EXACTNESS OF SEMIDEFINITE RELAXATIONS FOR NONLINEAR OPTIMIZATION PROBLEMS WITH UNDERLYING GRAPH STRUCTURE

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Abstract. This work is concerned with finding a global optimization technique for a broad class of nonlinear optimization problems, including quadratic and polynomial optimization problems. The main objective of this paper is to investigate how the (hidden) structure of a given real/complex-valued optimization problem makes it easy to solve. To this end, three conic relaxations are proposed. Necessary and sufficient conditions are derived for the exactness of each of these relaxations, and it is shown that these conditions are satisfied if the optimization problem is highly structured. More precisely, the structure of the optimization problem is mapped into a generalized weighted graph, where each edge is associated with a weight set extracted from the coefficients of the optimization problem. In the real-valued case, it is shown that the relaxations are all exact if each weight set is sign definite and in addition a condition is satisfied for each cycle of the graph. It is also proved that if some of these conditions are violated, the relaxations still provide a low-rank solution for weakly cyclic graphs. In the complex-valued case, the notion of "sign definite complex sets" is introduced for complex weight sets. It is then shown that the relaxations are exact if each weight set is sign definite (with respect to complex numbers) and the graph is acyclic. Three other structural properties are derived for the generalized weighted graph in the complex case, each of which guarantees the exactness of some of the proposed relaxations. This result is also generalized to graphs that can be decomposed as a union of edge-disjoint subgraphs, where each subgraph has certain structural properties. As an application, it is proved that a relatively large class of real and complex optimization problems over power networks are polynomial-time solvable (with an arbitrary accuracy) due to the passivity of transmission lines and transformers.

Key words. global optimization, convex optimization, semidefinite programming, graph theory, complex optimization, power systems, optimal power flow

AMS subject classifications. 90C25, 90C22, 90C30, 52A41, 90C35, 90C90

1. Introduction.

Several classes of optimization problems, including polynomial optimization problems and quadratically-constrained quadratic programs (QCQPs) as a special case, are nonlinear/non-convex and NP-hard in the worst case. The paper [15] provides a survey on the computational complexity of optimizing various classes of continuous functions over some simple constraint sets. Due to the complexity of such problems, several convex relaxations based on semidefinite programming (SDP) and second-order cone programming (SOCP) have gained popularity [5, 6]. These techniques enlarge the possibly non-convex feasible set into a convex set characterizable via convex functions, and then provide the exact value or a lower bound on the optimal objective value. The paper [8] shows how SDP relaxation can be used to find better approximations for maximum cut (MAX CUT) and maximum 2-satisfiability (MAX 2SAT) problems. Another approach is proposed in [9] to solve the max-3-cut problem via complex SDP. The approaches in [8] and [9] have been generalized in several papers, including [20, 30, 29, 32, 33, 19, 12, 11].

The SDP relaxation converts an optimization problem with a vector variable to a convex optimization problem with a matrix variable, via a lifting technique. The exactness of the relaxation can then be interpreted as the existence of a low-rank (e.g., rank-1) solution for the SDP relaxation. Several papers have studied the

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existence of a low-rank solution to matrix optimization problems with linear matrix inequality (LMI) constraints [7, 24]. The papers [2] and [23] provide an upper bound on the lowest rank among all solutions of a feasible LMI problem. A rank-1 matrix decomposition technique is developed in [27] to find a rank-1 solution whenever the number of constraints is small. This technique is extended in [13] to the complex SDP problem. The paper [1] presents a polynomial-time algorithm for finding an approximate low-rank solution.

This work is motivated by the fact that real-world optimization problems are highly structured in many ways and their structures could in principle help reduce the computational complexity. For example, transmission lines and transformers used in power networks are passive devices, and as a result optimization problems defined over electrical power networks have certain structures which distinguish them from abstract optimization problems with random coefficients. The high-level objective of this paper is to understand how the computational complexity of a given nonlinear optimization problem is related to its (hidden) structure. This work is concerned with a broad class of nonlinear real/complex optimization problems, including QCQPs. The main feature of this class is that the argument of each objective and constraint function is quadratic (as opposed to linear) in the optimization variable and the goal is to use three conic relaxations (SDP, reduced SDP and SOCP) to convexify the argument of the optimization problem.

In this work, the structure of the nonlinear optimization problem is mapped into a generalized weighted graph, where each edge is associated with a weight set constructed from the known parameters of the optimization problem (e.g., the coefficients). This generalized weighted graph captures both the sparsity of the optimization problem and possible patterns in the coefficients. First, it is shown that the proposed relaxations are exact for real-valued optimization problems, provided a set of conditions is satisfied. These conditions need each weight set to be sign definite and each cycle of the graph to have an even number of positive weight sets. It is also shown that if some of these conditions are not satisfied, the SDP relaxation is guaranteed to have a rank-2 solution for weakly cyclic graphs, from which an approximate rank-1 solution may be recovered. To study the complex-valued case, the notion of "sign-definite complex weight sets" is introduced and it is then proved that the relaxations are exact for a complex optimization problem if the graph is acyclic with sign definite weight sets (with respect to complex numbers). The complex case is further studied and it is proved that the SDP relaxation is tight for four types of graphs as well as any acyclic combination of these types of graphs. As an application, it is also shown that a large class of energy optimization problems may be convexified due to the physics of power networks. The results of this paper extend the recent works on energy optimization problems [17, 16, 25, 26, 18, 31] and general quadratic optimization problems [14, 4].

In the next section, we formally state the optimization problem and then survey two related works. The main contributions of the paper are outlined in Section 2.4, where the plan for the rest of the paper is also given.

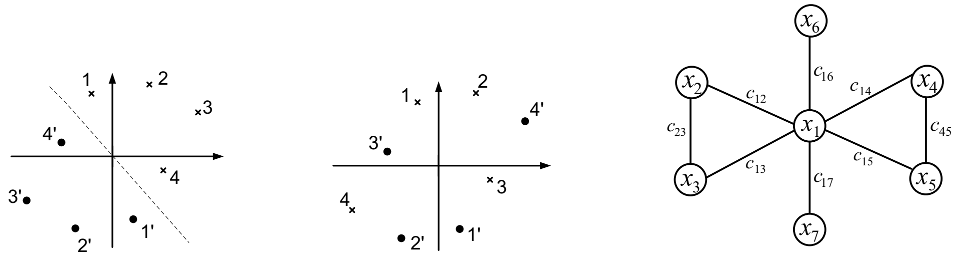
2. Problem Statement and Contributions.

Before introducing the problem, we need to make several notations and definitions.

2.1. Notations.

Essential notations and definitions will be provided below.

NOTATION 1. *In this work, scalars, vectors and matrices will be shown by lowercase, bold lowercase and uppercase letters (e.g.,* x*,* $\mathbf{x}$ *and* X*). Furthermore,* x_i *denotes*



(a) (b) (c)

FIG. 2.1. *In Figure (a), there exists a line separating the points marked as $\times$ (elements of $\mathcal{T}$) from the points marked as $\bullet$ (elements of $-\mathcal{T}$) so the set $\mathcal{T}$ is sign definite. In Figure (b), this is not the case. Figure (c) shows the weighted graph $\mathcal{G}$ studied in Example 2.*

the i^{th} entry of a vector $\mathbf{x}$, and X_{ij} denotes the $(i,j)^{th}$ entry of a matrix X.

NOTATION 2. *$\mathcal{R}$, $\mathcal{C}$, $\mathcal{S}^n$ and $\mathcal{H}^n$ denote the sets of real numbers, complex numbers, $n \times n$ symmetric matrices and $n \times n$ Hermitian matrices, respectively.*

NOTATION 3. *$Re\{M\}$, $Im\{M\}$, M^H, $rank\{M\}$ and $trace\{M\}$ denote the real part, imaginary part, conjugate transpose, rank and trace of a given scalar/matrix M, respectively. The notation $M \succeq 0$ means that M is symmetric/Hermitian and positive semidefinite.*

NOTATION 4. *The symbol $\measuredangle(x)$ represents the phase of a complex number x. The imaginary unit is denoted as "i", while "i" is used for indexing.*

NOTATION 5. *Given an undirected graph $\mathcal{G}$, the notation $i \in \mathcal{G}$ means that i is a vertex of $\mathcal{G}$. Moreover, the notation $(i,j) \in \mathcal{G}$ means that (i,j) is an edge of $\mathcal{G}$ and besides $i < j$.*

NOTATION 6. *Given a set $\mathcal{T}$, $|\mathcal{T}|$ denotes its cardinality. Given a graph $\mathcal{G}$, $|\mathcal{G}|$ shows the number of its vertices. Given a number (vector) $\mathbf{x}$, $|\mathbf{x}|$ denotes its absolute value (2-norm).*

DEFINITION 1. *A finite set $\mathcal{T} \subset \mathcal{R}$ is said to be* sign definite with respect to $\mathcal{R}$ *if its elements are either all negative or all nonnegative. $\mathcal{T}$ is called negative if its elements are negative and is called positive if its elements are nonnegative.*

DEFINITION 2. *A finite set $\mathcal{T} \subset \mathcal{C}$ is said to be* sign definite with respect to $\mathcal{C}$ *if when the sets $\mathcal{T}$ and $-\mathcal{T}$ are mapped into two collections of points in $\mathcal{R}^2$, then there exists a line separating the two sets (note that any or all elements of the sets $\mathcal{T}$ and $-\mathcal{T}$ are allowed to lie on the separating line).*

To illustrate Definition 2, consider a complex set $\mathcal{T}$ with four elements, whose corresponding points are labeled as 1, 2, 3 and 4 in Figure 2.1(a). The points corresponding to $-\mathcal{T}$ are labeled as 1', 2', 3' and 4' in the same picture. Since there exists a line separating the points marked as $\times$ (elements of $\mathcal{T}$) from the points marked as $\bullet$ (elements of $-\mathcal{T}$), the set $\mathcal{T}$ is sign definite. In contrast, if the elements of $\mathcal{T}$ are distributed according to Figure 2.1(b), the set will no longer be sign definite. Note that Definition 2 is inspired by the fact that a real set $\mathcal{T}$ is sign definite with respect to $\mathcal{R}$ if $\mathcal{T}$ and $-\mathcal{T}$ are separable via a point (on the horizontal axis).

DEFINITION 3. *Given a graph $\mathcal{G}$, a cycle space is the set of all possible cycles in the graph. An arbitrary basis for this cycle space is called a "cycle basis".*

DEFINITION 4. *In this work, a graph $\mathcal{G}$ is called weakly cyclic if every edge of the graph belongs to at most one cycle in $\mathcal{G}$ (i.e., the cycles of $\mathcal{G}$ are all edge-disjoint).*

DEFINITION 5. *Consider a graph $\mathcal{G}$, a subgraph $\mathcal{G}_s$ of this graph and a matrix*

$X \in \mathcal{C}^{|\mathcal{G}|\times|\mathcal{G}|}$. Define $X\{\mathcal{G}_s\}$ as a sub-matrix of X obtained by picking every row and column whose index belongs to the vertex set of $\mathcal{G}_s$. For instance, $X\{(i,j)\}$, where $(i,j) \in \mathcal{G}$, has rows i,j and columns i,j of X.

