Convexification of Optimal Power Flow Problem by Means of Phase Shifters

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***Abstract*—This paper is concerned with the convexification of the optimal power flow (OPF) problem. We have previously shown that this highly nonconvex problem can be solved efficiently via a convex relaxation after two approximations: (i) adding a sufficient number of virtual phase shifters to the network topology, and (ii) relaxing the power balance equations to inequality constraints. The objective of the present paper is to first provide a better understanding of the implications of Approximation (i) and then remove Approximation (ii). To this end, we investigate the effect of virtual phase shifters on the feasible set of OPF by thoroughly examining a cyclic system. We then show that OPF can be convexified under only Approximation (i), provided some mild assumptions are satisfied. Although this paper mainly focuses on OPF, the results developed here can be applied to several OPF-based emerging optimizations for future electrical grids.**

I. Introduction

The real-time operation of a power network depends heavily on various large-scale optimization problems solved from every few minutes to every several months. State estimation, optimal power flow (OPF), security-constrained OPF, and transmission planning are some fundamental operations solved for transmission networks. Although most of the energy-related optimizations are traditionally solved at transmission level, there are a few optimizations associated with distribution systems, e.g., sizing of capacitor banks and network reconfiguration. Each of these problems has the power flow equations embedded in it. With the exception of the security-constrained OPF, these problems often have less than 10,000 variables. Even though the number of variables in these optimizations is modest compared to many real-world optimizations involving millions of variables, it is very challenging to solve energy-related optimization problems efficiently. This is in part due to the nonlinearities imposed by the laws of physics.

The OPF problem is the most fundamental optimization for power systems, which aims to find an optimal operating point of the power network minimizing a certain objective function (e.g., power loss or generation cost) subject to network and physical constraints [1], [2]. OPF has a nonconvex/disconnected feasibility region in general [3], and its nonlinearity has been studied since 1962 [4], [5]. The paper [6] proposes a convex relaxation based on semidefinite programming (SDP) to solve OPF and it shows that this method works for IEEE benchmark systems in addition to several randomly generated networks. This technique is the first one to date that has the potential to find a provably global solution of practical OPF problems.

To further study the SDP relaxation derived in [6], the paper [7] proves that this relaxation is exact for acyclic network, provided the power balance equations are relaxed from equality constraints to inequality constraints (similar result was also derived in [8] and [9]). The work [7] also shows that the OPF problem can be solved for cyclic networks through a second-order cone program (SOCP) relaxation after two approximations: (i) relaxing the power balance equations as before, (ii) relaxing the angle constraints by placing a sufficient number of virtual phase shifters in the network. The objective of the present work is to provide a better understating of the SOCP relaxation for mesh networks.

In this work, we first provide a detailed analysis on a three-bus system to understand how adding a virtual phase shifter reduces the computational complexity of OPF and how this approximation affects the true solution of the original problem. This analysis is based on investigating the feasible set of the OPF problem. Then, we prove that the SOCP relaxation finds the optimal solution of a general OPF problem after adding virtual phase shifters—without further relaxing the power balance equations—provided that some mild assumptions are satisfied. These assumptions require that the lower and upper bounds on the voltage magnitudes be close to each other and that the angle difference across each line not be excessively large. The main conclusion of this paper is that real-world energy-related optimizations based on OPF become tractable after eliminating the angle constraints via incorporating virtual phase shifters [10].

The SDP/SOCP relaxation for OPF has attracted much attention due to its ability to find a global solution in polynomial time, and it has been applied to various applications in power systems including: voltage regulation in distribution systems [11], state estimation [12], calculation of voltage stability margin [13], economic dispatch in unbalanced distribution networks [14], power management under time-varying conditions [15], distributed energy management [16], and OPF with storage integration [17]. The results of this work enhance the theoretical foundation of the convex relaxation for all of the above problems as well as emerging applications related to renewable energy, distributed generation, and demand response.

Notations: $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{C}$ and $\mathbb{H}^n$ denote the sets of real numbers, positive real numbers, complex numbers, and $n \times n$ positive semidefinite Hermitian matrices, respectively. $\mathrm{Re}\{\mathbf{W}\}$, $\mathrm{Im}\{\mathbf{W}\}$, and $\mathrm{rank}\{\mathbf{W}\}$ denote the real part, imaginary part, and rank of a given scalar/matrix $\mathbf{W}$, respectively. The notation $\mathbf{W} \succeq 0$ means that $\mathbf{W}$ is Hermitian and positive semidefinite. The notation "i" is reserved for the imaginary unit. The notation $\measuredangle x$ denotes the angle of a complex number x. The symbol "*" represents the conjugate transpose operator. Given a matrix $\mathbf{W}$, its (l,m) entry is denoted as W_{lm}. The superscript $(\cdot)^{\rm opt}$ is used to show the optimal value of an optimization parameter.

