - \sum_{l\in N(k)} [\mathrm{Re}(W_{kl})\mathrm{Im}(y_{kl}^*) + \mathrm{Im}(W_{kl})\mathrm{Re}(y_{kl}^*)] \end{aligned} \tag{45}$$

$$\begin{aligned} P_{G_k} = P_{D_k} + & \sum_{l\in N(k)} W_{kk}\mathrm{Re}(y_{kl}^*) \\ & - \sum_{l\in N(k)} [\mathrm{Re}(W_{kl})\mathrm{Re}(y_{kl}^*) - \mathrm{Im}(W_{kl})\mathrm{Im}(y_{kl}^*)] \end{aligned} \tag{46}$$

Since $\hat{\mathbf{W}}$ is a solution of the problem for a lossless network, one can write:

$$Q_{G_k}^{opt} = Q_{D_k} + \sum_{l\in N(k)} \hat{W}_{kk}\text{Im}(y_{kl}^*) - \sum_{l\in N(k)} \text{Re}(\hat{W}_{kl})\text{Im}(y_{kl}^*) \tag{47}$$

Also, since all principal minors of $\hat{\mathbf{W}} \in \mathbf{H}_+^n$ are nonnegative, it yields that

$$\hat{W}_{kk}\hat{W}_{ll} \geq \hat{W}_{kl}\hat{W}_{lk} \tag{48}$$

In addition, the relation $\hat{W}_{kl} = \hat{W}_{lk}^*$ leads to

$$\hat{W}_{kl}\hat{W}_{lk} = [\text{Re}(\hat{W}_{kl})]^2 + [\text{Im}(\hat{W}_{kl})]^2 \tag{49}$$

From (44), (48) and (49) one can conclude that

$$\text{Re}(\bar{W}_{kl}) = \sqrt{\hat{W}_{kk}\hat{W}_{ll}} \geq \sqrt{[\text{Re}(\hat{W}_{kl})]^2 + [\text{Im}(\hat{W}_{kl})]^2} \geq |\text{Re}(\hat{W}_{kl})| \tag{50}$$

On the other hands,

$$\bar{Q}_{G_k} = Q_{D_k} + \sum_{l\in N(k)} \bar{W}_{kk}\text{Im}(y_{kl}^*) - \sum_{l\in N(k)} \text{Re}(\bar{W}_{kl})\text{Im}(y_{kl}^*) \tag{51}$$

From (44), (50) and (51) one can arrive at:

$$\bar{Q}_{G_k} \leq Q_{G_k}^{opt} \tag{52}$$

Since $\text{Im}(\bar{W}_{kl}) = 0$ for all $(l,m) \in \mathcal{L}$ and also the network is lossless, it results from (46) that

$$\bar{P}_{G_k} = P_{D_k} \tag{53}$$

Also, since $\hat{\mathbf{W}}$ is a solution of the problem, we have

$$\hat{W}_{ll} + \hat{W}_{mm} - \hat{W}_{lm} - \hat{W}_{ml} = \hat{W}_{ll} + \hat{W}_{mm} - 2\text{Re}(\hat{W}_{lm}) \leq (\Delta V_{lm}^{max})^2 \tag{54}$$

From (44), (50) and (54) one can arrive at:

$$\bar{W}_{ll} + \bar{W}_{mm} - \bar{W}_{lm} - \bar{W}_{ml} \leq \hat{W}_{ll} + \hat{W}_{ll} - 2\text{Re}(\hat{W}_{lm}) \leq (\Delta V_{lm}^{max})^2 \tag{55}$$

Finally, since $f_k(Q_{G_k})$ is an increasing function for all $k \in \mathcal{N}$, one can conclude from (44), (52), (53) and (55) that $\bar{\mathbf{W}}$ is a rank-1 solution of the OPF problem. ■

The proof of the above theorem is based on designing a set of voltages with zero phases corresponding to a rank-1 SDP solution. It is known that angles are small in practice [29]. This implies that the sum of reactive powers acts as a penalty to promote a rank-1 solution for a classic OPF minimizing a function of active powers [12], where this penalty mostly affects voltage magnitudes as opposed to angles due to Theorem 5.