2.2. Problem Statement. Consider an undirected graph $\mathcal{G}$ with n vertices (nodes), where each edge $(i,j) \in \mathcal{G}$ has been assigned a nonzero edge weight set $\{c^1_{ij}, c^2_{ij}, ..., c^k_{ij}\}$ with k real/complex numbers (note that the superscripts in the weights are not exponents). This graph is called a *generalized weighted graph* as every edge is associated with a set of weights as opposed to a single weight. Consider an unknown vector $\mathbf{x} = \begin{bmatrix} x_1 & \cdots x_n \end{bmatrix}$ belonging to $\mathcal{D}^n$, where $\mathcal{D}$ is either $\mathcal{R}$ or $\mathcal{C}$. For every $i \in \mathcal{G}$, x_i is a variable associated with node i of the graph $\mathcal{G}$. Define:

$$\mathbf{y} = \left\{|x_i|^2 \;\middle|\; \forall i \in \mathcal{G}\right\}, \quad \mathbf{z} = \left\{\mathrm{Re}\left\{c^t_{ij} x_i x_j^H\right\} \;\middle|\; \forall (i,j) \in \mathcal{G},\ t \in \{1,...,k\}\right\}$$

Note that according to Notation 5, $(i,j) \in \mathcal{G}$ means that (i,j) is an edge of the graph and that $i < j$. The sets $\mathbf{y}$ and $\mathbf{z}$ can be regarded as two vectors, where

- $\mathbf{y}$ collects the quadratic terms $|x_i|^2$'s (one term for each vertex).
- $\mathbf{z}$ collects the cross terms $\mathrm{Re}\{c^t_{ij} x_i x_j^H\}$'s ($k$ terms for each edge).

Although the above formulation deals with $\mathrm{Re}\{c^t_{ij} x_i x_j^H\}$ whenever $(i,j) \in \mathcal{G}$, it can handle terms of the form $\mathrm{Re}\{\alpha x_j x_i^H\}$ and $\mathrm{Im}\{\alpha x_i x_j^H\}$ for a complex weight α. This can be carried out using the following transformations:

$$\mathrm{Re}\{\alpha x_j x_i^H\} = \mathrm{Re}\{(\alpha^H) x_i x_j^H\}, \qquad \mathrm{Im}\{\alpha x_i x_j^H\} = \mathrm{Re}\{(-\alpha \mathrm{i}) x_i x_j^H\}$$

This work is concerned with the optimization problem:

$$\begin{aligned} \min_{\mathbf{x} \in \mathcal{D}^n} \quad & f_0(\mathbf{y}, \mathbf{z}) \\ \text{subject to} \quad & f_j(\mathbf{y}, \mathbf{z}) \leq 0, \quad j = 1, 2, ..., m \end{aligned} \tag{2.1}$$

for given functions $f_0, ..., f_m$. The computational complexity of the above optimization problem depends in part on the structure of the functions f_j's. Regardless of these functions, the optimization problem (2.1) is intrinsically hard to solve (NP-hard in the worst case) because $\mathbf{y}$ and $\mathbf{z}$ are both nonlinear functions of $\mathbf{x}$. The objective is to convexify the second-order nonlinearity embedded in $\mathbf{y}$ and $\mathbf{z}$. To this end, notice that there exist two linear functions $l_1 : \mathcal{C}^{n\times n} \to \mathcal{R}^n$ and $l_2 : \mathcal{C}^{n\times n} \to \mathcal{R}^{k\tau}$ such that $\mathbf{y} = l_1\left(\mathbf{x}\mathbf{x}^H\right)$ and $\mathbf{z} = l_2\left(\mathbf{x}\mathbf{x}^H\right)$, where τ denotes the number of edges in $\mathcal{G}$. Motivated by the above observation, if $\mathbf{x}\mathbf{x}^H$ is replaced by a new matrix variable X, then $\mathbf{y}$ and $\mathbf{z}$ both become linear in X. This implies that the non-convexity induced by the quadratic terms $\mathrm{Re}\{c^t_{ij} x_i x_j\}$'s and $|x_i|$'s all disappear if the optimization problem (2.1) is reformulated in terms of X. However, the optimal solution X may not be decomposable as $\mathbf{x}\mathbf{x}^H$ unless some additional constraints are imposed on X. It is straightforward to verify that the optimization problem (2.1) is equivalent to

$$\min_{X} \quad f_0(l_1(X), l_2(X)) \tag{2.2a}$$

$$\text{s.t.} \quad f_j(l_1(X), l_2(X)) \leq 0, \quad j = 1, ..., m \tag{2.2b}$$

$$X \succeq 0, \tag{2.2c}$$

$$\mathrm{rank}\{X\} = 1 \tag{2.2d}$$

where there is an implicit constraint that $X \in \mathcal{S}^n$ if $\mathcal{D} = \mathcal{R}$ and $X \in \mathcal{H}^n$ if $\mathcal{D} = \mathcal{C}$. To reduce the computational complexity of the above problem, two actions can be taken: (i) removing the nonconvex constraint (2.2d), (ii) relaxing the convex, but computationally-expensive, constraint (2.2c) to a set of simpler constraints on certain low-order submatrices of X. Based on this methodology, three relaxations will be proposed for the optimization problem (2.1) next.

SDP relaxation: This optimization problem is defined as

$$\min_{X} \quad f_0(l_1(X), l_2(X)) \tag{2.3a}$$
$$\text{s.t.} \quad f_j(l_1(X), l_2(X)) \leq 0, \quad j = 1, ..., m \tag{2.3b}$$
$$X \succeq 0 \tag{2.3c}$$

Reduced SDP relaxation: Choose a set of cycles $\mathcal{O}_1, ..., \mathcal{O}_p$ in the graph $\mathcal{G}$ such that they form a cycle basis. Let Ω denote the set of all subgraphs $\mathcal{O}_1, ..., \mathcal{O}_p$ as well as all edges of $\mathcal{G}$ that do not belong to any cycle in the graph (i.e., bridge edges). The reduced SDP relaxation is defined as

$$\min_{X} \quad f_0(l_1(X), l_2(X)) \tag{2.4a}$$
$$\text{s.t.} \quad f_j(l_1(X), l_2(X)) \leq 0, \qquad j = 1, ..., m \tag{2.4b}$$
$$X\{\mathcal{G}_s\} \succeq 0, \qquad \forall \mathcal{G}_s \in \Omega \tag{2.4c}$$

SOCP relaxation: This optimization problem is defined as

$$\min_{X} \quad f_0(l_1(X), l_2(X)) \tag{2.5a}$$
$$\text{s.t.} \quad f_j(l_1(X), l_2(X)) \leq 0, \qquad j = 1, ..., m \tag{2.5b}$$
$$X\{(i,j)\} \succeq 0, \qquad \forall (i,j) \in \mathcal{G} \tag{2.5c}$$

The reason why the above optimization problem is called an SOCP problem is that the condition $X\{(i,j)\} \succeq 0$ can be replaced by the linear and norm constraints

$$X_{ii}, X_{jj} \geq 0, \quad X_{ii} + X_{jj} \geq \left\| \begin{bmatrix} X_{ii} & X_{jj} & \sqrt{2}X_{ij} \end{bmatrix} \right\|$$

The main idea behind the introduction of the above SDP, reduced SDP and SOCP relaxations is to remove the non-convexity caused by the nonlinear relationship between $\mathbf{x}$ and $(\mathbf{y}, \mathbf{z})$. Note that these optimization problems are convex relaxations only when the functions $f_0, ..., f_m$ are convex. If any of these functions is nonconvex, additional relaxations might be needed to convexify the SDP, reduced SDP or SOCP optimization problem. Define $f^*, f^*_{\text{SDP}}, f^*_{\text{r-SDP}}$ and f^*_{SOCP} as the optimal solutions of the optimization problems (2.2), (2.3), (2.4) and (2.5), respectively. By comparing the feasible sets of these optimization problems, it can be concluded that

$$f^*_{\text{SOCP}} \leq f^*_{\text{r-SDP}} \leq f^*_{\text{SDP}} \leq f^* \tag{2.6}$$

Given a particular optimization problem of the form (2.1), if any of the above inequalities for f^* turns into an equality, the associated relaxation could be used to find the solution of the original optimization problem. In this case, it is said that the relaxation is "tight" or "exact". The objective of this paper is to relate the exactness of the proposed relaxations to the topology of the graph $\mathcal{G}$ and its weight sets $\{c^1_{ij}, c^2_{ij}, ..., c^k_{ij}\}$'s.

It is noteworthy that the aforementioned problem formulation can be easily generalized in two directions:

- *Allowance of weight sets with different cardinalities:* The above problem formulation assumes that every edge weight set has k elements. However, if the weight sets have different sizes, the trivial weight 0 can be added to each set multiple times in such a way that all expanded sets reach the same cardinality.
- *Inclusion of linear terms in* $\mathbf{x}$*:* The optimization problem (2.1) is formulated with respect to $\mathbf{x}\mathbf{x}^H$, but with no linear term in $\mathbf{x}$. This issue can be fixed by defining an expanded vector $\tilde{\mathbf{x}}$ as $\left[\begin{array}{cc} 1 & \mathbf{x}^H \end{array}\right]^H$. Then, the matrix $\tilde{\mathbf{x}}\tilde{\mathbf{x}}^H$ needs to be replaced by a new matrix variable $\tilde{X}$ under the constraint $\tilde{X}_{11} = 1$.

