Promises of Conic Relaxations in Optimal Transmission Switching of Power Systems

Salar Fattahi, Javad Lavaei, and Alper Atamtürk

Industrial Engineering and Operations Research, University of California, Berkeley

Abstract— **This paper studies the problem of optimal transmission switching (OTS) for power systems. The goal is to identify a topology of the power grid that minimizes the cost of the system operation while satisfying the physical and operational constraints. Most of the existing methods are based on converting the OTS problem into a mixed-integer linear program (MILP), and then iteratively solving a series of convex problems. The performance of these methods depends heavily on the strength of the MILP formulation. In this paper, we first show that finding the strongest variable upper bounds to be used in the MILP formulation of the OTS problem based on the big-M method is NP-hard. Then, we propose a convex conic relaxation of the big-M MILP formulation based on a semidefinite program (SDP). Strong valid inequalities using the reformulation-linearization technique (RLT) are proposed to strengthen the SDP relaxation by multiplying different linear constraints and then convexifying them in a lifted space. We extensively evaluate the performance of the proposed method on IEEE benchmarks systems.**

I. Introduction

In power systems, transmission lines have traditionally been considered uncontrollable devices in the infrastructure, except in the case of an outage or maintenance. However, due to the pressing needs to boost the sustainability, reliability and efficiency of power systems, power operators tend to leverage the flexibility in the topology of the grid and co-optimize the topology to improve the dispatch. For example, removing a few lines from the network could increase the power transfer. This would, in theory, stand in contrast with a conventionally accepted fact in the area of network flows; removing edges from the network would deteriorate the performance or increase the cost of the optimal flows. This seemingly inconsistent phenomenon stems from the underlying physical laws, which delineates a fundamental difference between power systems and classical network flow models.

The concept of optimally switching the lines of a transmission network was introduced by O'Neill *et al.* [1]. Later on, it was shown in a series of papers that the incorporation of controllable transmission switches in a grid could positively impact the voltage profiles [2], relieve network congestions [3], reduce operational costs [4], improve the reliability of the system [5] and enhance the economic efficiency of power markets [6]. However, the identification of an optimal topology, namely optimal transmission switching (OTS), is a nonconvex combinatorial optimization problem that is known to be NP-hard [7]. Therefore, brute-force search algorithms for finding the optimal topology of a power system are often inefficient due to the curse of dimensionality of real-world problems. Most of the existing methods are either based on heuristics or iterative relaxations of the problem. These methods include, but are not restricted to, Benders decomposition [4], [8], branch-and-bound and cutting-plane methods [9], [10], genetic algorithms [2], and line ranking [11], [12].

In this work, the focus is on the design of an efficient convex formulation of the OTS problem. The power flow equations are modeled using the well-known DC approximation, which is the backbone of the operation of power systems. The OTS problem consists of disjunctive constraints that are nonlinear in the original formulation. The aim of this work is twofold. First, we consider an MILP formulation of the OTS problem based on the big-M method [13], which is referred to as the *linearized OTS* problem in this paper. A question arises as to how difficult it is to find the optimal parameters of the big-M method in the linearized OTS problem. An optimal choice of these parameters is important for two reasons: 1) they would result in stronger convex relaxations of the problem, and hence, fewer iterations in branch-and-bound or cutting-plane methods, and 2) a conservative choice of these parameters would cause numerical and convergence issues.

Hedman *et al.* [5] point out that finding the optimal parameters of the linearized OTS problem may be cumbersome in general, and so they impose restrictive constraints on the absolute angles of voltages at different buses to overcome this issue at the expense of a possible shrinkage of the feasible region. In this work, we formally prove the difficulty of finding the optimal parameters of the big-M MILP formulation of the OTS problem by showing that it amounts to an NP-hard problem. Moreover, it is proved that there does not exist any polynomial-time algorithm to approximate these parameters within a constant factor. This new result adds a new dimension to the difficulty of the OTS problem; not only is solving the OTS problem as a mixed-integer nonlinear program difficult, but finding a good big-M MILP reformulation of this problem is NP-hard as well. Despite this negative result, we prove that one can find non-

Email: fattahi@berkeley.edu, lavaei@berkeley.edu, and atamturk@berkeley.edu

This work was supported by the ONR YIP Award, DARPA YFA Award, AFOSR YIP Award, NSF CAREER Award, and NSF EPCN Award. A. Atamtürk was supported, in part, by grant FA9550-10-1-0168 from the Office of the Assistant Secretary of Defense for Research and Engineering.

conservative values for the parameters of the linearized OTS problem for those power systems that have a fixed inflexible connected subgraph.

To develop an efficient technique, we adopt a lift-and-project approach to solve the OTS problem using an SDP relaxation, combined with valid inequalities that are based on the Sherali-Adams reformulation-linearization technique (RLT) [14]. We introduce a set of valid inequalities to attain a stronger relaxation of the feasible region of the OTS problem. In order to obtain the above-mentioned inequalities, we use RLT to generate valid nonconvex quadratic inequalities and then relax them to valid convex inequalities in a lifted space. The proposed convex program is called a strengthened SDP. Recently, we have shown that a similar relaxation can be utilized to find the optimal solution of the unit commitment (UC) problem in power systems [15]. In this work, we will demonstrate through simulations that the strengthened SDP problem is able to find globally optimal solutions of the OTS problem for different configurations of benchmark systems. One major benefit of the proposed method, compared to other state-of-the-art methods, is that it can be used in the convex hull pricing technique in energy markets due to its convexification property [16], [17]. Another advantage of the developed results is that, unlike the branch-and-bound and cutting-plane methods, they can be readily generalized to AC models of power systems by combining the proposed method for handling the discrete variables and the convexification technique introduced by Lavaei *et al.* [18].