Theorem 6. *Consider the OPF problem with the objective function $\sum_{k\in\mathcal{N}} \alpha_i P_{G_k}$ and the line constraint (1h) for a lossless cycle of size $n = 2k+1$. Let the graph representing the network be denoted by $\mathcal{C}_n$. The SDP relaxation of OPF for this network is exact if $Q_k^{min} = -\infty$, $Q_k^{max} = +\infty$ and the optimal lagrange multiplier matrix fits $\mathcal{C}_n$.*

Proof: In a lossless cyclic network of size n, inequality (2h) can be written in the following matrix form:

$$\mathbf{X}_{lm} \bullet \mathbf{W} \leq P_{lm}^{max} \quad \forall (l,m) \in \mathcal{L} \tag{56}$$

where $\mathbf{X}_{lm}$ is given by the following equation:

$$[\mathbf{X}_{lm}]_{ij} = \begin{cases} -\frac{\mathbf{i}}{2}\text{Im}(y_{lm}^*) & ij = lm \\ +\frac{\mathbf{i}}{2}\text{Im}(y_{lm}^*) & ij = ml \\ 0 & \text{otherwise} \end{cases} \tag{57}$$

Also, the objective function $\sum_{k\in\mathcal{N}} \alpha_i P_{G_k}$ can be written as

$$\sum_{k\in\mathcal{N}} \alpha_i P_{G_k} = \mathbf{T} \bullet \mathbf{W} + \sum_{k\in\mathcal{N}} P_{D_k} \tag{58}$$

where $\mathbf{T}$ is given by the following equation for a lossless cyclic network of size n:

$$[\mathbf{T}]_{ij} = \begin{cases} -\frac{\mathbf{i}}{2}(\alpha_i - \alpha_j)\text{Im}(y_{ij}^*) & |i-j| \in \{1, n-1\} \\ 0 & \text{otherwise} \end{cases} \tag{59}$$

Since $Q_k^{min} = -\infty$ and $Q_k^{max} = +\infty$, equation (3b) is always satisfied. Let ν_k^a, μ_k^a and ν_k^b, μ_k^b denote the lagrange multipliers corresponding to the upper bounding constraints (3a),(3c) and the lower bounding constraints (3a),(3c), respectively. In addition, the lagrange multipliers corresponding to (56) and inequality $\mathbf{W} \succeq 0$ are denoted by ξ_{lm} and $\mathbf{A} \in \mathbf{H}_+^n$.

By obtaining the lagrangian of the given optimization problem, it can be concluded that

$$A_{lm} = -\frac{\mathbf{i}}{2}\text{Im}(y_{lm}^*)[(\alpha_l - \alpha_m) + (\nu_l^a - \nu_m^a - \nu_l^b + \nu_m^b) + \xi_{lm}] \tag{60}$$

for all $(l,m) \in \mathcal{L}$, and

$$A_{lm} = 0 \quad \forall (l,m) \notin \mathcal{L} \tag{61}$$

The matrix $\mathbf{A}$ is in the form of (9), and also inequality (10) holds at optimality. Hence, in order to prove that $\mathbf{W}^{opt}$ has rank 1, it suffices to show that rank$(\mathbf{A}) = n-1$. We prove this by contradiction. Assume taht rank$(\mathbf{A}) < n-1$. Therefore, the determinant of each $(n-1)\times(n-1)$ submatrix $\mathbf{A}$ must be zero. In particular,

$$\begin{aligned} det\big(A_{\{n\},\{1\}}\big) =& A_{12}A_{23}\dots A_{(n-1)n} \\ &+ (-1)^n A_{1n} det\big(A_{\{1,n\},\{1,n\}}\big) = 0 \end{aligned} \tag{62}$$

Since the matrix $\mathbf{A}$ fits $\mathcal{C}_n$, the equation (60) yields that all non-zero off diagonal entries of $\mathbf{A}$ are imaginary numbers. As a result, for $n = 2k+1$ one can arrive at:

$$\text{Im}(A_{12}A_{23}\dots A_{(n-1)n}) = 0 \tag{63}$$

$$\text{Re}(A_{12}A_{23}\dots A_{(n-1)n}) \neq 0 \tag{64}$$

Since all principal minors of a Hermitian positive semidefinite matrix are nonnegative, we have $det\big(A_{\{1,n\},\{1,n\}}\big) \geq 0$. If $det\big(A_{\{1,n\},\{1,n\}}\big) > 0$, then

$$\text{Im}[(-1)^n A_{1n} det\big(A_{\{1,n\},\{1,n\}}\big)] \neq 0 \tag{65}$$

and if $det\big(A_{\{1,n\},\{1,n\}}\big) = 0$, then

$$\text{Re}[(-1)^n A_{1n} det\big(A_{\{1,n\},\{1,n\}}\big)] = 0 \tag{66}$$

Equations (63) and (65) contradict (62). Equations (64) and (66) also contradict (62). This completes the proof. ■