REMARK 1. *A cycle basis is required in order to construct the reduced SDP relaxation. To design such a basis, consider an arbitrary spanning forest of the graph $\mathcal{G}$. If any edge of the graph that does not belong to the spanning forest is added to the forest, a unique cycle will be created. The union of all those cycles forms a cycle basis [28]. Note that there may exist an exponential number of cycle bases and the solution of the reduced SDP relaxation depends on the choice of the cycle basis. For example, if the graph $\mathcal{G}$ contains a Hamiltonian cycle and this cycle belongs to the selected cycle basis, then the SDP and reduced SDP relaxations will lead to the same solution (because the constraint $X \succeq 0$ corresponding to the Hamiltonian cycle will be a part of the reduced SDP relaxation). Hence, the gap between the optimal objective values of the SDP and reduced SDP relaxations could, in principle, change with the choice of the cycle basis.*

REMARK 2. *A main contributor to the computational complexity of each of the abovementioned relaxations is the number of "important" variables, where an important variable is defined as an entry of X that has a nonzero coefficient in either the objective function or one of the constraints of the optimization (the cost of evaluating the functions f_j's and the number of constraints are considered as secondary factors for simplicity). The SDP relaxation has $O(n^2)$ important variables, whereas the SOCP relaxation has only $O(n+\tau)$ important variables with τ defined as the number of edges of $\mathcal{G}$. In particular, the number of important variables of the SOCP relaxation is $O(n)$ if the graph $\mathcal{G}$ is planar, and this property makes the SOCP relaxation far more appealing than the SDP relaxation for planar graphs. The number of important variables of the reduced SDP relaxation depends on the lengths of the cycles $\mathcal{O}_1, ..., O_p$, as well as the overlaps between these cycles (for example, an edge (i,j) creates at most one variable even if it appears in multiple cycles). The number of important variables of the reduced SDP relaxation is upper bounded by the sum of the lengths of $\mathcal{O}_1, ..., O_p$. Note that a cycle basis with a minimum length sum is called "minimum cycle basis" and such a basis can be found in polynomial time [28].*

2.3. Related Work. Consider the QCQP optimization problem:

$$\min_{\mathbf{x}\in\mathcal{D}^n} \quad \mathbf{x}^H M_1 \mathbf{x} \qquad \text{s.t.} \;\; \mathbf{x}^H M_j \mathbf{x} \le 0 \quad j = 2, ..., k \tag{2.7}$$

for given matrices $M_1, ..., M_k \in \mathcal{H}^n$. This problem is a special case of the optimization problem (2.1), where its generalized weighted graph $\mathcal{G}$ has two properties:

- Given two nodes $i, j \in \{1, ..., n\}$ such that $i < j$, there exists an edge between nodes i and j if and only if the (i,j) off-diagonal entry of at least one of the matrices $M_1, ..., M_k$ is nonzero.
- For every $(i,j) \in \mathcal{G}$, the weight set $\{c_{ij}^1, c_{ij}^2, ..., c_{ij}^k\}$ is the union of the $(i,j)^{\text{th}}$ entries of $M_1, ..., M_k$.

Due to the relation $\mathbf{x}^H M_i \mathbf{x} = \text{trace}\{M_i \mathbf{x}\mathbf{x}^H\}$ for $i = 1, ..., k$, the SDP relaxation of the optimization problem (2.7) turns out to be

$$\min_X \quad \text{trace}\{M_1 X\} \quad \text{s.t.} \quad \text{trace}\{M_j X\} \leq 0 \quad j = 2, ..., k, \quad X \succeq 0$$

The SOCP relaxation of the optimization problem (2.7) is obtained by replacing the constraint $X \succeq 0$ with $X\{(i,j)\} \succeq 0$ for every $(i,j) \in \mathcal{G}$. The relationship between the optimization problem (2.7) and its relaxations have been studied in two special cases in the literature:

- Consider the case $D = \mathcal{R}$. It has been shown in [14] that $f^*_{\text{SOCP}} = f^*_{\text{SDP}} = f^*$ if $-M_0, ..., -M_k$ are all Metzler matrices (a Metzler matrix is a matrix in which the off-diagonal entries are all nonnegative). This result implies that the proposed relaxations are all exact, independent of the topology of $\mathcal{G}$, as long as the set $\{c^1_{ij}, c^2_{ij}, ..., c^k_{ij}\}$ is negative for all $(i,j) \in \mathcal{G}$.
- Consider the case $D = \mathcal{C}$. It has been shown in the recent work [4] that $f^*_{\text{SDP}} = f^*$ if three conditions hold:
  1. $\mathcal{G}$ is a tree graph.
  2. M_1 is a positive semidefinite matrix.
  3. For every $(i,j) \in \mathcal{G}$, the origin $(0,0)$ is not an interior point of the convex hull of the 2-d polytope induced by the weight set $\{c^1_{ij}, c^2_{ij}, ..., c^k_{ij}\}$.

It can be shown that Condition (3) implies that the complex set $\{c^1_{ij}, c^2_{ij}, ..., c^k_{ij}\}$ is sign definite (see Definition 2).

As a special case of (2.7), the paper [21] studies an unconstrained quadratic zero-one program for which a quadratic objective is to be minimized subject to the constraint that each variable is either 0 or 1. The work [21] proves that this problem can be solved using a linear programming relaxation under various graph conditions (e.g., the acyclicity of the graph $\mathcal{G}$).

The above results all together suggest that the polynomial-time solvability (up to an arbitrary accuracy) of certain classes of QCQP problems might be inferred from some weak properties of their underlying generalized weighted graphs.

2.4. Contributions. Throughout this paper, we assume that $f_j(\mathbf{y}, \mathbf{z})$ is monotonic in every entry of $\mathbf{z}$ for $j = 0, 1, ..., m$ (but possibly nonconvex in $\mathbf{y}$ and $\mathbf{z}$). With no loss of generality, suppose that $f_j(\mathbf{y}, \mathbf{z})$ is an increasing function with respect to all entries of $\mathbf{z}$ (to ensure this property, it may be needed to change the sign of some edge weights and then redefine the functions). A few of the results to be developed in this work do not need this assumption, in which cases the name of the function f_j will be changed to g_j to avoid any confusion in the assumptions.

The objective of this paper is to study the interrelationship between f^*_{SOCP}, $f^*_{\text{r-SDP}}$, f^*_{SDP} and f^*. In particular, it is aimed to understand what properties the generalized weighted graph $\mathcal{G}$ should have to guarantee the exactness of some of the proposed relaxations. Another goal is to find out how low rank the solution of the SDP relaxation will be in the case when the relaxation is not exact.

In section 3, we derive necessary and sufficient conditions for the exactness of the each of the three aforementioned relaxations in both real and complex cases.

In Section 4, we consider the real-valued case $\mathcal{D} = \mathcal{R}$ and show that the SOCP, reduced SDP and SDP relaxations are all tight, provided each weight set $\{c^1_{ij}, ..., c^k_{ij}\}$ is sign definite with respect to $\mathcal{R}$ and

$$\prod_{(i,j)\in\mathcal{O}_r} \sigma_{ij} = (-1)^{|\mathcal{O}_r|}, \qquad \forall r \in \{1, ..., p\}$$

where σ_{ij} shows the sign of the weight set associated with the edge $(i,j) \in \mathcal{G}$. This condition is naturally satisfied in three special cases:

- $\mathcal{G}$ is acyclic with arbitrary sign definite edge sets.
- $\mathcal{G}$ is bipartite with positive weight sets.
- $\mathcal{G}$ is arbitrary with negative weight sets.

It is also shown that if the SDP relaxation is not exact, it still has a low rank (rank-2) solution in two cases:

- $\mathcal{G}$ is acyclic (but with potentially indefinite weight sets).
- $\mathcal{G}$ is a weakly-cyclic bipartite graph with sign definite edge sets.

In section 5, we consider the complex-valued case $\mathcal{D} = \mathcal{C}$ under the assumption that each edge set $\{c_{ij}^1, ..., c_{ij}^k\}$ is sign definite with respect to $\mathcal{C}$. This assumption is trivially met if $k \leq 2$ or the weight set contains only real (or imaginary) numbers. Some of the results developed in that section are:

- The SOCP, reduced SDP and SDP relaxations are all tight if $\mathcal{G}$ is acyclic.
- The SOCP, reduced SDP and SDP relaxations are tight if each weight set contains only real or imaginary numbers and

  $$\prod_{(i,j)\in\vec{\mathcal{O}}_r} \sigma_{ij} = (-1)^{|\mathcal{O}_r|}, \qquad \forall r \in \{1, ..., p\}$$

  where $\sigma_{ij} \in \{0, \pm 1, \pm \mathrm{i}\}$ shows the sign of each weight set and $\vec{\mathcal{O}}_r$ denotes a directed cycle corresponding to $\mathcal{O}_r$.
- The reduced SDP and SDP relaxations (but not necessarily the SOCP relaxation) are exact if $\mathcal{G}$ is bipartite and weakly cyclic with positive or negative real weight sets.
- The reduced SDP and SDP relaxations (but not necessarily the SOCP relaxation) are exact if $\mathcal{G}$ is a weakly cyclic graph with imaginary weight sets and nonzero signs σ_{ij}'s.

We also show that if the graph $\mathcal{G}$ can be decomposed as a union of edge-disjoint subgraphs in an acyclic way such that each subgraph has one of the above four structural properties, then the SDP relaxation is exact.

In Section 6, a detailed discussion is given to demonstrate how the results of this paper can be used for optimization over power networks. Finally, five illustrative examples are provided in section 7.

3. SDP, Reduced-SDP and SOCP Relaxations. In this section, the objective is to derive necessary and sufficient conditions for the exactness of the SDP, reduced-SDP and SOCP Relaxations. For every $r \in \{1, 2, ..., p\}$, let $\vec{\mathcal{O}}_r$ denote a directed cycle corresponding to $\mathcal{O}_r$, meaning that all edges of the undirected cycle $\mathcal{O}_r$ has been oriented consistently.

THEOREM 1. *The following statements hold true in both real and complex cases $\mathcal{D} = \mathcal{R}$ and $\mathcal{D} = \mathcal{C}$:*

i) The SDP relaxation is exact (i.e., $f^*_{SDP} = f^*$) if and only if it has a rank-1 solution X^*.

ii) The reduced SDP relaxation is exact (i.e., $f^*_{r\text{-}SDP} = f^*$) if and only if it has a solution X^* such that

$$\mathrm{rank}\{X^*\{\mathcal{G}_s\}\} = 1, \qquad \forall \mathcal{G}_s \in \Omega \tag{3.1}$$

iii) The SOCP relaxation is exact (i.e., $f^*_{SOCP} = f^*$) if and only if it has a *solution X^* such that*

$$\text{rank}\{X^*\{(i,j)\}\} = 1, \qquad \forall (i,j) \in \mathcal{G}$$

and that

$$\sum \measuredangle X^*_{ij} = 0, \qquad \forall r \in \{1,2,...,p\} \tag{3.2}$$

where the sum is taken over all directed edges (i,j) of the oriented cycle $\vec{\mathcal{O}}_r$. Moreover, the same result holds even if the condition (3.2) is replaced by (3.1).

Proof of Part (i): The proof is omitted due to its simplicity.

Proof of Part (ii): To prove the "only if" part, let $\mathbf{x}^*$ denote an arbitrary solution of the optimization problem (2.1). If $f^*_{\text{r-SDP}} = f^*$, then $X^* = (\mathbf{x}^*)(\mathbf{x}^*)^H$ is a solution of the reduced SDP relaxation, which satisfies the condition (3.1).