Notations: The notation $(\cdot)^\top$ denotes the transpose operator. Vectors and matrices are represented by bold lower case and bold upper case letters, respectively. The subscript ij denotes either the $(i,j)^{\text{th}}$ entry of a matrix $\mathbf{W}$ or the argument corresponding to the line (i,j). The notation w_i represents the i^{th} entry of a vector $\mathbf{w}$. The symbols $\mathbb{R}$ and $\mathbb{S}^n$ represent the sets of real numbers and $n\times n$ real symmetric matrices, respectively. The relation $\mathbf{u}\geq\mathbf{v}$ indicates that each element of vector $\mathbf{v}$ is less than or equal to the corresponding elements of vector $\mathbf{u}$, and the same notation is used for matrices. The notation $\mathbf{W}\succeq 0$ for a symmetric matrix $\mathbf{W}$ states that $\mathbf{W}$ is positive-semidefinite.

II. Problem Formulation

We consider a power network with n_b buses, n_g generators, and n_l lines. This network can be visualized by a general graph, denoted by $G(\mathcal{B},\mathcal{L})$, where $\mathcal{B}$ is the set of buses indexed from 1 to n_b and $\mathcal{L}$ is the set of lines that contains a pair (i,j) if there is a line between buses i and j. In order to streamline the presentation, we assume an arbitrary direction for each line of the power system. Define $\mathcal{N}_l^+(i)$ as the set of endpoints of the outgoing lines at bus i. In particular, $\mathcal{N}_l^+(i)$ is defined as $\{j\in\mathcal{B}|(i,j)\in\mathcal{L}\}$. Similarly, define $\mathcal{N}^-(i)\triangleq\{j\in\mathcal{B}|(j,i)\in\mathcal{L}\}$ as the set of endpoints of the incoming lines at bus i. Denote $\mathcal{G}=\{1,2,...,n_g\}$ as the set of all generators in the system. Furthermore, let $\mathcal{N}_g(i)$ collect the indices of all generators that are connected to bus i. Notice that $\mathcal{N}_g(i)$ may be empty for a bus i. The variable p_i corresponds to the production of generator $i\in\mathcal{G}$ and the variable θ_i is the voltage angle at bus $i\in\mathcal{B}$. For every $(i,j)\in\mathcal{L}$, the variable f_{ij} denotes the flow from bus i to bus j. Consider the set of lines $\mathcal{S}\subset\mathcal{L}$ that are equipped with on/off switches and define the decision variable x_{ij} for every $(i,j)\in\mathcal{S}$ as the status of the line (i,j). Let n_s denote the cardinality of this set. We refer to the lines belonging to $\mathcal{S}$ as *flexible* lines and the remaining lines as *inflexible* lines. Notice that the decision variables p_i, θ_i, and f_{ij} are continuous, whereas x_{ij} is binary. For simplicity of notations, define the variable vectors

$$\mathbf{p}\triangleq[p_1,p_2,...,p_{n_g}]^\top, \tag{1}$$
$$\Theta\triangleq[\theta_1,\theta_2,...,\theta_{n_b}]^\top, \tag{2}$$
$$\mathbf{f}\triangleq[f_{i_1j_1},f_{i_2j_2},...,f_{i_{n_l}j_{n_l}}]^\top, \tag{3}$$
$$\mathbf{x}\triangleq[x_{i'_1j'_1},x_{i'_2j'_2},...,x_{i'_{n_s}j'_{n_s}}]^\top, \tag{4}$$

where an arbitrary ordering is considered for the lines in $\mathcal{L}$. The objective function of the OTS problem is defined as $\sum_{i=1}^{n_b}g_i(p_i)$, where $g_i(p_i)$ takes the quadratic form

$$g_i(p_i)=a_i\times p_i^2+b_i\times p_i. \tag{5}$$

for some constants $a_i,b_i\geq 0$. The above objective function is the sum of the production costs of all generators. Every operating power system needs to satisfy some operational constraints. These operational constraints arise from physical and security limitations. As for the physical limitations, there are constraints on the unit and line capacities. Furthermore, the power system must satisfy the power balance equations. On the security side, there may be a cardinality constraint on the maximum number of flexible lines that can be switched off in order to ensure the reliability of the system during its normal operation. Let the vector $\mathbf{d}=[d_1,d_2,....,d_{n_b}]^\top$ collect the set of demands at different buses. Moreover, define $p_i^{\min}$ and $p_i^{\max}$ as the lower and upper bounds on the production level of generator i, and $f_{ij}^{\max}$ as the capacity of line $(i,j)\in\mathcal{L}$. Each line $(i,j)\in\mathcal{L}$ is associated with a susceptance that is represented by B_{ij}.

Based on these definitions, the OTS problem can be formulated as

$$\underset{\substack{\{f_{ij}\}_{(i,j)\in\mathcal{L}}\\ \{x_{ij}\}_{(i,j)\in\mathcal{S}}\\ \{p_i,\theta_i\}_{i\in\mathcal{G}}}}{\text{minimize}}\quad\sum_{i\in\mathcal{G}}g_i(p_i) \tag{6a}$$

$$\begin{aligned}
\text{s.t.}\quad & x_{ij}\in\{0,1\}, && \forall(i,j)\in\mathcal{S} && \text{(6b)}\\
& p_k^{\min}\leq p_k\leq p_k^{\max}, && \forall k\in\mathcal{G} && \text{(6c)}\\
& -f_{ij}^{\max}x_{ij}\leq f_{ij}\leq f_{ij}^{\max}x_{ij}, && \forall(i,j)\in\mathcal{S} && \text{(6d)}\\
& -f_{ij}^{\max}\leq f_{ij}\leq f_{ij}^{\max}, && \forall(i,j)\in\mathcal{L}\backslash\mathcal{S} && \text{(6e)}\\
& B_{ij}(\theta_i-\theta_j)x_{ij}=f_{ij}, && \forall(i,j)\in\mathcal{S} && \text{(6f)}\\
& B_{ij}(\theta_i-\theta_j)=f_{ij}, && \forall(i,j)\in\mathcal{L}\backslash\mathcal{S} && \text{(6g)}\\
& \sum_{k\in\mathcal{N}_g(i)}p_k-d_i=\sum_{j\in\mathcal{N}_l^+(i)}f_{ij}-\sum_{j\in\mathcal{N}_l^-(i)}f_{ji}, && \forall i\in\mathcal{B} && \text{(6h)}\\
& \sum_{(i,j)\in\mathcal{S}}x_{ij}\geq|\mathcal{L}|-r, && && \text{(6i)}
\end{aligned}$$