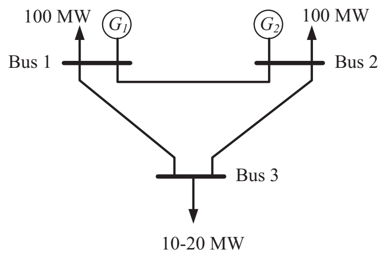


Fig. 1: The three-bus power network studied in Section II.

Definition 1: The Pareto front (face) of a set $\mathcal{S}$ is defined as the collection of all points $\mathbf{x} \in \mathcal{S}$ for which there does not exist a different point $\mathbf{y} \in \mathcal{S}$ such that $\mathbf{y} \leq \mathbf{x}$ (componentwise).

II. Illustrative Example

Consider the three-bus network depicted in Figure 1 with the node set $\mathcal{N} = \{1, 2, 3\}$, the line set $\mathcal{L} = \{(1, 2), (2, 1), (2, 3), (3, 2), (3, 1), (1, 3)\}$, and the line impedances (z):

$$z_{12} = 0.3 + \mathrm{i}, \quad z_{23} = 0.4 + \mathrm{i}, \quad z_{31} = 0.5 + \mathrm{i}$$

where $z_{lm} = z_{ml}$ for every $(l, m) \in \mathcal{L}$. Let $y_{lm} \triangleq \frac{1}{z_{lm}}$ denote the admittance of the line (l, m). In this network, the loads at buses 1 and 2 are fixed at the value 100MW, whereas the load at bus 3 is flexible and can accept any amount of power in the range [10MW,20MW]. Define V_i as the complex voltage at bus k and θ_k as its phase for every $k \in \mathcal{N}$. Define also P_{G_k} as the unknown active-power output of generator $k \in \{1, 2\}$, which is associated with a production cost $f_k(P_{G_k})$ for a strictly increasing function $f_k(\cdot)$. The goal is to find an output vector (P_{G_1}, P_{G_2}) minimizing the production cost $f_1(P_{G_1}) + f_2(P_{G_2})$ subject to:

- *Voltage constraints:* $|V_1| = |V_2| = |V_3| = 10$.
- *Angle constraints:* $|\theta_{lm}| \triangleq |\theta_l - \theta_m| \leq \theta_{lm}^{\max}$ for every line $(l, m) \in \mathcal{L}$.

Assume that $\theta_{12}^{\max} = 40°$, $\theta_{23}^{\max} = 50°$ and $\theta_{31}^{\max} = 20°$. Note that the angle constraint $|\theta_{lm}| \leq \theta_{lm}^{\max}$ can be regarded as the flow constraints $P_{lm}, P_{ml} \leq P_{lm}^{\max} = P_{ml}^{\max}$, where P_{lm} and P_{ml} denote the flows entering the line (l, m) from its l and m endpoints, and

$$P_{12}^{\max} = 71.29, \quad P_{23}^{\max} = 90.89, \quad P_{31}^{\max} = 37.21 \tag{3}$$

This optimal power flow (OPF) problem is formulated in (1), which has two vector variables $(P_{G_1}, P_{G_2}) \in \mathbb{R}^2$ and $(\theta_{12}, \theta_{23}, \theta_{31}) \in \mathbb{R}^3$ under the implicit assumption that $\theta_{lm} = -\theta_{ml}$ for every $(l, m) \in \mathcal{L}$. Notice that

- (1a)-(1c) account for the power balance equations.
- (1e) reflects the laws of physics.
- (1f) accounts for the fact that there must exist $\theta_1, \theta_2, \theta_3$ with the property that $\theta_{lm} = \theta_l - \theta_m$ for every line (l, m).

The OPF problem (1) can be described in a compact form as

$$\min_{(P_{G_1}, P_{G_2}) \in \mathcal{P}} f_1(P_{G_1}) + f_2(P_{G_2}) \tag{4}$$

where $\mathcal{P}$ represents the projection of the feasible set of OPF onto the space for the production vector (P_{G_1}, P_{G_2}). The blue region in Figure 2(a) depicts the nonconvex set $\mathcal{P}$. Since $\mathcal{P}$ and its convex hull share the same Pareto face (lower boundary), the solution of the above optimization does not change if $\mathcal{P}$ is replaced by its convex hull. However, the challenge is to find an *algebraic* convex representation of the convex hull of $\mathcal{P}$.