To prove the "if" part, consider a matrix X^* satisfying (3.1). For every $r \in \{1,...,p\}$, since $X\{\mathcal{O}_r\}$ is positive semidefinite and rank-1, it can be written as the product of a vector and its transpose. This yields that

$$\sum \measuredangle X^*_{ij} = 0, \qquad \forall r \in \{1,2,...,p\} \tag{3.3}$$

where the sum is taken over all directed edges (i,j) of the oriented cycle $\vec{\mathcal{O}}_r$. Let $\mathcal{T}$ be an arbitrary spanning tree of $\mathcal{G}$. The vertices of $\mathcal{T}$ can be iteratively labeled by some real numbers (angles) $\theta_1,...,\theta_n$ in such a way that $\theta_i - \theta_j = \measuredangle X^*_{ij}$, $\forall (i,j) \in \mathcal{T}$, and that these numbers belong to the discrete set $\{0, 180^0\}$ in the case $\mathcal{C} = \mathcal{R}$. It can be inferred from (3.3) that $\theta_i - \theta_j = \measuredangle X^*_{ij}$ for every $(i,j) \in \mathcal{G}$. Now, define $\mathbf{x}^*$ as

$$\begin{bmatrix} \sqrt{X_{11}}e^{-\theta_1 \mathrm{i}} & \sqrt{X_{22}}e^{-\theta_2 \mathrm{i}} & \cdots & \sqrt{X_{nn}}e^{-\theta_n \mathrm{i}} \end{bmatrix}^H$$

Observe that $(\mathbf{x}^*)(\mathbf{x}^*)^H$ and $\mathbf{X}^*$ are the same on the diagonal and have identical off-diagonal $(i,j)^{\text{th}}$ entries for every $(i,j) \in \mathcal{G}$. This implies that $(\mathbf{x}^*)(\mathbf{x}^*)^H$ is a rank-1 solution of the reduced SDP relaxation. Therefore, the relaxation is exact.

Proof of Part (iii): The proof is omitted due to its similarity to the proof of Part (ii) provided above. ■

Theorem 1 provides necessary and sufficient conditions for the exactness of the SDP, reduced SDP and SOCP relaxations. As mentioned before, one can write $f^*_{\text{SOCP}} \leq f^*_{\text{r-SDP}} \leq f^*_{\text{SDP}} \leq f^*$. Using the matrix completion theorem and chordal extension, two conclusions can be made [10]:

- If $\mathcal{G}$ is an acyclic graph, then the relation $f^*_{\text{SOCP}} = f^*_{\text{r-SDP}} = f^*_{\text{SDP}}$ holds, even in the case where $f^*_{\text{SDP}} \neq f^*$.
- Expand the graph $\mathcal{G}$ by connecting all vertices inside each cycle $\mathcal{O}_r$ to each other for $r = 1,2,...,p$. Then, the relation $f^*_{\text{r-SDP}} = f^*_{\text{SDP}}$ holds (independent of whether or not $f^*_{\text{SDP}} = f^*$) if the expanded graph is chordal and every maximal clique of this graph corresponds to a single edge of $\mathcal{G}$ or one of the cycles $\mathcal{O}_1,...,\mathcal{O}_p$. This condition is met for weakly cyclic graphs as well as a broad class of planar graphs.

Part (iii) of Theorem 1 shows that the SOCP relaxation is exact if two conditions are satisfied for an optimal solution X^* of this optimization problem: (1) every 2×2 edge submatrix $X^*\{(i,j)\}$ loses rank, and (2) if the phase of X^*_{ij} is assigned to the edge (i,j) of the graph $\mathcal{G}$ for every $(i,j) \in \mathcal{G}$, then the sum of the edge phases becomes zero for every cycle in the cycle basis. As will be shown throughout this paper, Condition (1) is satisfied by imposing a sign definiteness constraint on each edge weight set. In contrast, Condition (2) is strongly related to the graph topology and weakly related to the structure of each edge weight set.

Remark 3. *Condition* (3.2) *requires that the edge angles around the oriented cycle* $\vec{\mathcal{O}}_r$ *add up to zero. However, since* $sin(\theta_{ij})$ *is a periodic function of* θ_{ij}*, equation* (3.2) *can be replaced by*

$$\sum \measuredangle X^*_{ij} = 2l_r\pi \tag{3.4}$$

for every integer number l_r*. More precisely, the number "zero" in the right side of equation* (3.2) *can be interpreted as "zero angle" and therefore it can take any value of the form* $2l_r\pi$*. To prove this statement, let* θ_{ij} *denote the phase of* X^*_{ij}*. Then,* $\measuredangle X^*_{ij}$ *can be considered as* $\theta_{ij} + 2l\pi$ *for any integer number* l*, and as a result there is some degree of freedom in defining* $\measuredangle X^*_{ij}$*. It is straightforward to show that if* (3.4) *is satisfied for* $r = 1, ..., p$*, then* $\measuredangle X^*_{ij}$*'s can all be redefined (using their periodic nature) to meet Condition* (3.2)*. Hence, "zero angle" means* $2l\pi$ *throughout this paper.*

4. Real-valued Optimization Problems. In this section, the optimization problem (2.1) will be studied in the real-valued case $\mathcal{D} = \mathcal{R}$. Since $\mathbf{x} \in \mathcal{R}^n$, one can write $\text{Re}\left\{c^t_{ij}x_i x^H_j\right\} = \text{Re}\left\{\text{Re}\{c^t_{ij}\}x_i x^H_j\right\}$, for all $(i,j) \in \mathcal{G}$ and $t \in \{1, ..., k\}$. Hence, changing the complex weight c^t_{ij} to $\text{Re}\{c^t_{ij}\}$ does not affect the optimization problem. Therefore, with no loss of generality, assume that the edge weights are all real numbers. For every edge $(i,j) \in \mathcal{G}$, define the edge sign σ_{ij} as follows:

$$\sigma_{ij} = \begin{cases} 1 & \text{if} \quad c^1_{ij}, ..., c^k_{ij} \geq 0 \\ -1 & \text{if} \quad c^1_{ij}, ..., c^k_{ij} \leq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.1}$$

By convention, we define $\sigma_{ij} = -1$ if $c^1_{ij} = \cdots = c^k_{ij} = 0$.

Theorem 2. *The relations* $f^*_{SOCP} = f^*_{r\text{-}SDP} = f^*_{SDP} = f^*$ *hold for the optimization problem (2.1) in the real-valued case* $\mathcal{D} = \mathcal{R}$ *if*

$$\sigma_{ij} \neq 0, \qquad \forall (i,j) \in \mathcal{G} \tag{4.2a}$$

$$\prod_{(i,j)\in\mathcal{O}_r} \sigma_{ij} = (-1)^{|\mathcal{O}_r|}, \qquad \forall r \in \{1, ..., p\} \tag{4.2b}$$

Proof: In light of the relation $f^*_{\text{SOCP}} \leq f^*_{\text{r-SDP}} \leq f^*_{\text{SDP}} \leq f^*$, it suffices to prove that $f^* \leq f^*_{\text{SOCP}}$. Consider an arbitrary feasible point X of the optimization problems (2.5). It is enough to show the existence of a feasible point $\mathbf{x}$ for the optimization problem (2.1) with the property that the objective value of this optimization problem at $\mathbf{x}$ is lower than or equal to the objective value of the SOCP relaxation at the point X. For this purpose, choose an arbitrary spanning tree $\mathcal{T}$ of the graph $\mathcal{G}$. A set of ± 1 numbers $\sigma_1, \sigma_2, ..., \sigma_n$ can be iteratively assigned to the vertices of this tree in such a way that $\sigma_i\sigma_j = -\sigma_{ij}$ for every $(i,j) \in \mathcal{T}$ (this is due to (4.2a)). Now, it can be deduced from (4.2b) that

$$\sigma_i\sigma_j = -\sigma_{ij}, \qquad \forall (i,j) \in \mathcal{G}$$

Corresponding to the feasible point X of the SOCP relaxation, define the vector $\mathbf{x}$ as

$$\begin{bmatrix} \sigma_1\sqrt{X_{11}} & \sigma_2\sqrt{X_{22}} & \cdots & \sigma_n\sqrt{X_{nn}} \end{bmatrix}^H$$

(note that $X_{11}, ..., X_{nn} \geq 0$ due to the conditions $X\{(i,j)\} \succeq 0$ for every $(i,j) \in \mathcal{G}$). Observe that

$$|x_i|^2 = X_{ii}, \qquad i = 1, ..., n \tag{4.3}$$

On the other hand, (2.5c) yields

$$|X_{ij}| \leq \sqrt{X_{ii}}\sqrt{X_{jj}}, \qquad \forall (i,j) \in \mathcal{G}$$

and therefore

$$\begin{aligned} c_{ij}^t X_{ij} &\geq -|c_{ij}^t|\sqrt{X_{ii}}\sqrt{X_{jj}} = -c_{ij}^t \sigma_{ij}\sqrt{X_{ii}}\sqrt{X_{jj}} \\ &= c_{ij}^t \sigma_i \sigma_j \sqrt{X_{ii}}\sqrt{X_{jj}} = c_{ij}^t x_i x_j, \quad \forall (i,j) \in \mathcal{G} \end{aligned} \tag{4.4}$$

for $t = 1, 2, ..., k$. It can be concluded from (4.3) and (4.4) that

$$l_1\left(\mathbf{x}\mathbf{x}^H\right) = l_1(X), \qquad l_2\left(\mathbf{x}\mathbf{x}^H\right) \leq l_2(X)$$

Hence, since $f_0(\cdot,\cdot)$ is increasing in its second vector argument, one can write:

$$f_j(\mathbf{y}, \mathbf{z}) \leq f_j(l_1(X), l_2(X))$$

for $j = 0, 1, ..., m$, where $\mathbf{y} = l_1\left(\mathbf{x}\mathbf{x}^H\right)$ and $\mathbf{z} = l_2\left(\mathbf{x}\mathbf{x}^H\right)$. This implies that $\mathbf{x}$ is a feasible point of the optimization problem (2.1) whose corresponding objective value is smaller than or equal to the objective value for the feasible point X of the optimization problem (2.5). This proves the claim $f^* \leq f^*_{\text{SOCP}}$ and thus completes the proof. ■

Condition (4.2a) ensures that each edge weight set is sign definite. Theorem 2 states that the SDP, reduced SDP and SOCP relaxations are exact for the original optimization problem (2.1) under the above sign definite condition, provided that each cycle in the cycle basis has an even number of edges with positive signs. Note that the exactness of the SDP relaxation does not imply that the relaxation has a unique rank-1 solution. In particular, if a sample of the SDP relaxation is solved numerically, the obtained solution may be high rank. In this case, a rank-1 solution X^* is hidden and needs to be recovered (following the constructive proof of the theorem). The conditions offered in Theorem 2 hold true in three important special cases, as explained below.

Corollary 1. *The relations $f^*_{SOCP} = f^*_{r\text{-}SDP} = f^*_{SDP} = f^*$ hold for the optimization problem (2.1) in the case $\mathcal{D} = \mathcal{R}$ if one of the following happens:*

1) *$\mathcal{G}$ is acyclic with arbitrary sign definite edge sets (with respect to $\mathcal{R}$).*
2) *$\mathcal{G}$ is bipartite with positive weight sets.*
3) *$\mathcal{G}$ is arbitrary with negative weight sets.*

Proof: The proof follows immediately from Theorem 2 by noting that a bipartite graph has no odd cycle. ■

Assume that the edge sets of the graph $\mathcal{G}$ are all sign definite. Corollary 1 implies a trade-off between the topology and the edge signs σ_{ij}'s. On one extreme, the edge signs could be arbitrary as long as the graph has a very sparse topology. On the other extreme, the graph topology could be arbitrary (sparse or dense) as long as the edge signs are all negative. The following theorem proves that if σ_{ij}'s are zero, the optimization problem (2.1) becomes NP-hard even for an acyclic graph $\mathcal{G}$.