where

- (6b) states that the status of each line must be binary;
- (6c) imposes lower and upper bounds on the generation of generating units;
- (6d) and (6e) state that the flow over a flexible or inflexible line must be within the line capacities when its switch is on, and it should be zero otherwise;
- (6f) and (6g) relate the flow over each line to the voltage angles of the two endpoints of the line if it is in service, and it sets the flow to zero otherwise;
- (6h) requires that the power balance equation be satisfied at every bus;
- (6i) enforces at most r number of flexible lines to be switched off during the operation.

Due to space restrictions, we consider only one time slot of the system operation. However, the techniques to be developed in this paper could be used for the OTS problem over multiple time slots with coupling constraints, such as ramping limits on the productions of the generators. As another generalization, one can consider a combined unit commitment and optimal transmission switching problem, as formulated in Hedman *et al.* [4] (the commitment parameters could be handled using the method explained in Fattahi *et al.* [15]). Henceforth, the term "optimal solution" refers to a globally optimal solution rather than a local solution.

III. Linearization of OTS Problem

The aforementioned formulation of the OTS problem belongs to the class of mixed-integer nonlinear programs. The nonlinearity of this optimization problem is, in part, caused by the multiplication of the binary variable x_{ij} and the continuous variables θ_i and θ_j in (6f). However, since this nonlinear constraint has a disjunctive nature, one can use the big-M reformulation technique to formulate it in a linear way. In particular, we can re-write (6f) for each flexible line (i,j) in the form

$$B_{ij}(\theta_i-\theta_j)-M_{ij}(1-x_{ij}) \leq f_{ij} \leq B_{ij}(\theta_i-\theta_j)+M_{ij}(1-x_{ij}) \tag{7}$$

for a *large enough* M_{ij}, which we refer to as a *linearized OTS formulation* (note that the binary variables still act as nonlinear equations). The above inequalities imply that if x_{ij} is equal to 0, then (7) (and hence (6f)) is redundant as it is dominated by (6d). On the other hand, if x_{ij} is equal to 1, the line is in service and it needs to satisfy the physical constraint $f_{ij} = B_{ij}(\theta_i - \theta_j)$. The term "large enough" for M_{ij} is ambiguous, and indeed the design of an effective M_{ij} is a challenging task that will be studied below.

Definition 1: For every $(i,j) \in \mathcal{S}$, it is said that M_{ij} is feasible for the OTS problem if it preserves the equivalence between (7) and (6f) in the OTS problem. The smallest feasible M_{ij} is denoted by M_{ij}^*.

Based on the structure of the problem, one can pick extremely conservative values for M_{ij} to make (7) equivalent to (6f). For instance, it can be easily verified that, for every flexible line (i,j), the number $\sum_{(r,k)\in\mathcal{L}} f_{rk}^{\max}/B_{rk}$ is a feasible choice for M_{ij}. The question of interest is the following: Can we devise an algorithm that efficiently finds the smallest feasible M_{ij} for every flexible line (i,j)? The reason behind this question is twofold:

1. Most of the methods for solving MILP problems, such as cutting-plane and branch-and-bound algorithms, are based on iterative linear relaxations of the constraints. Therefore, while a sufficiently large value for M_{ij} does not change the feasible region of the OTS problem after replacing (6f) with (7), it may have a significant impact on the feasible region of the problem after a convex relaxation. In other words, small values for M_{ij} can lead to tighter relaxations.
2. Large values for M_{ij} may cause numerical issues for the types of convex relaxations proposed in this work.

In what follows, it will be shown that one cannot devise an algorithm that efficiently finds the smallest feasible M_{ij} because it amounts to an NP-hard problem. Furthermore, we will prove the impossibility of any constant approximation of M_{ij} in the linearized OTS problem.

Theorem 1: Consider an instance of the OTS problem and pick a flexible line (i,j). Unless $P = NP$, we have:

- (NP-hardness) There is no polynomial-time algorithm for finding the smallest feasible value of M_{ij}.
- (APX-completeness) There is no polynomial-time constant-factor approximation algorithm for finding the smallest feasible value of M_{ij}.

Proof: To prove the NP-hardness of the problem, we show that there exists a polynomial reduction from the *longest path* problem in unweighted graphs that is well-known to be NP-hard. The longest path problem is defined as follows: Given an undirected graph $G(\mathcal{V},\mathcal{E})$, where $\mathcal{V}$ and $\mathcal{E}$ stand for the set of vertices and edges, respectively, what is the longest simple path between the given vertices i and j in $\mathcal{V}$? Let the length of the longest path be denoted by P^{opt}. We construct an instance of the OTS problem in the following way: Consider $|\mathcal{V}|$ buses and, for every $(r,l) \in \mathcal{E}$, connect buses r and l through a line with an arbitrary direction that is equipped with a switch (note that $\mathcal{S} = \mathcal{E}$ in this case). For each line $(r,l) \in \mathcal{E}$, we set its susceptance and flow capacity equal to 1. For every bus $s \notin \{i,j\}$ in the system, we associate a load with demand d_s and a generator with the lower and upper bounds $p_s^{\min}$ and $p_s^{\max}$, where $d_s = p_s^{\min} = p_s^{\max} = 1$. Connect a generator with $p_i^{\min} = 0$ and $p_i^{\max} = 1$ to bus i. We pick a number $\epsilon \in (0,1)$ and connect a load with the demand $d_j = 1+\epsilon$ and a generator with $p_j^{\min} = 0$ and $p_j^{\max} = 1$ to bus j. Finally, set $r = |\mathcal{V}|$.