A. Reformulated OPF Problem

The OPF problem (1) is nonconvex in light of the nonlinear term $e^{\theta_{lm}\mathrm{i}}$ in the constraint (1e). To convexify this term, consider a vector $\begin{bmatrix} W_{12} & W_{23} & W_{31} \end{bmatrix} \in \mathbb{C}^3$ such that

$$\begin{bmatrix} 1 & W_{lm} \\ W_{ml} & 1 \end{bmatrix} = \text{Rank 1 \& Positive Semidefinite}$$

for every $(l, m) \in \mathcal{L}$, where $W_{ml} = W_{lm}^*$. The above relation is equivalent to the property $|W_{ij}| = 1$, implying that there exists an angle vector $\begin{bmatrix} \theta_{12} & \theta_{23} & \theta_{31} \end{bmatrix}$ such that $W_{lm} = e^{\theta_{lm}\mathrm{i}}$ for every $(l, m) \in \mathcal{L}$. This means that the OPF problem can be reformulated as (2) with two vector variables $(P_{G_1}, P_{G_2}) \in \mathbb{R}^2$ and $(W_{12}, W_{23}, W_{31}) \in \mathbb{C}^3$ under the implicit assumption $W_{lm} = W_{ml}^*$ for every $(l, m) \in \mathcal{L}$. This reformulation transforms the nonlinear constraint (1e) into the linear constraint (2e) at the cost of creating two nonlinear equations: (i) angle constraint (2f), and (ii) rank constraint (2g). The reformulated OPF problem can be convexified by removing these two problematic constraints. The rest of this section aims to delve into the consequences of this convexification.

B. Removal of Angle Constraint

Assume that the topology of the network given in Figure 1 has been modified by adding a phase shifter to one of the lines $(1, 2)$, $(2, 3)$ and $(3, 1)$. Assume also that the phase shift of this device, denoted as γ, can be optimized in addition to P_{G_1} and P_{G_2}. To account for this modification in the optimization problem, the constraint (2f) needs to be replaced by:

$$\measuredangle W_{12} + \measuredangle W_{23} + \measuredangle W_{31} = \gamma \tag{5}$$

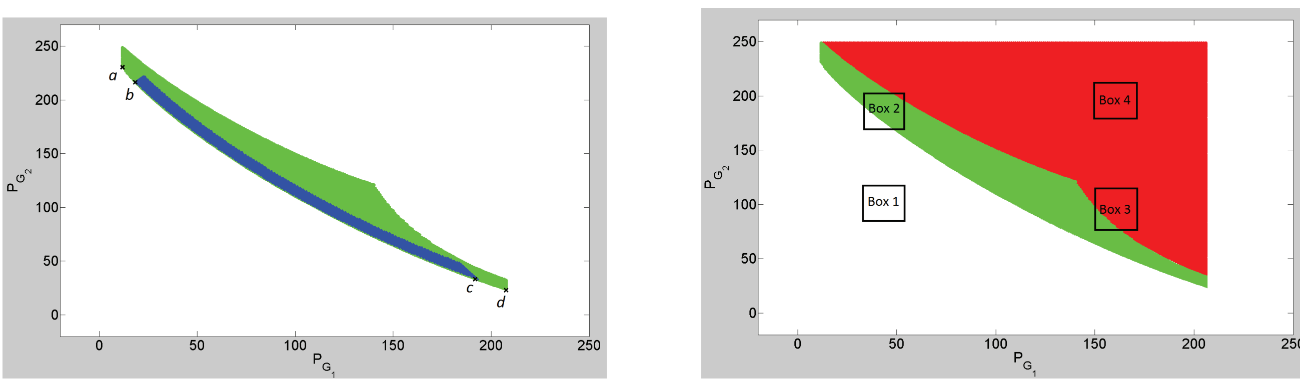
Since γ is a free parameter, it can be argued that the above constraint is redundant in the optimization. As a result, if the three-bus network has a controllable phase shifter, its OPF problem does not need the constraint (2f).

The above argument signifies that the elimination of the undesirable constraint (2f) from the reformulated OPF problem (2) is equivalent to the incorporation of a controllable phase shifter into the power network. Define $\mathcal{P}_s$ as the projection of the feasible set of Optimization (2) onto the space for (P_{G_1}, P_{G_2}) after removing the angle constraint (2f). The set $\mathcal{P}_s$ is depicted in Figure 2(a), which has two components: (i) the blue part $\mathcal{P}$, and (ii) the green part created by the elimination of the angle constraint. Similar to $\mathcal{P}$, the set $\mathcal{P}_s$ is non-convex. Nevertheless, it will be shown in the next subsection that the Pareto front of $\mathcal{P}_s$ can be characterized efficiently.

C. Removal of Angle and Rank Constraints

To convexify the reformulated OPF problem (2), we remove the nonlinear constraints (2f) and (2g). Let $\mathcal{P}_s^{(c)}$ denote the