Theorem 3. *Finding an optimal solution of the optimization problem (2.1) is an NP-hard problem for an acyclic $\mathcal{G}$ with sign-indefinite weight sets (even if $k = 2$).*

Proof: Given a set of real numbers $\{\omega_1, ..., \omega_t\}$, the number partitioning problem (NPP) aims to find out whether there exists a sign set $\{s_1, ..., s_t\}$ with the property

$$\sum_{i=1}^{t} s_i\omega_i = 0, \qquad s_1, ..., s_t \in \{-1, 1\} \tag{4.5}$$

This decision problem is known to be NP-complete. NPP can be written as the following quadratic optimization problem:

$$\min_{s_1,...,s_{t+1}} \ 0 \qquad \text{s.t.} \ \ s_{t+1} \times \sum_{i=1}^{t} s_i\omega_i = 0, \quad s_1^2 = \cdots = s_{t+1}^2 = 1,$$

where s_{t+1} is a new slack variable, which is either -1 or 1 and has been introduced to make the first constraint of the above optimization problem quadratic. By defining n as $t+1$ and $\mathbf{x}$ as $\left[\begin{array}{cccc} s_1 & s_2 & \cdots & s_{t+1} \end{array}\right]$, the above optimization problem reduces to:

$$\min_{\mathbf{x}} \ 0 \qquad \text{s.t.} \ \ \sum_{i=1}^{n-1} x_i x_n \omega_i \leq 0, \quad \sum_{i=1}^{n-1} x_i x_n(-\omega_i) \leq 0, \quad x_1^2 = \cdots = x_n^2 = 1$$

Since NPP is NP-hard, solving the above optimization problem is NP-hard as well. On the other hand, the generalized weighted graph for the above optimization problem has the following form: node n is connected to node i with the weight set $\{\omega_i, -\omega_i\}$ for $i = 1, ..., n-1$. Hence, optimization over this acyclic graph is NP-hard. ■

Theorem 3 states that optimization over a very sparse generalized weighted graph (acyclic graph with only two elements in each weight set) is still hard unless the weight sets are sign definite. However, it will be shown in the next subsection that the SDP relaxation always has a rank-2 solution for this type of graph, which may be used to find an approximate solution to the original problem.

4.1. Low-Rank Solution for SDP Relaxation. Suppose that the conditions stated in Theorem 2 do not hold. The SDP relaxation may still be exact (depending on the coefficients of the optimization problem (2.1)), in which case the relaxation has a rank-1 solution X^*. A question arises as to whether the rank of X^* is yet small whenever the relaxation is inexact. The objective of this subsection is to address this problem in two important scenarios. Given the graph $\mathcal{G}$ and the parameters $\mathbf{x}, \mathbf{y}, \mathbf{z}$ introduced earlier, consider the optimization problem

$$\min_{\mathbf{x}\in\mathcal{R}^n} \ g_0(\mathbf{y}, \mathbf{z}) \qquad \text{s.t.} \ \ g_j(\mathbf{y}, \mathbf{z}) \leq 0, \quad j = 1, 2, ..., m \tag{4.6}$$

for arbitrary functions $g_i(\cdot, \cdot)$, $i = 0, 1, ..., m$. The difference between the above optimization problem and (2.1) is that the functions $g_i(\cdot, \cdot)$'s may not be increasing in $\mathbf{z}$. In line with the technique used in Section 2 for the nonconvex optimization problem (2.1), an SDP relaxation can be defined for the above optimization problem. As expected, this relaxation may not have a rank-1 solution, in which case the relaxation is not exact. Nevertheless, it is beneficial to find out how small the rank of an optimal solution of this relaxation could be. This problem will be addressed next for an acyclic graph $\mathcal{G}$.

Theorem 4. *Assume that the graph $\mathcal{G}$ is acyclic. The SDP relaxation for the optimization problem (4.6) always has a solution X^* whose rank is at most 2.*

Proof: The SDP relaxation for the optimization problem (4.6) is as follows:

$$\min_{X\in\mathcal{S}^n} \quad g_0(l_1(X),l_2(X)) \quad \text{s.t.} \quad g_j(l_1(X),l_2(X))\le 0 \quad j=1,...,m, \quad X\succeq 0 \tag{4.7}$$

This is indeed a real-valued SDP relaxation. One can consider a complex-valued SDP relaxation as

$$\min_{\tilde{X}\in\mathcal{H}^n} \quad g_0(l_1(\tilde{X}),l_2(\tilde{X})) \quad \text{s.t.} \quad g_i(l_1(\tilde{X}),l_2(\tilde{X}))\le 0 \quad j=1,...,m, \quad \tilde{X}\succeq 0 \tag{4.8}$$

where its matrix variable, denoted as $\tilde{X}$, is complex. Observe that $l_1(\tilde{X})=l_1(\text{Re}\{\tilde{X}\})$ and $l_2(\tilde{X})=l_2(\text{Re}\{\tilde{X}\})$ for every arbitrary Hermitian matrix $\tilde{X}$, due to the fact that the edge weights of the graph $\mathcal{G}$ are all real. This implies that the real and complex SDP relaxations have the same optimal objective value (note that $\text{Re}\{\tilde{X}\}\succeq 0$ if $\tilde{X}\succeq 0$). In particular, if $\tilde{X}^*$ denotes an optimal solution of the complex SDP relaxation, $\text{Re}\{\tilde{X}^*\}$ will be an optimal solution of the real SDP relaxation. As will be shown later in Theorem 7, the optimization problem (4.8) has a rank-1 solution $\tilde{X}^*$. Therefore, $\tilde{X}^*$ can be decomposed as $(\tilde{\mathbf{x}}^*)(\tilde{\mathbf{x}}^*)^H$ for some complex vector $\tilde{\mathbf{x}}^*$. Now, one can write:

$$\text{Re}\{\tilde{X}^*\}=\text{Re}\{\tilde{\mathbf{x}}\}\text{Re}\{\tilde{\mathbf{x}}\}^H+\text{Im}\{\tilde{\mathbf{x}}\}\text{Im}\{\tilde{\mathbf{x}}\}^H$$

Hence, $\text{Re}\{\tilde{X}^*\}$ is a real-valued matrix with rank at most 2 (as it is the sum of two rank-1 matrices), which is also a solution of the real SDP relaxation. ■

Theorem 4 states that the SDP relaxation of the general optimization problem (4.6) always has a rank 1 or 2 solution if its sparsity can be captured by an acyclic graph. This result makes no assumptions on the monotonicity of the functions $g_j(\cdot,\cdot)$'s. Note that the SDP relaxation for the optimization problem (4.6) may not have a unique solution, but a solution with rank at most 2 may be found using the constructive proof developed in the theorem.

If the functions $g_j(\cdot,\cdot)$'s are convex, then the SDP relaxation becomes a convex program. In this case, a low-rank solution X^* can be found with an arbitrary accuracy in polynomial time. If X^* has rank-1, then the relaxation is exact. Otherwise, X^* has rank 2 from which an approximate rank-1 solution may be found by making the smallest nonzero eigenvalue of X^* equal to 0. A more powerful strategy is to force the undesirable nonzero eigenvalue towards zero by penalizing the objective function of the SDP relaxation via a regularization term such as $\alpha\times\text{trace}\{X\}$ for an appropriate value of α. The graph of the penalized SDP relaxation is still acyclic and therefore the penalized optimization problem will have a rank-1 or 2 solution. Since X^* has only one undesirable eigenvalue that needs to be eliminated (converted to zero), the wealth of results in the literature of compressed sensing justifies that the above penalization technique might be an effective heuristic method.

Theorem 4 studies the SDP relaxation for only acyclic graphs. Partial results will be provided below for cyclic graphs.

THEOREM 5. *Assume that $\mathcal{G}$ is a weakly-cyclic bipartite graph, and that*

$$\sigma_{ij}\neq 0 \qquad \forall (i,j)\in\mathcal{O}_1\cup\mathcal{O}_2\cup\cdots\cup\mathcal{O}_p$$

The SDP relaxation (2.3) for the optimization problem (2.1) in the real-valued case $\mathcal{D}=\mathcal{R}$ has a solution X^* whose rank is at most 2.

Proof: Consider the complex-valued SDP relaxation:

$$\min_{\tilde{X}\in\mathcal{H}^n} \quad f_0(l_1(\tilde{X}), l_2(\tilde{X})) \tag{4.9a}$$

$$\text{s.t.} \quad f_j(l_1(\tilde{X}), l_2(\tilde{X})) \leq 0, \quad j = 1, ..., m \tag{4.9b}$$

$$\tilde{X} \succeq 0 \tag{4.9c}$$

As discussed in the proof of Theorem 4, three properties hold:

- The real and complex SDP relaxations have the same optimal objective value.
- If $\tilde{X}^*$ denotes an optimal solution of the complex SDP relaxation, $\text{Re}\{\tilde{X}^*\}$ turns out to be an optimal solution of the real SDP relaxation
- If $\tilde{X}^*$ is positive semidefinite and rank-1, its real part $\text{Re}\{\tilde{X}^*\}$ is positive semidefinite and rank 1 or 2.

Hence, to prove the theorem, it suffices to show that the complex-valued optimization problem (4.9) has a rank-1 solution. Since every cycle of $\mathcal{G}$ has an even number of vertices (as it is bipartite), a diagonal matrix T with entries from the set $\{0, 1, \mathrm{i}\}$ can be designed in such a way that

$$T_{ii} \times T_{jj} = \mathrm{i}, \qquad \forall (i,j) \in \mathcal{G} \tag{4.10}$$

The next step is to change the variable $\tilde{X}$ in the optimization problem (4.9) to $T\bar{X}T^H$, where $\bar{X}$ is a Hermitian matrix variable. Equation (4.10) yields

$$\tilde{X}_{ii} = \bar{X}_{ii}, \qquad \forall i \in \mathcal{G} \tag{4.11a}$$

$$\tilde{X}_{ij} = \alpha_{ij}\bar{X}_{ij}, \qquad \forall (i,j) \in \mathcal{G} \tag{4.11b}$$

where $\alpha_{ij} \in \{-\mathrm{i}, \mathrm{i}\}$. Therefore, by defining $\bar{c}_{ij}^t$ as $\alpha_{ij}c_{ij}^t$, one can write:

$$\text{Re}\{c_{ij}^t\tilde{X}_{ij}\} = \text{Re}\{\bar{c}_{ij}^t\bar{X}_{ij}\} \tag{4.12}$$

for every $t \in \{1, 2..., k\}$. It results from (4.11a) and (4.12) that if the complex-valued SDP relaxation (4.9) is reformulated in terms of $\bar{X}$, its underlying graph looks like $\mathcal{G}$ with the only difference that the weights c_{ij}^t's are replaced by $\bar{c}_{ij}^t$'s. On the other hand, since c_{ij}^t is a real number, $\bar{c}_{ij}^t$ is purely imaginary. Hence, it follows from Theorem 11 (stated later in the paper) that the reformulated complex SDP relaxation has a rank-1 solution $\bar{X}^*$ because its graph is weakly cyclic with purely imaginary weights. Now, $\tilde{X}^* = T\bar{X}^*T^H$ becomes rank one. In other words, the complex SDP relaxation has a rank-1 solution $\tilde{X}^*$. This completes the proof. ■

There are several applications, where the goal is to find a low-rank positive semidefinite matrix X satisfying a set of constraints (such as linear matrix inequalities). Theorems 4 and 5 provide sufficient conditions under which the feasibility problem

$$\begin{aligned} &f_j(l_1(X), l_2(X)) \leq 0, \quad j = 1, ..., m \\ &X \succeq 0, \end{aligned} \tag{4.13}$$

has a low rank solution, where the rank does not depend on the size of the problem.