Note that the designed instance is indeed feasible if and only if there is a simple path between nodes i and j in G. Furthermore, the size of the constructed instance of the OTS problem is polynomial in the size of the instance of longest path problem. Denote the feasible region of the designed instance of the OTS problem as $\mathcal{F}$. To be able to substitute (6f) with (7) for some line $(i,j) \in \mathcal{S}$ without affecting $\mathcal{F}$, the number M_{ij}^* has to be equal to $\max_{\mathcal{F}}\{|\theta_i-\theta_j|\}$. Without loss of generality, we drop the absolute value in the rest of the proof. According to the defined characteristics of the loads and the generators in the system, for any feasible solution of the OTS problem, there should be at least one simple path from bus i to j consisting of the lines that are switched on. Therefore, for every $(\mathbf{f}^*, \Theta^*, \mathbf{x}^*, \mathbf{p}^*) \in \arg\max_{\mathcal{F}}\{\theta_i - \theta_j\}$, there exists a path $\mathcal{P}^* = \{(i, v_1), (v_1, v_2), ..., (v_k, j)\}$ with $x^*_{rk} = 1$ for all $(r, k) \in \mathcal{P}^*$. With no loss of generality, assume that the assigned directions of the lines respect the directions in $\mathcal{P}^*$. One can write

$$M^*_{ij} = \theta^*_i - \theta^*_j = \sum_{(r,k)\in\mathcal{P}^*} \theta^*_r - \theta^*_k = \sum_{(r,k)\in\mathcal{P}^*} f^*_{rk} \leq \sum_{(r,k)\in\mathcal{P}^*} f^{\max}_{rk} \leq P^{\text{opt}} \tag{8}$$

Now, it is desirable to construct a feasible solution $(\bar{\mathbf{f}}, \bar{\Theta}, \bar{\mathbf{x}}, \bar{\mathbf{p}}) \in \mathcal{F}$ that includes a simple path with lines that are switched on from buses i to j whose length is P^{opt}. To this end, consider the instance of the longest path problem and suppose that $\mathcal{P}^{\text{opt}} = \{(i, u_1), (u_1, u_2), ..., (u_l, j)\}$ defines the longest simple path in G between nodes i and j. For every flexible line (i, j) in the corresponding instance of the OTS problem, we set $\bar{x}_{ij}$ to 1 if this line belongs to $\mathcal{P}^{\text{opt}}$ and set to 0 otherwise. Moreover, we set $\bar{\theta}_j$ to 0 and define $\bar{\theta}_k = P^{\text{opt}}_{kj}$ for every bus k in $\mathcal{P}^{\text{opt}}$, where P^{opt}_{kj} is the length of the unique path between buses k and j in $\mathcal{P}^{\text{opt}}$. This yields that $\bar{f}_{rl}$ is equal to 1 for every line (r, l) in $\mathcal{P}^{\text{opt}}$. Furthermore, for every flexible line (t, s) that does not belong to $\mathcal{P}^{\text{opt}}$, we set $\bar{f}_{ts}$ to 0. To satisfy (6h), set $\bar{p}_s = 1$ for every $s \neq j$ and $\bar{p}_j = \epsilon$. So far, we have constructed a feasible solution $(\bar{\mathbf{f}}, \bar{\Theta}, \bar{\mathbf{x}}, \bar{\mathbf{p}})$ that satisfies the following property:

$$M^*_{ij} = \theta^*_i - \theta^*_j \geq \bar{\theta}_i - \bar{\theta}_j = \bar{\theta}_i = P^{\text{opt}} \tag{9}$$

Inequality (9) together with (8) establishes the proof of the NP-hardness of finding M^*_{ij}. The APX-completeness of the problem follows from the fact that, unless $P = NP$, there is no polynomial-time constant-factor approximation algorithm for determining the longest path between nodes i and j in G. ■

Remark 1: The decision version of the OTS problem is known to be NP-complete [10]. One may speculate that the NP-hardness of finding the best M_{ij} for every $(i, j) \in \mathcal{S}$ may follow directly from this seminal result. However, notice that there are some well-known problems with disjunctive constraints, such as the minimization of total tardiness on a single machine, which are known to be NP-hard [19] and yet there are efficient methods to find the optimal parameters of their big-M reformulation [20]. Theorem 1 shows that not only is finding the best M_{ij} for the OTS problem NP-hard, but one cannot hope for obtaining a strong big-M MILP formulation.

Although finding a good approximation of M^*_{ij} is a daunting task in general, one can find a non-conservative values for M_{ij} for certain structures in power systems. In what follows, it will be shown that those power systems whose inflexible lines form a connected subgraph are within this class of structures.

IV. OTS Problem with Fixed Connected Subgraph

Consider a power system where the set of inflexible lines contains a spanning tree of the power system. Under such circumstances, the objective is to show that a non-trivial upper bound on M_{ij} can be efficiently derived by solving a shortest path problem. Furthermore, it will be proved that this upper bound is tight in the sense that there exist some instances of the OTS problem with a fixed connected subgraph for which this upper bound is equal to M^*_{ij}. Before presenting this result, it is desirable to state that the OTS problem is hard to solve even under the assumption of a fixed connected subgraph.

Theorem 2: The OTS problem with a fixed connected subgraph is NP-hard.

Proof: The proof follows from a slightly modified argument made in the proof of Theorem 3.1 in [10]. ■

Denote the undirected weighted subgraph induced by the inflexible lines in the power system as $G_{\mathcal{I}}(\mathcal{B}_{\mathcal{I}}, \mathcal{L}_{\mathcal{I}}, \mathcal{W}_{\mathcal{I}})$, where $\mathcal{L}_{\mathcal{I}} = \mathcal{L}\backslash\mathcal{S}$ and $\mathcal{B}_{\mathcal{I}} = \mathcal{B}$. Furthermore, we set the weight of each edge $(i, j) \in \mathcal{L}_{\mathcal{I}}$ equal to $f^{\max}_{ij}/B_{ij}$ and let $\mathcal{W}_{\mathcal{I}}$ denote the set of all these weights.