OPF Problem			Reformulated OPF Problem		
Min $f_1(P_{G_1}) + f_2(P_{G_2})$ subject to:			Min $f_1(P_{G_1}) + f_2(P_{G_2})$ subject to:		
$P_{G_1} - 100 = P_{12} + P_{13}$		(1a)	$P_{G_1} - 100 = P_{12} + P_{13}$		(2a)
$P_{G_2} - 100 = P_{21} + P_{23}$		(1b)	$P_{G_2} - 100 = P_{21} + P_{23}$		(2b)
$10 \leq P_{31} + P_{32} \leq 20$		(1c)	$10 \leq P_{31} + P_{32} \leq 20$		(2c)
$P_{lm} \leq P_{lm}^{\max}$,	$(l,m) \in \mathcal{L}$	(1d)	$P_{lm} \leq P_{lm}^{\max}$,	$(l,m) \in \mathcal{L}$	(2d)
$P_{lm} = \text{Re}\{100(1 - e^{\theta_{lm}\mathrm{i}})y_{lm}^*\}$,	$(l,m) \in \mathcal{L}$	(1e)	$P_{lm} = \text{Re}\{100(1 - W_{lm})y_{lm}^*\}$,	$(l,m) \in \mathcal{L}$	(2e)
$\theta_{12} + \theta_{23} + \theta_{31} = 0$		(1f)	$\measuredangle W_{12} + \measuredangle W_{23} + \measuredangle W_{31} = 0$		(2f)
			$\begin{bmatrix} 1 & W_{lm} \\ W_{ml} & 1 \end{bmatrix} = \text{Rank } 1$,	$(l,m) \in \mathcal{L}$	(2g)
			$\begin{bmatrix} 1 & W_{lm} \\ W_{ml} & 1 \end{bmatrix} \succeq 0$,	$(l,m) \in \mathcal{L}$	(2h)



(a) (b)

Fig. 2: (a): Feasible set $\mathcal{P}$ (blue area) and feasible set $\mathcal{P}_s$ (blue and green areas); (b) Feasible set $\mathcal{P}_s$ (green area) and feasible set $\mathcal{P}_s^{(c)}$ (green and red areas);

projection of the feasible set of this convexified problem onto the space for (P_{G_1}, P_{G_2}). As shown in Figure 2(b), $\mathcal{P}_s^{(c)}$ expands the non-convex region $\mathcal{P}_s$ into a convex set by adding the red part. Observe that $\mathcal{P}_s$ and $\mathcal{P}_s^{(c)}$ share the same Pareto front (lower boundary). As a result, the non-convex optimization

$$\min_{(P_{G_1}, P_{G_2}) \in \mathcal{P}_s} f_1(P_{G_1}) + f_2(P_{G_2}) \tag{6}$$

and its convexified counterpart

$$\min_{(P_{G_1}, P_{G_2}) \in \mathcal{P}_s^{(c)}} f_1(P_{G_1}) + f_2(P_{G_2}) \tag{7}$$

have the same solution $(P_{G_1}^{\text{opt}}, P_{G_2}^{\text{opt}})$. This implies that the OPF problem can be transformed into a convex optimization whenever the power network has a controllable phase shifter.

D. Convexification via Virtual/Actual Phase Shifter

So far, it has been shown that the OPF problem can be solved efficiently, provided that the network drawn in Figure 1 contains a phase shifter on one of its lines. Assume that the power network has no such phase shifter in its circuit. Under this circumstance, one may add a virtual (fictitious) phase shifter to the network and then solve the convexified OPF problem accordingly. A question arises as to whether the obtained solution has any connection to the solution of the original OPF problem. This question will be investigated below.

Consider the feasible sets $\mathcal{P}$ and $\mathcal{P}_s$ plotted in Figure 2(a), corresponding to the OPF problem without and with phase shifter, respectively. Four points have been marked on the Pareto front of $\mathcal{P}_s$ as a, b, c and d. Notice that the Pareto front of $\mathcal{P}_s$ has three segments:

- *Segment with the endpoints b and c:* This segment "almost" overlaps the Pareto front of $\mathcal{P}$. Indeed, there is a very little gap between this segment and the front of $\mathcal{P}$.
- *Segment with the endpoints a and b:* This segment extends the Pareto front of $\mathcal{P}$ from the top.
- *Segment with the endpoints c and d:* This segment extends the Pareto front of $\mathcal{P}$ from the bottom.

The gap between the Pareto front of $\mathcal{P}$ and a subset of the Pareto front of $\mathcal{P}_s$ with the endpoints b and c can be unveiled by performing some simulations. For instance, assume that $f_1(P_{G_1}) = P_{G_1}$ and $f_1(P_{G_2}) = 1.2P_{G_2}$. Two OPF problems will be solved next:

- *OPF without phase shifter:* The solution turns out to be $(P_{G_1}^{\text{opt}}, P_{G_2}^{\text{opt}}) = (145.56, 68.18)$ corresponding to the optimal cost \$227.37.

- *OPF with phase shifter:* The solution turns out to be $(P_{G_1}^{\text{opt}}, P_{G_2}^{\text{opt}}, \gamma^{\text{opt}}) = (144.27, 69.39, 6.02^\circ)$ corresponding to the optimal cost \$227.53.

Although the optimal value of γ is not negligible, the optimal production $(P_{G_1}^{\text{opt}}, P_{G_2}^{\text{opt}})$ has very similar values in the above cases. Due to the closeness of these two values in the space for (P_{G_1}, P_{G_2}), the solution of the OPF problem with phase shifter can be fed into a local search algorithm in order to find the global solution of the OPF problem with no phase shifter.