5. Complex-Valued Optimization Problems. In this section, the optimization problem (2.1) will be studied in the complex-valued case $\mathcal{D} = \mathcal{C}$. Several scenarios will be explored below.

5.1. Acyclic Graph with Complex Edge Weights. Consider the case where each edge weight set is complex and sign definite with respect to $\mathcal{C}$.

THEOREM 6. *The relations $f^*_{SOCP} = f^*_{r\text{-}SDP} = f^*_{SDP} = f^*$ hold in the complex-valued case $\mathcal{D} = \mathcal{C}$, provided that the graph $\mathcal{G}$ is acyclic and the weight set $\{c^1_{ij}, c^2_{ij}, ..., c^k_{ij}\}$ is sign definite with respect to $\mathcal{C}$ for all $(i,j) \in \mathcal{G}$.*

Proof: The decomposition technique developed in [25] will be deployed to prove this theorem. Similar to Theorem 2, it is enough to show that $f^* \leq f^*_{\text{SOCP}}$. To this end, consider an arbitrary feasible solution of the optimization problem (2.5), denoted as X. Given an edge $(i,j) \in \mathcal{G}$, since the set $\{c^1_{ij}, c^2_{ij}, ..., c^k_{ij}\}$ is sign definite, it follows from the hyperplane separation theorem that there exists a nonzero real vector $(\alpha_{ij}, \beta_{ij})$ such that

$$\text{Re}\{c^t_{ij}(\alpha_{ij} + \beta_{ij}\text{i})\} = \text{Re}\{c^t_{ij}\}\alpha_{ij} - \text{Im}\{c^t_{ij}\}\beta_{ij} \leq 0 \tag{5.1}$$

for every $t \in \{1, 2, ..., k\}$. On the other hand, (2.5c) yields

$$|X_{ij}| \leq \sqrt{X_{ii}}\sqrt{X_{jj}}, \qquad \forall (i,j) \in \mathcal{G} \tag{5.2}$$

Consider the function

$$\left|X_{ij} + \gamma_{ij}(\alpha_{ij} + \beta_{ij}\text{i})\right|^2 - X_{ii}X_{jj}$$

in which γ_{ij} is an unknown real number. This function is negative at $\gamma = 0$ (because of (5.2)) and positive at $\gamma = +\infty$. Hence, due to the continuity of this function, there exists a positive number γ_{ij} such that

$$\left|X_{ij} + \gamma_{ij}(\alpha_{ij} + \beta_{ij}\text{i})\right|^2 = X_{ii}X_{jj} \tag{5.3}$$

Define θ_{ij} as the phase of the complex number $X_{ij} + \gamma_{ij}(\alpha_{ij} + \beta_{ij}\text{i})$. A set of angles $\{\theta_1, \theta_2, ..., \theta_n\}$ can be found iteratively by exploiting the tree topology of the graph $\mathcal{G}$ in such a way that

$$\theta_i - \theta_j = \theta_{ij}, \quad \forall (i,j) \in \mathcal{G} \tag{5.4}$$

Define the vector $\mathbf{x}$ as

$$\begin{bmatrix} \sqrt{X_{11}}e^{-\theta_1\text{i}} & \sqrt{X_{22}}e^{-\theta_2\text{i}} & \cdots & \sqrt{X_{nn}}e^{-\theta_n\text{i}} \end{bmatrix}^H \tag{5.5}$$

Using (5.1), (5.3) and (5.4), one can write:

$$\begin{aligned} \text{Re}\{c^t_{ij}x_ix^H_j\} &= \text{Re}\left\{c^t_{ij}\sqrt{X_{ii}}\sqrt{X_{jj}}e^{(\theta_i-\theta_j)\text{i}}\right\} = \text{Re}\left\{c^t_{ij}\sqrt{X_{ii}}\sqrt{X_{jj}}e^{\theta_{ij}\text{i}}\right\} \\ &= \text{Re}\big\{c^t_{ij}(X_{ij} + \gamma_{ij}(\alpha_{ij} + \beta_{ij}\text{i}))\big\} \\ &= \text{Re}\{c^t_{ij}X_{ij}\} + \gamma_{ij}\text{Re}\big\{c^t_{ij}(\alpha_{ij} + \beta_{ij}\text{i})\big\} \leq \text{Re}\{c^t_{ij}X_{ij}\} \end{aligned}$$

for every $t \in \{1, 2, ..., k\}$. Having shown the above relation, the rest of the proof is in line with the proof of Theorem 2. More precisely, the above inequality implies that

$$l_1\left(\mathbf{x}\mathbf{x}^H\right) = l_1(X), \qquad l_2\left(\mathbf{x}\mathbf{x}^H\right) \leq l_2(X)$$

and therefore

$$f_j(\mathbf{y}, \mathbf{z}) \leq f_i(l_1(X), l_2(X)), \qquad j = 0, 1, ..., m$$

where $\mathbf{y} = l_1\left(\mathbf{x}\mathbf{x}^H\right)$ and $\mathbf{z} = l_2\left(\mathbf{x}\mathbf{x}^H\right)$. Hence, $\mathbf{x}$ is a feasible point of the optimization problem (2.1) whose corresponding objective value is smaller than or equal to the objective value for the feasible point X of the optimization problem (2.5). Consequently, $f^* \leq f^*_{\text{SOCP}}$. This completes the proof. ■

The quadratically-constrained quadratic program (2.7) is a special case of the optimization problem (2.1). Hence, the SDP relaxation is tight for this QCQP problem if $\mathcal{G}$ is acyclic with sign definite weight sets. This result improves upon the result developed in [4] by removing the assumption $M_0 \succeq 0$ (see Section 2.3).

Corollary 2. *The relations $f^*_{SOCP} = f^*_{r\text{-}SDP} = f^*_{SDP} = f^*$ hold in the complex-valued case $\mathcal{D} = \mathcal{C}$ if the graph $\mathcal{G}$ is acyclic and $k \leq 2$.*

Proof: The proof is an immediate consequence of Theorem 6 and the fact that every complex set with one or two elements is sign definite. ■

Corollary 2 states that the optimization problem (2.1) in the complex-valued case can be solved through three relaxations if its structure can be captured by an acyclic graph with at most two weights on each of its edges.

5.2. Weakly Cyclic Graph with Real Edge Weights. It is shown in the preceding subsection that the SDP relaxation is exact, provided $\mathcal{G}$ is acyclic and each weight set is sign definite with respect to $\mathcal{C}$. This result requires the assumption of monotonicity of $f_j(\mathbf{y}, \mathbf{z})$ in $\mathbf{z}$ for $j = 0, 1, ..., m$. The first objective of this part is to show that this assumption is not needed as long as the weight sets are real. To this end, consider the optimization problem

$$\min_{\mathbf{x} \in \mathcal{C}^n} \quad g_0(\mathbf{y}, \mathbf{z}) \qquad \text{s.t.} \quad g_j(\mathbf{y}, \mathbf{z}) \leq 0, \quad j = 1, 2, ..., m \tag{5.6}$$

for arbitrary functions $g_i(\cdot, \cdot)$, $i = 0, 1, ..., m$. The difference between the above optimization problem and (2.1) is that the functions $g_j(\cdot, \cdot)$'s may not be increasing in $\mathbf{z}$. One can derive the SDP, reduced SDP and SOCP relaxations for the above optimization problem by replacing $f_0, ..., f_m$ with $g_0, ..., g_m$ in (2.3)-(2.5). This part aims to investigate the case when the edge weights are all real numbers, while the unknown parameter $\mathbf{x}$ is complex.

Theorem 7. *Consider the complex-valued case $\mathcal{D} = \mathcal{C}$ and assume that the edge weights of $\mathcal{G}$ are all real numbers. The SDP, reduced SDP and SOCP relaxations associated with the optimization problem (5.6) are all exact if the graph $\mathcal{G}$ is acyclic.*

Proof: It is straightforward to show that every real set is sign definite with respect to $\mathcal{C}$. Therefore, the edge weight sets of $\mathcal{G}$ are all sign definite. Let X denote an arbitrary feasible point of the SOCP relaxation. Define $(\alpha_{ij}, \beta_{ij})$ as $(0, 1)$ for every $(i, j) \in \mathcal{G}$. Then,

$$\text{Re}\{c^t_{ij}(\alpha_{ij} + \beta_{ij}\text{i})\} = \text{Re}\{c^t_{ij}\}\alpha_{ij} - \text{Im}\{c^t_{ij}\}\beta_{ij} = 0$$

for every $t \in \{1, ..., k\}$ (note that $c^t_{ij} \in \mathcal{R}$ by assumption). Following the proof of Theorem 6, define $\mathbf{x}$ as the vector given in (5.5). Therefore,

$$\text{Re}\{c^t_{ij} x_i x_j^H\} = \text{Re}\{c^t_{ij} X_{ij}\} + \gamma_{ij}\text{Re}\big\{c^t_{ij}(\alpha_{ij} + \beta_{ij}\text{i})\big\} = \text{Re}\{c^t_{ij} X_{ij}\}$$

Now, the rest of the proof is in line with the proof of Theorem 6. More precisely,

$$l_1\left(\mathbf{x}\mathbf{x}^H\right) = l_1(X), \qquad l_2\left(\mathbf{x}\mathbf{x}^H\right) = l_2(X)$$

Given an arbitrary feasible point X for the SOCP relaxation, the above equality implies that $\mathbf{x}$ is a feasible point of the original optimization problem (5.6) and that X and $\mathbf{x}$ both give rise to the same objective value. This completes the proof. ■

Consider the general optimization problem (5.6) in the case when $\mathcal{G}$ is acyclic with real edge weights. As discussed before, the associated SDP relaxation may not be tight if its variable $\mathbf{x}$ is restricted to real numbers. However, Theorem 7 shows that the relaxation is exact if $\mathbf{x}$ is a complex-valued variable. In what follows, the results of Theorem 7 will be generalized to cyclic graphs for the optimization problem (2.1).