Theorem 3: For every flexible line (i, j), the number M^*_{ij} is upper bounded by the multiplication of B_{ij} and the length of the shortest path between nodes i and j in $G_{\mathcal{I}}$. Moreover, this upper bound is tight in the worst case.

Proof: Using the notations defined in the proof of Theorem 1, we have $M^*_{ij} = B_{ij}(\theta^*_i - \theta^*_j)$, which is equal to

$$B_{ij} \sum_{(r,l)\in\mathcal{P}^*_{\mathcal{I}}} (\theta^*_r - \theta^*_l), \tag{10}$$

where $\mathcal{P}^*_{\mathcal{I}}$ is the shortest path from nodes i to j in $G_{\mathcal{I}}$. Notice that the above equation is due to the existence of inflexible lines in the network. The equation (10) yields that

$$M^*_{ij} \leq B_{ij} \sum_{(r,l)\in\mathcal{P}^*_{\mathcal{I}}} \frac{f^{\max}_{rl}}{B_{rl}}. \tag{11}$$

It can be observed that the sum in the right-hand-side of (11) is equal to the length of the shortest path between nodes i and j in $G_{\mathcal{I}}$. Furthermore, a similar feasible solution that is derived in the second part of the proof of Theorem 1 can be used to show the tightness of this upper bound. ■

Notice that the existence of a fixed connected subgraph in power systems is a practical assumption since power operators should guarantee the reliability of the system by ensuring the connectivity of the power network. Therefore, due to Theorem 3, we can design relatively small values for M_{ij}'s in order to strengthen its relaxation and to avoid numerical issues that may arise in the proposed method. In the remainder of the paper, we will focus on the OTS problem with a fixed connected subgraph. However, it is worthwhile to mention that the proposed convex model can be readily applied to the OTS problem without assuming the existence of such fixed connected subgraph, as long as a relatively good upper bound on M^*_{ij} is known *a priori* for every flexible line (i, j).

V. Conic Relaxation of OTS Problem

In what follows, different convex relaxations of the OTS problem will be introduced and analyzed.

A. SDP Relaxation

Consider the binary constraint (6b) on the status of each flexible line in the linearized OTS formulation. If we relax (6b) to $0 \leq x_{ij} \leq 1$ for every flexible line, the resulted formulation is called **quadratic programming (QP) relaxation of the OTS problem**. However, as will be shown later, this relaxation is almost never exact. Therefore, the objective of this section is to devise a strong convex model of the OTS problem that is exact for many practical instances. Define the variable vector

$$\mathbf{w} = [\mathbf{x}^\top, \mathbf{p}^\top, \Theta^\top, \mathbf{f}^\top]^\top. \tag{12}$$

All constraints of the linearized OTS problem, except for (6b), can be merged into a single linear vector inequality $\mathbf{A}\mathbf{w} \geq \mathbf{b}$, for some constant matrix $\mathbf{A}$ and some constant vector $\mathbf{b}$. Furthermore, (6b) can be written as $x_{ij}(x_{ij}-1) = 0$ for every flexible line (i, j). Therefore, the OTS problem can be reformulated as

$$\underset{\mathbf{w}\in\mathbb{R}^{n_s+n_g+n_b+n_l}}{\text{minimize}} \quad c(\mathbf{w}) \tag{13a}$$

$$\text{s.t.} \quad \mathbf{A}\mathbf{w} \geq \mathbf{b}, \tag{13b}$$

$$w_k(w_k - 1) = 0, \quad k = 1, \ldots, n_s \tag{13c}$$

where $c(\mathbf{w})$ is the quadratic cost function in terms of $\mathbf{w}$. To address the nonconvexity of (13c), define a new matrix variable $\mathbf{W}$ and set it equal to $\mathbf{w}\mathbf{w}^\top$. Using this new variable, the objective function and the constraints in (13) can be written linearly in terms of $\mathbf{w}$ and $\mathbf{W}$. Relaxing the only nonconvex constraint $\mathbf{W} = \mathbf{w}\mathbf{w}^\top$ to the conic constraint $\mathbf{W} \succeq \mathbf{w}\mathbf{w}^\top$ gives rise to the SDP relaxation of the OTS problem

$$\underset{\substack{\mathbf{w}\in\mathbb{R}^{n_s+n_g+n_b+n_l}\\ \mathbf{W}\in\mathbb{S}^{n_s+n_g+n_b+n_l}}}{\text{minimize}} \quad c_r(\mathbf{w}, \mathbf{W}) \tag{14a}$$

$$\text{s.t.} \quad \mathbf{A}\mathbf{w} \geq \mathbf{b}, \tag{14b}$$

$$W_{kk} - w_k = 0, \quad k = 1, \ldots, n_s \tag{14c}$$

$$\mathbf{W} \succeq \mathbf{w}\mathbf{w}^\top, \tag{14d}$$

where

$$c_r(\mathbf{w}, \mathbf{W}) = \sum_{i\in\mathcal{G}} a_i \times W_{n_s+i,n_s+i} + b_i \times w_{n_s+i}. \tag{15}$$

Remark 2: Note that (14) is indeed a relaxation of the OTS problem. This is due to the fact that if $\mathbf{w}$, defined in (12), is an optimal solution of the OTS problem, then $(\mathbf{w}, \mathbf{w}\mathbf{w}^T)$ is feasible for (14) and has the same objective value as the optimal cost of the OTS problem. Furthermore, the proposed SDP relaxation solves the OTS problem if and only if it has an optimal solution $(\mathbf{w}^*, \mathbf{W}^*)$ for which the matrix

$$\left[\begin{array}{c|c} 1 & {\mathbf{w}^*}^\top \\ \hline \mathbf{w}^* & \mathbf{W}^* \end{array}\right]$$

has rank equal to 1. From a different perspective, this relaxation is exact only if x_{ij}'s are all binary numbers at an optimal solution of (14).