V. Conclusions

Recently, it has been shown that a global solution of different classes of the OPF problem can be obtained via SDP relaxation methods. Although an SDP relaxation is exact for radial networks under certain conditions, this approach may fail when there are cycles in the network. Inspired by this fact, we study the OPF problem in very special cyclic networks and provide sufficient conditions under which the SDP relaxation is exact. When the goal is to minimize a linear function of active powers, we prove that the SDP relaxation is exact for odd cycles if the optimal lagrange multiplier matrix fits the graph representing the network. The exactness of the SDP relaxation for a cyclic network of size 3 and 4 is also proved under certain conditions. We also investigate the OPF problem in weakly-cyclic networks and provide sufficient conditions for the existence of rank-1 or -2 SDP solutions. Furthermore, when the objective function is a summation of increasing functions of reactive powers, we show that the SDP relaxation is exact under technical conditions. This result justifies why the sum of reactive power is a low-rank-promoting penalty for the SDP relaxation of a classic OPF problem.

References

[1] Z. Miao, L. Fan, H. G. Aghamolki, and B. Zeng, "Least squares estimation based SDP cuts for SOCP relaxation of AC OPF," *IEEE Transactions on Automatic Control*, vol. 63, no. 1, pp. 241–248, 2018.

[2] S. Magnusson, P. C. Weeraddana, and C. Fischione, "A distributed approach for the optimal power-flow problem based on ADMM and sequential convex approximations," *IEEE Transactions on Control of Network Systems*, vol. 2, no. 3, pp. 238–253, 2015.

[3] J. A. Momoh, M. E. El-Hawary, and R. Adapa, "A review of selected optimal power flow literature to 1993. part i: Nonlinear and quadratic programming approaches," *IEEE Transactions on Power Systems*, vol. 14, no. 1, pp. 96–104, 1999.

[4] D. K. Molzahn, F. Dorfler, H. Sandberg, S. H. Low, S. Chakrabarti, R. Baldick, and J. Lavaei, "A survey of distributed optimization and control algorithms for electric power systems," *IEEE Transactions on Smart Grid*, vol. 8, no. 6, pp. 2941–2962, 2017.

[5] H.-D. Chiang and C.-Y. Jiang, "Feasible region of optimal power flow: Characterization and applications," *IEEE Transactions on Power Systems*, vol. 33, no. 1, pp. 236–244, 2018.

[6] S. Kar, G. Hug, J. Mohammadi, and J. M. F. Moura, "Distributed state estimation and energy management in smart grids: A consensus innovations approach," *IEEE Journal of Selected Topics in Signal Processing*, vol. 8, no. 6, pp. 1022–1038, 2014.

[7] R. A. Abarghooee, V. Terzija, F. Golestaneh, and A. Roosta, "Multi-objective dynamic optimal power flowconsidering fuzzy-based smart utilization of mobile electric vehicles," *IEEE Transactions on Industrial Informatics*, vol. 12, no. 2, pp. 503–514, 2016.

[8] A. Castillo, C. Laird, C. A. Silva-Monroy, J.-P. Watson, and R. P. ONeill, "The unit commitment problem with AC optimal power flow constraints," *IEEE Transactions on Power Systems*, vol. 31, no. 6, pp. 4853–4866, 2016.

[9] D. K. Molzahn, B. C. Lesieutre, and C. L. DeMarco, "A sufficient condition for power flow insolvability with applications to voltage stability margins," *IEEE Transactions on Power Systems*, vol. 28, no. 3, pp. 2592–2601, 2013.

[10] A. Y. S. Lam, B. Zhang, A. Dominguez-Garcia, and D. Tse, "An optimal and distributed method for voltage regulation in power distribution systems," *IEEE Transactions on Power Systems*, vol. 30, no. 4, pp. 1714–1726, 2015.