THEOREM 8. *Assume that $\{c_{ij}^1, ..., c_{ij}^k\}$ is a positive or negative real set for every $(i,j) \in \mathcal{G}$. The relations $f_{r\text{-}SDP}^* = f_{SDP}^* = f^*$ hold for the optimization problem (2.1) in the complex-valued case $\mathcal{D} = \mathcal{C}$ if the graph $\mathcal{G}$ is bipartite and weakly cyclic.*

Proof: Following the proof of Theorem 5, consider the matrix T defined in (4.10), and change the variable X in the SDP relaxation to $\bar{X}$ through the relation $X = T\bar{X}T^H$. This implies that the real weights c_{ij}^t's will change to the imaginary weights $\bar{c}_{ij}^t$'s defined in the proof of Theorem 5. Hence, the reformulated SDP optimization problem is over a graph with purely imaginary weights. The existence of a rank-1 solution $\bar{X}^*$ (and hence a rank-1 matrix X^*) is guaranteed by Theorem 10. ■

Note that the SOCP relaxation may not be exact under the assumptions of Theorem 8. As a direct application of this theorem, the class of quadratic optimization problems proposed later in Example 3 is polynomial-time solvable with an arbitrary precision.

5.3. Cyclic Graph with Real and Imaginary Edge Weights. In this part, there is no specific assumption on the topology of the graph $\mathcal{G}$, but it is assumed that each edge weight is either real or purely imaginary. The definition of the edge sign σ_{ij} introduced earlier for real-valued weight sets can be extended as follows:

$$\sigma_{ij} = \begin{cases} 1 & \text{if} \quad c_{ij}^1, ..., c_{ij}^k \geq 0 \\ -1 & \text{if} \quad c_{ij}^1, ..., c_{ij}^k \leq 0 \\ \mathrm{i} & \text{if} \quad c_{ij}^1 \times \mathrm{i}, ..., c_{ij}^k \times \mathrm{i} \geq 0 \\ -\mathrm{i} & \text{if} \quad c_{ij}^1 \times \mathrm{i}, ..., c_{ij}^k \times \mathrm{i} \leq 0 \\ 0 & \text{otherwise} \end{cases}, \qquad \forall (i,j) \in \mathcal{G}$$

By convention, $\sigma_{ij} = -1$ if $c_{ij}^{(1)} = \cdots = c_{ij}^{(k)} = 0$. Define also σ_{ji} as σ_{ij}^H for every $(i,j) \in \mathcal{G}$. The parameter σ_{ij} being nonzero implies that the elements of each edge weight set $\{c_{ij}^1, ..., c_{ij}^k\}$ are homogeneous in type (real or imaginary) and in sign (positive or negative).

THEOREM 9. *The relations $f_{SOCP}^* = f_{r\text{-}SDP}^* = f_{SDP}^* = f^*$ hold for Optimization (2.1) in the complex-valued case $\mathcal{D} = \mathcal{C}$ with real and purely imaginary edge weight sets if*

$$\sigma_{ij} \neq 0, \qquad \forall (i,j) \in \mathcal{G} \tag{5.7a}$$

$$\prod_{(i,j)\in\vec{\mathcal{O}}_r} \sigma_{ij} = (-1)^{|\mathcal{O}_r|}, \qquad \forall r \in \{1, ..., p\} \tag{5.7b}$$

Proof: Consider an arbitrary feasible point X for the SOCP relaxation. Choose a spanning tree of $\mathcal{G}$ and denote it as $\mathcal{T}$. In light of (5.7a), n numbers $\sigma_1, \sigma_2, ..., \sigma_n$ belonging to the set $\{\pm 1, \pm \mathrm{i}\}$ can be iteratively designed with the property that

$$\sigma_i \sigma_j^H = -\sigma_{ij}, \qquad \forall (i,j) \in \mathcal{T} \tag{5.8}$$

This relation together with (5.7b) yields that $\sigma_i \sigma_j^H = -\sigma_{ij}$ for every $(i,j) \in \mathcal{G}$. Now, define $\mathbf{x}$ as

$$\begin{bmatrix} \sigma_1^H \sqrt{X_{11}} & \sigma_2^H \sqrt{X_{22}} & \cdots & \sigma_n^H \sqrt{X_{nn}} \end{bmatrix}^H \tag{5.9}$$

As before, it can be shown that $l_1\left(\mathbf{x}\mathbf{x}^H\right) = l_1(X)$ and $l_2\left(\mathbf{x}\mathbf{x}^H\right) \leq l_2(X)$. Therefore, it holds that $f_j(\mathbf{y},\mathbf{z}) \leq f_j(l_1(X), l_2(X))$ for $j = 0, 1, ..., m$, where $\mathbf{y} = l_1\left(\mathbf{x}\mathbf{x}^H\right)$ and $\mathbf{z} = l_2\left(\mathbf{x}\mathbf{x}^H\right)$. This means that corresponding to every feasible point X of the SOCP relaxation, the original optimization has a feasible point $\mathbf{x}$ with a lower or equal objective value. Therefore, $f^* \leq f^*_{\text{SOCP}}$. The proof is completes by combining this inequality with $f^*_{\text{SOCP}} \leq f^*_{\text{r-SDP}} \leq f^*_{\text{SDP}} \leq f^*$. ■

5.4. Weakly Cyclic Graph with Imaginary Edge Weights. If $\mathcal{G}$ has at least one odd cycle whose edge weight sets consist only of imaginary numbers, then the conditions given in Theorem 9 are violated. The reason is that the product of an odd number of imaginary numbers (edge signs) can never become a real number. The high-level goal of this part is to show that the SDP relaxation can still be tight in presence of such cycles, while the SOCP relaxation is not guaranteed to be exact. In this subsection, we assume that $\mathcal{G}$ is weakly cyclic.

To proceed with the paper, a new SOCP relaxation needs to be introduced. This optimization problem assigns one real scalar variable q_i to every vertex $i \in \mathcal{G}$ and one 2×2 block matrix variable

$$\begin{bmatrix} U(\mathcal{G}_s) & V(\mathcal{G}_s) \\ V(\mathcal{G}_s)^H & W(\mathcal{G}_s) \end{bmatrix}$$

to every subgraph $\mathcal{G}_s \in \Omega$, where $U(\mathcal{G}_s), W(\mathcal{G}_s) \in \mathcal{S}^{|\mathcal{G}_s|}$ and $V(\mathcal{G}_s) \in \mathcal{R}^{|\mathcal{G}_s| \times |\mathcal{G}_s|}$. Let U, V and W denote the parameter sets $\{U(\mathcal{G}_s) \mid \forall \mathcal{G}_s \in \Omega\}$, $\{V(\mathcal{G}_s) \mid \forall \mathcal{G}_s \in \Omega\}$ and $\{W(\mathcal{G}_s) \mid \forall \mathcal{G}_s \in \Omega\}$, respectively.

NOTATION 7. *For every $\mathcal{G}_s \in \Omega$, we arrange the elements in the vertex set of $\mathcal{G}_s$ in an increasing order. Then, we index the rows and columns of each of the matrices $U(\mathcal{G}_s), V(\mathcal{G}_s), V(\mathcal{G}_s)$ according to the ordered vertex set of $\mathcal{G}_s$. For example, if $\mathcal{G}_s$ has three vertices $5, 7, 1$, the ordered set becomes $\{1, 5, 7\}$, and therefore the three rows of $U(\mathcal{G}_s)$ are called row 1, row 5 and row 7. As an example, $U_{17}(\mathcal{G}_s)$ refers to the last entry on the first row of $U(\mathcal{G}_s)$.*

For every $r \in \{1, 2, ..., p\}$, let μ_r denote the largest index in the vertex set of $\mathcal{O}_r$. Define $\mathbf{q}$ as the vector corresponding to the set $\{q_1, ..., q_n\}$. Recall that

$$l_2(\mathbf{x}\mathbf{x}^H) = \left\{\text{Re}\left\{c^t_{ij} x_i x_j^H\right\} \;\middle|\; \forall (i,j) \in \mathcal{G},\; t \in \{1, ..., k\}\right\}$$

Define $\bar{l}(V)$ as a vector obtained from $l_2(\mathbf{x}\mathbf{x}^H)$ by replacing each entry $\text{Re}\left\{c^t_{ij} x_i x_j^H\right\}$ with a new term $\text{Im}\{c^t_{ij}\} \times (V_{ij}(\mathcal{G}_s) - V_{ji}(\mathcal{G}_s))$, where $\mathcal{G}_s$ denotes the unique subgraph in Ω containing the edge (i,j) (the uniqueness of such subgraph is guaranteed by the weakly cyclic property of $\mathcal{G}$).

Expanded SOCP: This optimization problem is defined as

$$\min_{\mathbf{q},U,V,W} \quad f_0(\mathbf{q}, \bar{l}(V)) \tag{5.10a}$$

subject to:

$$f_j(\mathbf{q}, \bar{l}(V)) \leq 0, \qquad j = 1, 2, ..., m \tag{5.10b}$$

$$U_{ii}(\mathcal{G}_s) + W_{ii}(\mathcal{G}_s) = q_i, \quad \forall \mathcal{G}_s \in \Omega,\; i \in \mathcal{G}_s \tag{5.10c}$$

$$\begin{bmatrix} U_{ii}(\mathcal{G}_s) & V_{ij}(\mathcal{G}_s) \\ V_{ij}(\mathcal{G}_s) & W_{jj}(\mathcal{G}_s) \end{bmatrix} \succeq 0,\; \forall \mathcal{G}_s \in \Omega,\; (i,j) \in \mathcal{G}_s \tag{5.10d}$$

$$\begin{bmatrix} U_{jj}(\mathcal{G}_s) & V_{ji}(\mathcal{G}_s) \\ V_{ji}(\mathcal{G}_s) & W_{ii}(\mathcal{G}_s) \end{bmatrix} \succeq 0,\; \forall \mathcal{G}_s \in \Omega,\; (i,j) \in \mathcal{G}_s \tag{5.10e}$$

$$W_{\mu_r \mu_r}(\mathcal{O}_r) = 0, \qquad r = 1, 2, ..., p \tag{5.10f}$$

Similar to the argument made for the SOCP relaxation (2.5), the above optimization problem is in the form of an SOCP program because its constraints (5.10d) and (5.10e) can be replaced by linear and norm constraints. Moreover, this optimization problem can be regarded as an expanded version of the SOCP relaxation (2.5). Denote the optimal objective value of this optimization problem as $f^*_{\text{e-SOCP}}$.

THEOREM 10. *Consider the optimization problem (2.1) in the complex-valued case $\mathcal{D} = \mathcal{C}$, and assume that the graph $\mathcal{G}$ is weakly cyclic with only purely imaginary edge weights. The following statements hold:*

i) The expanded SOCP is a relaxation for the optimization problem (2.1), meaning that $f^*_{e\text{-}SOCP} \leq f^*$.

ii) The expanded SOCP relaxation is exact if and only if it has a solution $(\mathbf{q}^*, U^*, V^*, W^*)$ for which all 2×2 matrices given in (5.10d) and (5.10e) have rank 1.

iii) $f^*_{SOCP} \leq f^*_{e\text{-}SOCP}$.

iv) $f^*_{e\text{-}SOCP} \leq f^*_{r\text{-}SDP}$.

v) The relations $f^*_{e\text{-}SOCP} = f^*_{r\text{-}SDP} = f^*_{SDP} = f^*$ hold if $\sigma_{ij} \neq 0$ for every $(i,j) \in \mathcal{G}$.