The next theorem shows the inefficiency of the SDP relaxation for the OTS problem.

Theorem 4: For the linearized OTS problem, the following statements hold:

1. The optimal objective values of the QP and SDP relaxations are the same.
2. Suppose that the optimal solution of the QP relaxation is unique. Then, the SDP and QP relaxations are not exact if there exists a flexible line (i, j) whose flow in the optimal solution of the OTS problem is not equal to 0, $-f_{ij}^{\max}$ or $f_{ij}^{\max}$.

Proof: The proof of the first part follows from the same argument made in the proof of Theorem 1 in [15]. To prove the second part by contradiction, assume that $\mathbf{w}^* = [{\mathbf{x}^*}^\top, {\mathbf{p}^*}^\top, {\Theta^*}^\top, {\mathbf{f}^*}^\top]^\top$ is an optimal solution of the OTS problem and its QP relaxation. Suppose that there exists a flexible line (i, j) for which f_{ij}^* is not equal to 0, $-f_{ij}^{\max}$ or $f_{ij}^{\max}$. Without loss of generality, we only consider the case $f_{ij}^* > 0$. Define $\bar{\mathbf{x}}$ as $\bar{x}_{rl} = x_{rl}^*$ for every line $(r, l) \neq (i, j)$ and $\bar{x}_{ij} = x_{ij}^* + \epsilon$, where

$$\frac{f_{ij}^*}{f_{ij}^{\max}} - 1 \leq \epsilon < 0 \tag{16}$$

One can easily verify that $\mathbf{w}^* = [\bar{\mathbf{x}}^\top, {\mathbf{p}^*}^\top, {\Theta^*}^\top, {\mathbf{f}^*}^\top]^\top$ is also an optimal solution to the QP relaxation of the OTS problem. This contradicts with the uniqueness of the optimal solution of the QP relaxation. ■

Indeed, Theorem 4 suggests that the QP and SDP relaxations are almost always inexact for generic OTS problems.

B. Valid Inequalities

In this subsection, we will design valid inequalities to strengthen the SDP relaxation of the OTS problem. Let $\mathcal{F}$ denote the feasible region of the OTS problem. An inequality is called *valid* if it is satisfied by all feasible points in $\mathcal{F}$. To design such valid inequalities, consider two linear inequalities in the OTS problem

$$\mathbf{n}^\top\mathbf{w} - b_1 \geq 0, \quad \mathbf{m}^\top\mathbf{w} - b_2 \geq 0 \tag{17}$$

for some fixed vectors $\mathbf{n}$ and $\mathbf{m}$ and some scalars b_1 and b_2. The quadratic inequality

$$\mathbf{n}^\top\mathbf{w}\mathbf{w}^\top\mathbf{m} - (\mathbf{m}^\top b_1 + \mathbf{n}^\top b_2)\mathbf{w} + b_1 b_2 \geq 0, \tag{18}$$

generated by the multiplication of the two inequalities in (17) is also valid for all points in $\mathcal{F}$. This quadratic and potentially nonconvex valid inequality can be relaxed to

$$\mathbf{n}^\top\mathbf{W}^\top\mathbf{m} - (\mathbf{m}^\top b_1 + \mathbf{n}^\top b_2)\mathbf{w} + b_1 b_2 \geq 0, \tag{19}$$

which is linear in terms of $(\mathbf{w}, \mathbf{W})$.

C. Strengthened SDP Relaxation

Based on the method delineated in subsection V-B, one can design the set of quadratic valid inequalities

$$(\mathbf{A}\mathbf{w} - \mathbf{b})(\mathbf{A}\mathbf{w} - \mathbf{b})^\top \geq 0 \tag{20}$$

for the OTS problem and then relax it to

$$\mathbf{A}\mathbf{W}\mathbf{A}^\top - \mathbf{b}\mathbf{w}^\top\mathbf{A}^\top - \mathbf{A}\mathbf{w}\mathbf{b}^\top + \mathbf{b}\mathbf{b}^\top \geq 0. \quad (21)$$

Replacing the nonconvex constraint (13c) with (14c) leads to the **Reformulation-Linearization Technique (RLT) relaxation of the OTS problem**. As delineated by Sherali *et al.* [14], the RLT relaxation outperforms the QP relaxation since the feasible region of the QP relaxation always includes that of the RLT relaxation. However, as will later be shown in simulations, both of these methods often fail to generate feasible solutions to the OTS problem.

To generate a better convex relaxation of the OTS problem, we incorporate (21) into the SDP relaxation of the OTS problem:

$$\begin{aligned} \underset{\substack{\mathbf{w}\in\mathbb{R}^{n_s+n_g+n_b+n_l} \\ \mathbf{W}\in\mathbb{S}^{n_s+n_g+n_b+n_l}}}{\text{minimize}} \quad & c_r(\mathbf{w},\mathbf{W}) & (22a)\\ \text{s.t.} \quad & \mathbf{A}\mathbf{w} \geq \mathbf{b}, & (22b)\\ & \mathbf{A}\mathbf{W}\mathbf{A}^\top - \mathbf{b}\mathbf{w}^\top\mathbf{A}^\top - \mathbf{A}\mathbf{w}\mathbf{b}^\top + \mathbf{b}\mathbf{b}^\top \geq 0 & (22c)\\ & W_{kk} - w_k = 0, \quad k = 1,\ldots,n_s & (22d)\\ & \mathbf{W} \succeq \mathbf{w}\mathbf{w}^\top. & (22e) \end{aligned}$$

As will be demonstrated in simulations, this relaxation is exact in most of the test systems. However, in a few cases, this relaxation does not lead to an integral solution for the statuses of the flexible lines. This is due to the fact that there may not be a unique solution for the OTS problem in non-congested power systems. Therefore, in the strengthened SDP relaxation of the problem, any convex combination of these optimal solutions is also optimal. On the other hand, most SDP solvers use interior point methods to obtain the optimal solution. These methods often converge to a point that is in the interior of the set of optimal solutions, which would not be feasible for the OTS problem in the presence of multiple SDP solutions. In order to address this issue, we will incorporate the penalty function

$$q_r(\mathbf{w},\mathbf{W}) = \sum_{k=1}^{n_S} t_k \times W_{kk}, \quad (23)$$

in the objective function of the convex formulation, where t_k is a small randomly-generated positive number. This penalization could indirectly penalize the rank of $\mathbf{W}$, while having a negligible effect on the optimality of the optimal solution.