The solution of OPF with phase shifter in the above case lies on the segment between the endpoints b and c. If the solution belongs to the segment with the endpoints a and b, then the solution of OPF without phase shifter corresponds to the top corner of the Pareto front of $\mathcal{P}$. Similar remark can be made for the segment with the endpoints c and d.

It can be shown that the optimal values of γ corresponding to the points a and d are equal to 12.27° and 8.25°, respectively. Hence, the optimal phase shift is never greater than 12.27°. Nonetheless, a large phase shift as high as 6.02° still means that the OPF problems with and without phase shifter have very similar optimal costs.

E. Effect of Production Limits

The OPF problem (1) does not restrict the outputs P_{G_1} and P_{G_2}. To understand the effect of lower and upper bounds on these two parameters, assume that (P_{G_1}, P_{G_2}) must belong to a pre-specified box $\mathcal{B} \in \mathbb{R}^2$. As before, suppose that the power network is equipped with an actual/virtual phase shifter. The objective is to find out whether the non-convex OPF problem

$$\min_{(P_{G_1}, P_{G_2}) \in \mathcal{P}_s \cap \mathcal{B}} f_1(P_{G_1}) + f_2(P_{G_2}) \tag{8}$$

and its convexified counterpart

$$\min_{(P_{G_1}, P_{G_2}) \in \mathcal{P}_s^{(c)} \cap \mathcal{B}} f_1(P_{G_1}) + f_2(P_{G_2}) \tag{9}$$

still share the same optimal solution. To address this problem, four positions are considered in Figure 2(b) for the box $\mathcal{B}$:

- $\mathcal{B}$= *Box 1:* In this case, Optimizations (8) and (9) are both infeasible.
- $\mathcal{B}$= *Box 2:* In this case, Optimizations (8) and (9) share the same optimal solution, which lies on the lower boundary of $\mathcal{P}_s \cap \mathcal{B}$.
- $\mathcal{B}$= *Box 3:* In this case, Optimizations (8) and (9) share the same optimal solution, which corresponds to the lower left corner of the box $\mathcal{B}$.
- $\mathcal{B}$= *Box 4:* In this case, Optimizations (8) is infeasible while Optimization (9) has a solution.

It can be inferred from the above observations that as long as the nonconvex optimization (8) is feasible, it can be solved via the convex problem (9). Hence, the OPF problem can be solved efficiently even with production limits, provided that an actual/virtual phase shifter is incorporated in the network.

III. Main Results

A. Optimal Power Flow

Consider a power network with the set of buses $\mathcal{N} := \{1, 2, ..., n\}$, the set of generator buses $\mathcal{O} \subseteq \mathcal{N}$ and the set of flow lines $\mathcal{L} \subseteq \mathcal{N} \times \mathcal{N}$, where:

- A known active-power load with the value P_{D_k} is connected to each bus $k \in \mathcal{N}$.
- A generator with an unknown real output P_{G_k} is connected to each bus $k \in \mathcal{O}$.
- Each line $(l, m) \in \mathcal{L}$ of the network is modeled as a passive device with an admittance y_{lm} (the network can be modeled as a general admittance matrix).

The goal is to design the unknown real outputs of all generators in such a way that the load constraints are satisfied (see Remark 1 for the inclusion of reactive power in the problem). To formulate this OPF problem, define:

- V_k: Unknown complex voltage at bus $k \in \mathcal{N}$.
- P_{lm}: Unknown active power transferred from bus $l \in \mathcal{N}$ to the rest of the network through the line $(l, m) \in \mathcal{L}$.
- $f_k(P_{G_k})$: Known convex function representing the generation cost for generator $k \in \mathcal{O}$.

Define $\mathbf{V}$, $\mathbf{P}_G$ and $\mathbf{P}_D$ as the vectors $\{V_k\}_{k\in\mathcal{N}}$, $\{P_{G_k}\}_{k\in\mathcal{O}}$ and $\{P_{D_k}\}_{k\in\mathcal{N}}$, respectively. Given the known vector $\mathbf{P}_D$, OPF minimizes the total generation cost $\sum_{k\in\mathcal{O}} f_k(P_{G_k})$ over the unknown parameters $\mathbf{V}$ and $\mathbf{P}_G$ subject to the power balance equations at all buses and some network constraints. To simplify the presentation, with no loss of generality assume that $\mathcal{O} = \mathcal{N}$. The mathematical formulation of OPF is given in (10), where:

- (10a) is the power balance equation accounting for the conservation of energy at bus k.
- (10b) and (10c) restrict the active power and voltage magnitude at bus k, for the given limits $P_k^{\min}, P_k^{\max}, V_k^{\min}, V_k^{\max}$.
- (10d) limits the flow over the line (l, m), for the given upper bounds $P_{lm}^{\max} = P_{ml}^{\max}$.
- $\mathcal{N}(k)$ denotes the set of the neighboring nodes of bus k.