[11] B. Zhang and D. Tse, "Geometry of injection regions of power networks," *IEEE Transactions on Power Systems*, vol. 28, no. 2, pp. 788–797, 2013.

[12] R. Madani, S. Sojoudi, and J. Lavaei, "Convex relaxation for optimal power flow problem: Mesh networks," *IEEE Transactions on Power Systems*, vol. 30, no. 1, pp. 199–211, 2015.

[13] S. H. Low, "Convex relaxation of optimal power flow part I: Formulations and equivalence," *IEEE Transactions on Control of Network Systems*, vol. 1, no. 1, pp. 15–27, 2014.

[14] R. Jabr, "Radial distribution load flow using conic programming," *IEEE Transactions on Power Systems*, vol. 21, no. 3, pp. 1458–1459, 2006.

[15] D. K. Molzahn and I. A. Hiskens, "Convex relaxations of optimal power flow problems: An illustrative example," *IEEE Transactions on Circuits and Systems*, vol. 63, no. 5, pp. 650–660, 2016.

[16] X. Bai, H. Wei, K. Fujisawa, and Y. Wang, "Semidefinite programming for optimal power flow problems," *International Journal of Electrical Power and Energy Systems*, vol. 30, no. 6, pp. 383–392, 2008.

[17] J. Lavaei and H. Low, "Zero duality gap in optimal power flow problem," *IEEE Transactions on Power Systems*, vol. 27, no. 1, pp. 92–107, 2012.

[18] S. Sojoudi and J. Lavaei, "Physics of power networks makes hard optimization problems easy to solve," in *Power and Energy Society General Meeting, 2012 IEEE*. IEEE, 2012, pp. 1–8.

[19] ——, "Exactness of semidefinite relaxations for nonlinear optimization problems with underlying graph structure," *SIAM Journal on Optimization*, vol. 24, no. 4, pp. 1746–1778, 2014.

[20] S. Bahrami, F. Therrien, V. W. Wong, and J. Jatskevich, "Semidefinite relaxation of optimal power flow for AC-DC grids," *IEEE Transactions on Power Systems*, vol. 32, no. 1, pp. 289–304, 2017.

[21] D. K. Molzahn, B. C.Lesieutre, and C. L. D. Marco, "A sufficient condition for global optimality of solutions to the optimal power flow problem," *IEEE Transactions on Power Systems*, vol. 29, no. 2, pp. 978–979, 2014.

[22] R. Madani, S. Sojoudi, G. Fazelnia, and J. Lavaei, "Finding low-rank solutions of sparse linear matrix inequalities using convex optimization," *SIAM Journal on Optimization*, vol. 27, no. 2, pp. 725–758, 2017.

[23] S. Huang, Q. Wu, J. Wang, and H. Zhao, "A sufficient condition on convex relaxation of AC optimal power flow in distribution networks," *IEEE Transactions on Power Systems*, vol. 32, no. 2, pp. 1359–1368, 2017.

[24] L. Gan and S. H. Low, "An online gradient algorithm for optimal power flow on radial networks," *IEEE Journal on Selected Areas in Communications*, vol. 34, no. 3, pp. 625–638, 2016.

[25] B. Kocuk, S. S. Dey, and X. A. Sun, "Inexactness of SDP relaxation and valid inequalities for optimal power flow," *IEEE Transactions on Power Systems*, vol. 31, no. 1, pp. 642–651, 2016.

[26] Q. Peng and S. H. Low, "Distributed optimal power flow algorithm for radial networks, i: Balanced single phase case," *IEEE Transactions on Smart Grid*, vol. 9, no. 1, pp. 111–121, 2018.

[27] W. Wei, J. Wang, N. Li, and S. Mei, "Optimal power flow of radial networks and its variations: A sequential convex optimization approach," *IEEE Transactions on Smart Grid*, vol. 8, no. 6, pp. 625–638, 2017.

[28] R. Madani, M. Ashraphijuo, and J. Lavaei, "Promises of conic relaxation for contingency-constrained optimal power flow problem," *IEEE Transactions on Power Systems*, vol. 31, no. 2, pp. 1297–1307, 2016.

[29] B. Stott, J. Jardim, and O. Alsac, "Dc power flow revisited," *IEEE Transactions on Power Systems*, vol. 24, no. 3, pp. 1290–1300, 2009.