VI. Numerical Results

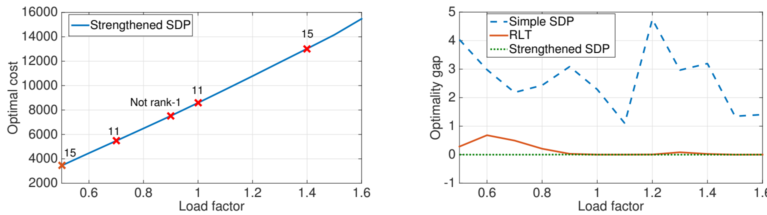
In this section, we evaluate and compare the performance of the proposed method for the OTS problem on different IEEE test case scenarios. In order to generate multiple instances of the OTS problem for each IEEE test system, we multiply the loads by a load factor α chosen from the set $\{\alpha_1, \alpha_2, ..., \alpha_k\}$. We also consider a uniform rating for all lines in the system. For each IEEE test case, we implement and compare the performance of the simple SDP, RLT, and strengthened SDP relaxations. To this goal, we plot two figures for each IEEE test case. The first figure shows the optimal cost of the OTS problem using the strengthened SDP relaxation accompanied with the optimal configurations of the flexible lines. The second figure shows the optimality gap using different relaxations compared to the optimal solution of the OTS problem, where the optimality gap is defined as

$$\text{Optimality gap} \triangleq \frac{\text{upper bound} - \text{lower bound}}{\text{upper bound}} \times 100.$$

The terms "upper bound" and "lower bound" denote the globally optimal cost of the OTS problem (found using the Gurobi solver) and the optimal cost of the relaxation, respectively. Furthermore, except for the last test case, the cardinality constraint (6i) is deactivated by setting r equal to the number of flexible lines.

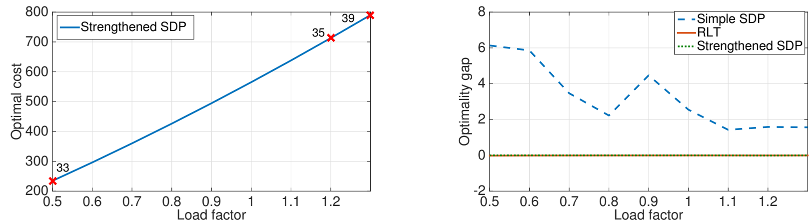
As the load factor changes from α_1 to α_k, the configuration of flexible lines in the optimal solution of the OTS problem may change. Whenever the statuses of the flexible lines for a load scenario vary from the previous ones, the corresponding scenario is marked on the optimal cost curve of the strengthened SDP relaxation by a red cross. Hence, if there is no mark on the curve for a particular load scenario, it means that the statuses of the flexible lines are the same as the ones in the previous load scenario. The number above each mark shows the optimal configuration of the flexible lines, when converted from base 10 to 2. For instance, if we have 3 flexible lines in the test case, the number 4 on the optimal cost indicates that the first and second flexible lines are switched off and only the third one is in service (note that $4 = (100)_2$). Notice that we only mark those scenarios for which there is a change in the optimal configuration. Furthermore, if the strengthened SDP relaxation cannot find the optimal configuration of the flexible lines, the corresponding scenario is marked with "Not rank-1" to indicate that an optimal solution of the strengthened SDP is not rank-1. In all test cases, an arbitrary spanning tree of the network is fixed with inflexible lines, and the flexible lines are randomly picked among the set of non-tree lines. In order to linearize the OTS problem (except for the binary constraints), we use the method delineated in section IV to find upper bounds on optimal values of M_{ij}'s for flexible lines. The simulations are run on a laptop computer with an Intel Core i7 quad-core 2.50 GHz CPU and 16GB RAM. The results reported in this section are for a serial implementation in MATLAB using the CVX framework and MOSEK solver.

Consider the IEEE 14-bus system with 5 generators and 20 lines. Among the non-tree lines, we randomly pick 5 lines and consider them as flexible lines. The load factors are $\alpha_i = 0.1 \times i$ for $i = 5, 6, ..., 16$. Figure 1a shows the optimal configuration of these 5 flexible lines for different load factors using the proposed strengthened SDP relaxation. It can be observed that, except for one case, the strengthened SDP relaxation can optimally find the correct statuses of all flexible lines. Figure 1b plots the the optimality gap using the SDP, RLT, and strengthened SDP relaxations. This figure shows that only the strengthened SDP relaxation has a zero optimality gap for different load scenarios. Notice that even



(a) Strengthened SDP (b) Optimality Gap

Fig. 1: 12 load scenarios for the IEEE 14-bus system with 5 flexible lines.



(a) Strengthened SDP (b) Optimality Gap

Fig. 2: 9 load scenarios for the IEEE 30-bus system with 7 flexible lines.

for the case where the optimal solution of the strengthened SDP relaxation is not rank-1, the optimality gap is almost zero. More precisely, the optimal cost using the strengthened SDP relaxation for this scenario is 7518.22, while the optimal cost using the Gurobi solver is 7518.58, and therefore the gap between these solutions is less than 0.0049%.