B. SOCP Relaxation of OPF

The OPF problem (10) is non-convex due to the nonlinear terms $|V_l|^2$'s and $V_l V_m^*$'s. Since this problem is NP-hard in the worst case, the paper [7] suggests solving a convex relaxation of OPF. To describe this relaxation, consider a line $(l, m) \in \mathcal{L}$. Notice that the nonlinear terms $|V_l|^2$, $|V_m|^2$ and $V_l V_m^*$ can be replaced by three linear terms $W_{ll} \in \mathbb{R}$, $W_{mm} \in \mathbb{R}$ and $W_{lm} \in \mathbb{C}$, respectively, where

$$\begin{bmatrix} W_{ll} & W_{lm} \\ W_{ml} & W_{mm} \end{bmatrix} = \begin{bmatrix} V_l \\ V_m \end{bmatrix} \begin{bmatrix} V_l^* & V_m^* \end{bmatrix} \tag{12}$$

Since the matrix in the left side of the above equation is the product of two vectors, it must be positive semidefinite and rank 1. The above equation can be relaxed as:

$$\begin{bmatrix} W_{ll} & W_{lm} \\ W_{ml} & W_{mm} \end{bmatrix} \succeq 0 \tag{13}$$

If the above matrix turns out be rank-1, then there exist two complex numbers V_l and V_m satisfying the equation (12). The relaxation from (12) to (13) is the key to the convexification of the OPF problem. Applying the above argument to OPF leads to the SOCP relaxation (11), which is obtained via three steps: (i) changing the vector variable $\mathbf{V} \in \mathbb{C}^n$ to a matrix variable $\mathbf{W} \in \mathbb{H}^n$, (ii) imposing a positive semidefinite constraint on

OPF Problem	SOCP Relaxation of OPF
Minimize $\sum_{k\in\mathcal{N}} f_k(P_{G_k})$ over $\mathbf{P}_G \in \mathbb{R}^n$ and $\mathbf{V} \in \mathbb{C}^n$ Subject to: $P_{G_k} - P_{D_k} = \sum_{l\in\mathcal{N}(k)} \text{Re}\{V_k(V_k^* - V_l^*)y_{kl}^*\}, \quad k \in \mathcal{N} \quad$ (10a) $P_k^{\min} \leq P_{G_k} \leq P_k^{\max}, \quad k \in \mathcal{N} \quad$ (10b) $V_k^{\min} \leq \|V_k\| \leq V_k^{\max}, \quad k \in \mathcal{N} \quad$ (10c) $\|P_{lm}\| \triangleq \|\text{Re}\{V_l(V_l^* - V_m^*)y_{lm}^*\}\| \leq P_{lm}^{\max}, \quad (l,m) \in \mathcal{L} \quad$ (10d)	Minimize $\sum_{k\in\mathcal{N}} f_k(P_{G_k})$ over $\mathbf{P}_G \in \mathbb{R}^n$, and $\mathbf{W} \in \mathbb{H}^n$ Subject to: $P_{G_k} - P_{D_k} = \sum_{l\in\mathcal{N}(k)} \text{Re}\{(W_{kk} - W_{kl})y_{kl}^*\}, \quad k \in \mathcal{N} \quad$ (11a) $P_k^{\min} \leq P_{G_k} \leq P_k^{\max}, \quad k \in \mathcal{N} \quad$ (11b) $(V_k^{\min})^2 \leq W_{kk} \leq (V_k^{\max})^2, \quad k \in \mathcal{N} \quad$ (11c) $\text{Re}\{(W_{ll} - W_{lm})y_{lm}^*\} \leq P_{lm}^{\max}, \quad (l,m) \in \mathcal{L} \quad$ (11d) $\begin{bmatrix} W_{ll} & W_{lm} \\ W_{ml} & W_{mm} \end{bmatrix} \succeq 0, \quad (l,m) \in \mathcal{L} \quad$ (11e)

every 2×2 submatrix of $\mathbf{W}$ corresponding to an existing line of the power network, and (iii) rewriting the quadratic terms $|V_l|^2$'s and $V_l V_m^*$ in terms of the entries of $\mathbf{W}$. The SOCP relaxation (11) is said to be *exact* if it has a solution $\mathbf{W}^{\text{opt}}$ with the property

$$\begin{bmatrix} W_{ll}^{\text{opt}} & W_{lm}^{\text{opt}} \\ W_{ml}^{\text{opt}} & W_{mm}^{\text{opt}} \end{bmatrix} = \text{rank } 1, \qquad \forall (l,m) \in \mathcal{L} \tag{14}$$

In order for the OPF problem and the SOCP relaxation to have the same solution, the relaxation must be exact. However, this is only a sufficient condition.

Lemma 1: The OPF problem (10) and the SOCP relaxation (11) have the same optimal objective value if and only if the SOCP relaxation has a rank-1 solution $\mathbf{W}^{\text{opt}}$.