In the second case study, consider the IEEE 30-bus system. Seven lines are chosen from the set of non-tree lines and equipped with switches. The load factors are $\alpha_i = 0.1 \times i$ for $i = 5, 6, ..., 13$. It can be observed that the strengthened SDP relaxation is exact for all tested load scenarios. It is worthwhile to mention that the RLT relaxation results in a nearly zero optimality gap for all load scenarios. While this shows that the RLT relaxation significantly outperforms the simple SDP relaxation, it does not produce integral values for the statuses of the flexible lines in any of the considered load factors.

In the third test scenario, consider the IEEE 57-bus system. After fixing a spanning tree of the network with inflexible lines, 10 of the non-tree lines are equipped with switches. The load factors are $\alpha_i = 0.15 + 0.05 \times i$ for $i = 1, 2, ..., 8$. As demonstrated in Figure 3a, except for two cases, the strengthened SDP relaxation can find the rank-1 solution, which leads to an optimal solution of the OTS problem. Figure 3b shows the optimality gap using different methods. It can be observed that for the two cases where the strengthened SDP relaxation is not exact, the optimality gap is close to zero (0.0067% on average). Furthermore, although the optimality gap of the RLT relaxation is close to zero, its optimal solution is never rank-1, which leads to the non-exactness of this relaxation.

Finally, consider the OTS problem for the IEEE 118-bus system. An arbitrary spanning tree of the network is fixed with inflexible lines, and 20 of the remaining lines are equipped with switches. Furthermore, we set r equal to 10 in the cardinality constraint (6i). This means that at most 10

Test case		SDP	RLT	SSDP
14-bus, LF $= 0.5$	Obj. value	5400.47	5448.31	5475.28
	Time (s)	0.50	1.21	1.01
30-bus, LF $= 0.7$	Obj. value	340.76	360.11	360.10
	Time (s)	1.59	1.66	2.18
57-bus, LF $= 0.5$	Obj. value	4808.63	17906.93	17918.50
	Time (s)	10.45	5.024	39.58
118-bus, LF $= 0.2$	Obj. value	172.96	174.98	175.20
	Time (s)	403.49	54.70	698.5

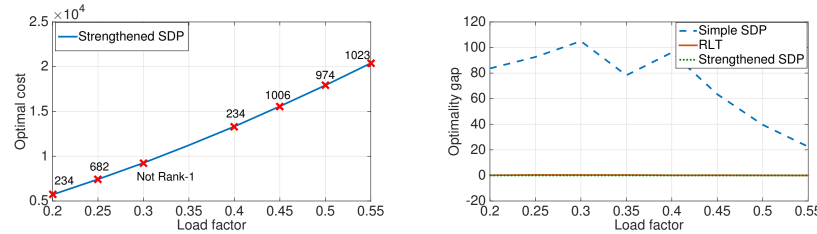
TABLE I: Running times and objective values of different test cases with given load factors (LF) using the SDP, RLT, Strengthened SDP (SSDP) relaxations.

of the flexible lines can be switched off at the same time. Consider the load factors as $\alpha_i = 0.1 \times i$ for $i = 1, 2, ..., 10$. Figure 4a shows the optimal cost of the strengthened SDP relaxation together with the schedule of the flexible lines for different load factors. It can be observed that the strengthened SDP relaxation does not obtain an exact solution only in two test scenarios. Moreover, the optimality gaps of the SDP, RLT, and strengthened SDP relaxations are compared in Figure 4b. For the case where the strengthened SDP relaxation does not produce a rank-1 solution, the optimality gap is almost zero. Similar to the previous test systems, the RLT relaxation significantly outperforms the simple SDP relaxation in terms of the optimality gap. However, this relaxation does not find a rank-1 (exact) solution in any of the test cases.

Table I presents the running time of different test cases using the proposed relaxations.

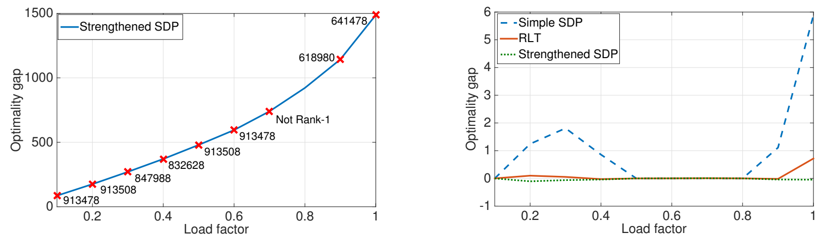
VII. Conclusion

Finding an optimal topology of a power system subject to operational and security constraints is a daunting task due to its inherent complexities. Common techniques for solving this problem are mostly based on a linear reformulation of this problem using the big-M method followed by iterative



(a) Strengthened SDP (b) Optimality Gap

Fig. 3: 8 load scenarios for the IEEE 57-bus system with 10 flexible lines.



(a) Strengthened SDP (b) Optimality Gap

Fig. 4: 10 load scenarios for the IEEE 118-bus system with 15 flexible lines and $r = 10$.

methods for solving mixed-integer linear programs (MILPs), such as branch-and-bound or cutting-plane methods. However, the performance of these methods partly relies on the strength of the convex relaxation of the big-M MILP reformulations. In this paper, we show that not only is solving the optimal transmission switching (OTS) problem hard, but one cannot hope for an efficient way to find the optimal parameters of its big-M MILP reformulation since it amounts to an NP-hard problem. Furthermore, we show the inapproximability of these parameters up to any constant factor. Moreover, we develop a convex conic relaxation of the problem based on a lift-and-project technique. The proposed convex model is strengthened by adding strong valid inequalities that are designed by multiplying different linear constraints in the problem and lifting them to a higher dimension to convexify them. The performance of the proposed method is demonstrated on different IEEE systems. Unlike branch-and-bound and cutting-planning methods, the technique developed in this paper can be readily generalized to AC models of power systems.

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