Proof: See [7] for the proof. ∎

It can be inferred from Lemma 1 that the exactness of the SOCP relaxation may not guarantee the equivalence between OPF and its convex relaxation. To better understand this fact, consider a closed path in the network, denoted as $j_1 \to j_2 \to \ldots \to j_k \to j_1$. The equation $\theta_{j_1 j_2} + \theta_{j_2 j_3} + \cdots + \theta_{j_k j_1} = 0$ is implicitly enforced in the OPF problem, but its equivalent constraint

$$\measuredangle W_{j_1 j_2} + \measuredangle W_{j_2 j_3} + \cdots + \measuredangle W_{j_k j_1} = 0 \tag{15}$$

has not been included in the SOCP relaxation due to its non-convexity. The absence of this nonlinear constraint in the SOCP relaxation introduces a gap between OPF and its relaxation even in the case when the relaxation is exact.

We choose an arbitrary spanning tree of the power network and add a controllable phase shifter to each line of the network not belonging to this tree. Consider the OPF problem (10) under the assumption that the phase shifts of these devices can all be optimized in addition to the two original parameters $\mathbf{P}_G \in \mathbb{R}^n$ and $\mathbf{V} \in \mathbb{C}^n$. We refer to this problem as *OPF with variable phase shifters*. As shown in [7], the solution of this problem is independent of the non-unique choice of the spanning tree.

Lemma 2: The OPF problem with variable phase shifters and the SOCP relaxation (11) have the same optimal objective value if and only if the SOCP relaxation is exact.

Proof: See [7] for the proof. ∎

Lemma 2 states that the SOCP relaxation (11) is a convex relaxation of OPF with variable phase shifters as opposed to the original OPF problem (10). As discussed in Section II, there is a good connection between the OPF problem and OPF with variable phase shifters. The rest of this paper aims to study under what conditions OPF with variable phase shifters can be solved efficiently. Alternatively, the objective is to investigate the exactness of the SOCP relaxation.

C. Exactness of SOCP Relaxation

Definition 2: Given an arbitrary line $(l,m) \in \mathcal{L}$ and two numbers $U_l, U_m \in \mathbb{R}_+$, define $\mathcal{P}_{lm}(U_l, U_m)$ as the set of all pairs (P_{lm}, P_{ml}) for which there exists an angle $\theta_{lm} \in [-180°, 180°]$ such that

$$P_{lm} = \text{Re}\left\{(U_l^2 - U_l U_m e^{\theta_{lm} \mathrm{i}})y_{lm}^*\right\} \tag{16a}$$
$$P_{ml} = \text{Re}\left\{(U_m^2 - U_l U_m e^{-\theta_{lm} \mathrm{i}})y_{lm}^*\right\} \tag{16b}$$
$$P_{lm}, P_{ml} \leq P_{lm}^{\max} \tag{16c}$$

Note that $\mathcal{P}_{lm}(U_l, U_m) = \mathcal{P}_{ml}(U_l, U_m)$ represents the feasible set for the flow vector (P_{lm}, P_{ml}) under the line flow constraints $P_{lm}, P_{ml} \leq P_{lm}^{\max}$ in the case where the voltage magnitudes are fixed according to the equation $(|V_l|, |V_m|) = (U_l, U_m)$.

Assumption 1: The set $\mathcal{P}_{lm}(U_l, U_m)$ forms a monotonically decreasing curve in $\mathbb{R}^2$, for every line $(l,m) \in \mathcal{L}$ and the pair $(U_l, U_m) \in [V_l^{\min}, V_l^{\max}] \times [V_m^{\min}, V_m^{\max}]$.

The set $\mathcal{P}_{lm}(U_l, U_m)$ is essentially the boundary of an ellipse in the absence of the flow constraints $P_{lm}, P_{ml} \leq P_{lm}^{\max}$ (i.e, in the case where $P_{lm}^{\max} = +\infty$). Hence, Assumption 1 requires that the intersection of the ellipse with the flow constraints gives rise to the Pareto front of the ellipse. To elaborate on this assumption, consider the fixed-voltage-magnitude case $V_l^{\min} = V_l^{\max} = V_m^{\min} = V_m^{\max}$. Assumption 1 is equivalent to the condition $\theta_{lm}^{\max} \leq \measuredangle y_{lm}^*$, where $\theta_{lm}^{\max}$ is an angle satisfying the equation

$$P_{lm}^{\max} = \text{Re}\left\{((V_l^{\max})^2 - (V_l^{\max})^2 e^{\theta_{lm}^{\max} \mathrm{i}})y_{lm}^*\right\} \tag{17}$$

Observe that $\theta_{lm}^{\max}$ is equal to $45°$, $78.6°$ and $90°$ for the inductance to resistance ratio of 1, 5 and ∞, respectively.

Assumption 2: For every $[U_1, U_2, ..., U_n] \in [V_1^{\min}, V_1^{\max}] \times \cdots \times [V_n^{\min}, V_n^{\max}]$, the OPF problem under the (additional) fixed-voltage-magnitude constraints $|V_k| = U_k$, $k = 1, ..., n$, is feasible.

Assumptions 1 and 2 are practical due to two reasons: (i) the angle difference across each line is barely more than 30° in practice due to the thermal and stability limits, and (ii) the voltage limits $V_k^{\min}$ and $V_k^{\max}$ are normally enforced to be less than 5-10% percent away from the nominal value of the voltage magnitude at bus $k \in \mathcal{N}$.

Theorem 1: Under Assumptions 1 and 2, the SOCP relaxation is exact.

Sketch of Proof: Let $(\mathbf{P}_G^{\text{opt}}, \mathbf{W}^{\text{opt}})$ denote an arbitrary solution of the SOCP relaxation. Consider the SOCP relaxation under the additional constraints

$$W_{kk} = W_{kk}^{\text{opt}}, \qquad k = 1, 2, ..., n \tag{18}$$

We refer to this problem as fixed-voltage-magnitude SOCP relaxation. First, it is obvious that $(\mathbf{P}_G^{\text{opt}}, \mathbf{W}^{\text{opt}})$ is a solution of this new problem. Second, the fixed-voltage-magnitude SOCP relaxation is indeed the SOCP relaxation of a fixed-voltage-magnitude OPF with variable phase shifters, which is defined as the minimization of the cost $\sum_{k\in\mathcal{N}} f_k(P_{G_k})$ subject to:

$$P_{G_k} - P_{D_k} = \sum_{l\in\mathcal{N}(k)} P_{kl}, \qquad k \in \mathcal{N} \tag{19a}$$

$$P_k^{\min} \leq P_{G_k} \leq P_k^{\max}, \qquad k \in \mathcal{N} \tag{19b}$$

$$P_{lm} \leq P_{lm}^{\max}, \qquad (l,m) \in \mathcal{L} \tag{19c}$$

$$(P_{lm}, P_{ml}) \in \mathcal{P}\left(\sqrt{W_{ll}^{\text{opt}}}, \sqrt{W_{mm}^{\text{opt}}}\right), \qquad (l,m) \in \mathcal{L} \tag{19d}$$

The variables of the above optimization are P_{G_k} and P_{lm} for every $k \in \mathcal{N}$ and $(l,m) \in \mathcal{L}$. By Assumptions 1 and 2, this optimization is feasible and moreover $P\left(\sqrt{W_{ll}^{\text{opt}}}, \sqrt{W_{mm}^{\text{opt}}}\right)$ is a monotonically decreasing curve in $\mathbb{R}^2$. Since this optimization is a generalized network flow problem, it follows from [18] that its corresponding SOCP relaxation is exact. ■

Theorem 2: Let $(\mathbf{P}_G^{\text{opt}}, \mathbf{W}^{\text{opt}})$ denote an arbitrary solution of the SOCP relaxation. Under Assumptions 1 and 2, the point $(\mathbf{P}_G^{\text{opt}}, \mathbf{V}^{\text{opt}})$ is an optimal solution of the OPF problem with variable phase shifters, for some vector $\mathbf{V}^{\text{opt}}$ with the property

$$|V_k^{\text{opt}}| = \sqrt{W_{kk}^{\text{opt}}}, \qquad k = 1, 2, ..., n \tag{20}$$

Proof: The proof is omitted due to space restrictions. ■

Theorem 2 states that the SOCP relaxation finds the optimal values of the objective function, generator outputs and bus voltage magnitudes for the OPF problem with variable phase shifters. Nonetheless, it may fail to find the correct phases. More precisely, if the problem has a unique solution, then $\measuredangle W_{lm}^{\text{opt}}$ plays the role of the angle difference θ_{lm}^{opt} for every $(l,m) \in \mathcal{L}$. If the problem has multiple solutions, a separate optimization may need to be solved in order to find a feasible set of phases.

Remark 1: Given a line $(l,m) \in \mathcal{L}$, the reactive power Q_{lm} over the line (l,m) can be written as a linear function of P_{lm}, P_{ml}, W_{ll} and W_{mm}. Hence, the inclusion of reactive-power constraints into the OPF problem (10) is equivalent to adding a set of linear constraints to the SOCP relaxation. In line with [11], it can be shown that the results of this paper are all valid in presence of the constraints $Q_k \leq Q_k^{\max}$, $k \in \mathcal{N}$, provided the inductance-to-resistance ratio of each line is greater than 1.

IV. Conclusions

This paper tackles the non-convexity of the optimal power flow (OPF) problem. We have recently shown that this problem can be convexified for mesh networks after two approximations: (i) relaxing the angle constraints by incorporating virtual phase shifters into the network, and (ii) relaxing the power balance equations to convex inequalities. In this work, we first explore the implications of Approximation (i) and its effect on the feasible set of the OPF problem. We then prove that Approximation (ii) is not required as long as some mild assumptions are satisfied. The main conclusion of this paper is that OPF can be solved efficiently after relaxing only some angle constraints. Although the main focus of the paper is placed on OPF, the results can be applied to emerging energy-related optimizations related to storage and renewable, distributed generation and demand response for smart grids.

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