Proof: Since the proof is long and involved, it has been moved to the appendix. ■

Assume that the graph $\mathcal{G}$ is weakly cyclic and its edge weights are all imaginary numbers. Theorem 10 shows that $f^*_{\text{SOCP}} \leq f^*_{\text{e-SOCP}} \leq f^*_{\text{r-SDP}} \leq f^*_{\text{SDP}} \leq f^*$, and that the relations $f^*_{\text{e-SOCP}} = f^*_{\text{r-SDP}} = f^*_{\text{SDP}} = f^*$ hold if each edge weight set has homogeneous elements ($\sigma_{ij} = \mathrm{i}$ or $-\mathrm{i}$). Note that the SOCP relaxation may not be exact, and one needs to use the expanded SOCP relaxation in this case. Interestingly, this result makes no assumption on the signs of the edges belonging to the same cycle in the cycle basis (unlike (5.7b)).

Although Theorem 10 deals with imaginary coefficients, some of the results derived in this paper for complex/real optimization problems with real coefficients are based on this powerful theorem. This is due to the fact that real numbers may be converted to imaginary numbers through a simple multiplication.

5.5. General Graph with Complex Edge Weight Sets. Given an arbitrary subgraph $\tilde{\mathcal{G}}_s$ of the graph $\mathcal{G}$, four important types will be defined for this subgraph in the following:

- **Type I:** $\tilde{\mathcal{G}}_s$ is acyclic with complex weight sets such that $\{c^1_{ij}, ..., c^k_{ij}\}$ is sign definite with respect to $\mathcal{C}$ for every $(i,j) \in \tilde{\mathcal{G}}_s$.
- **Type II:** $\tilde{\mathcal{G}}_s$ is weakly cyclic with imaginary weight sets and nonzero sign σ_{ij} (i.e., $\sigma_{ij} = \pm\mathrm{i}$) for every $(i,j) \in \tilde{\mathcal{G}}_s$.
- **Type III:** $\tilde{\mathcal{G}}_s$ is bipartite and weakly cyclic with the property that $\{c^1_{ij}, ..., c^k_{ij}\}$ is a real weight set with nonzero sign σ_{ij} (i.e., $\sigma_{ij} = \pm 1$) for every $(i,j) \in \tilde{\mathcal{G}}_s$.
- **Type IV:** $\tilde{\mathcal{G}}_s$ has only real and imaginary weights with the property that

$$\sigma_{ij} \neq 0, \qquad \forall (i,j) \in \tilde{\mathcal{G}}_s \tag{5.11a}$$

$$\prod_{(i,j)\in\vec{\mathcal{O}}_r} \sigma_{ij} = (-1)^{|\mathcal{O}_r|}, \quad \forall \mathcal{O}_r \in \{\mathcal{O}_1, ..., \mathcal{O}_p\} \cap \tilde{\mathcal{G}}_s \tag{5.11b}$$

By assuming $\tilde{\mathcal{G}}_s = \mathcal{G}_s$, it follows from the theorems developed earlier in this paper that the SDP relaxation is exact for the optimization problem (2.1) if $\mathcal{G}$ is Type I, II, III or IV. In this part, the objective is to show that the relaxation is still tight if $\mathcal{G}$

can be decomposed into a number of Type I-IV subgraphs in an acyclic way.

Theorem 11. *Assume that $\mathcal{G}$ can be decomposed as the union of a number of edge-disjoint subgraphs $\tilde{\mathcal{G}}_1, ..., \tilde{\mathcal{G}}_\omega$ in such a way that*

i) $\tilde{\mathcal{G}}_s$ is Type I, II, III or IV for every $s \in \{1, ..., \omega\}$.

ii) The cycle $\mathcal{O}_r$ is entirely inside one of the subgraphs $\tilde{\mathcal{G}}_1, ..., \tilde{\mathcal{G}}_\omega$ for every $r \in \{1, ..., p\}$.

Then, the relations $f^*_{r\text{-}SDP} = f^*_{SDP} = f^*$ hold for the optimization problem (2.1) in the complex-valued case $\mathcal{D} = \mathcal{C}$.

Proof: Given an arbitrary solution X^* of the reduced SDP relaxation, consider the optimization problem:

$$\min_{X} \quad f_0(l_1(X), l_2(X)) \tag{5.12a}$$

$$\text{s.t.} \quad f_j(l_1(X), l_2(X)) \leq 0, \qquad j = 1, ..., m \tag{5.12b}$$

$$X\{\mathcal{O}_r\} \succeq 0, \qquad r = 1, ..., p \tag{5.12c}$$

$$X\{(i,j)\} \succeq 0, \qquad \forall (i,j) \in \mathcal{G} \tag{5.12d}$$

$$X_{ii} = X^*_{ii}, \qquad \forall i \in \mathcal{G} \tag{5.12e}$$

$$X_{ij} = X^*_{ij}, \qquad \forall (i,j) \in \mathcal{G} \backslash \tilde{\mathcal{G}}_s \tag{5.12f}$$

for any subgraph $\tilde{\mathcal{G}}_s \in \{\tilde{\mathcal{G}}_1, ..., \tilde{\mathcal{G}}_\omega\}$ ($\mathcal{G} \backslash \tilde{\mathcal{G}}_s$ means to exclude the edges of $\tilde{\mathcal{G}}_s$ from $\mathcal{G}$). The above optimization problem is obtained from the reduced SDP relaxation by setting certain entries of the variable X equal to their optimal values extracted from X^*. More precisely, this optimization problem aims to optimize the off-diagonal entries of X corresponding to the edges of $\tilde{\mathcal{G}}_s$. It is obvious that $X = X^*$ is a solution of the above optimization problem. On the other hand, since $\tilde{\mathcal{G}}_s$ is Type I, II, III or IV, it follows from Theorems 6, 8, 9 and 10 that the above optimization problem has an optimal solution for which the matrices given in (5.12c) and (5.12d) become rank-1 for every (i,j) and $\mathcal{O}_r$ belonging to $\tilde{\mathcal{G}}_s$. By making this argument on all subgraphs $\tilde{\mathcal{G}}_1, ..., \tilde{\mathcal{G}}_\omega$ and using Property (ii) stated in the theorem, one can design a solution for the reduced SDP relaxation for which condition (3.1) holds. Therefore, the SDP and reduced SDP relaxations will both be exact in light of Theorem 1. ■

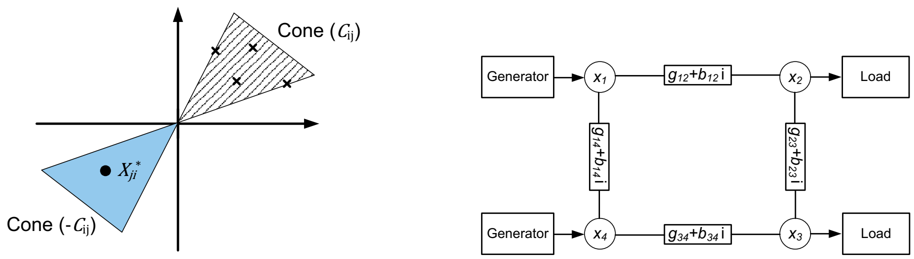
5.6. Roles of Graph Topology and Sign Definite Weight Sets. Part (iii) of Theorem 1 states that the optimization problem (2.1) is polynomial-time solvable with an arbitrary accuracy if the SOCP relaxation (2.5) has a solution X^* satisfying two conditions:

1) $X^*\{(i,j)\}$ has rank 1 for every $(i,j) \in \mathcal{G}$.
2) $\sum \measuredangle X^*_{ij}$ is equal to zero for every $r \in \{1, 2, ..., p\}$, where the sum is taken over all directed edges (i,j) of the oriented cycle $\vec{\mathcal{O}}_r$.

Since $X^*\{(i,j)\}$ is a 2×2 matrix corresponding to a single edge of the graph, Condition (1) is strongly related to the properties of the edge set $\{c^1_{ij}, ..., c^k_{ij}\}$. In contrast, the graph topology (namely its cycle basis) plays an important role in Condition (2). The goal of this part is to understand how these conditions are satisfied for various graphs studied earlier in the complex-valued case $\mathcal{D} = \mathcal{C}$.

To explore Condition (1), consider an edge $(i,j) \in \mathcal{G}$. Observe that the set $\{c^1_{ij}, ..., c^k_{ij}\}$ can be mapped into k vectors

$$\vec{c}^{\,t}_{ij} = \begin{bmatrix} \text{Re}\{c^t_{ij}\} & \text{Im}\{c^t_{ij}\} \end{bmatrix}^H, \qquad t = 1, 2, ..., k$$



(a) (b)

FIG. 5.1. *(a) This figure shows the cones $\mathcal{C}_{ij}$ and $-\mathcal{C}_{ij}$, in addition to the position of the complex point X^*_{ji}; (b) an example of the power circuit studied in Section 6.*

in $\mathcal{R}^2$. Define the following vector corresponding to X^*_{ji}:

$$\vec{X}^*_{ji} = \left[\begin{array}{cc} \mathrm{Re}\{X^*_{ij}\} & -\mathrm{Im}\{X^*_{ij}\} \end{array}\right]^H$$

Recall that X^*_{ij} plays the role of $(x^*_i)(x^*_j)^H$ whenever the SOCP relaxation is tight. Now, one can write

$$\mathrm{Re}\{c^t_{ij}X^*_{ij}\} = \vec{c}_{ij}^{\,t} \cdot \vec{X}^*_{ji} = |\vec{c}_{ij}^{\,t}||\vec{X}^*_{ji}|\cos(\measuredangle\vec{c}_{ij}^{\,t} - \measuredangle\vec{X}^*_{ji}) \tag{5.13}$$

where "·" stands for *inner product*. Define $\mathcal{C}_{ij}$ as the smallest convex cone in $\mathcal{R}^2$ containing the vectors $\vec{c}_{ij}^{\,1}, ..., \vec{c}_{ij}^{\,k}$. Let $\mathrm{B}\{\mathcal{C}_{ij}\}$ denote the boundary of the cone $\mathcal{C}_{ij}$. The set $\{c^1_{ij}, ..., c^k_{ij}\}$ being sign definite is equivalent to the condition

$$\{\mathcal{C}_{ij} \cap (-\mathcal{C}_{ij})\} \subseteq \mathrm{B}\{\mathcal{C}_{ij}\}, \tag{5.14}$$

meaning that $\mathcal{C}_{ij}$ and its mirror set can have common points only on their boundaries. This fact is illustrated in Figure 5.1(a). Suppose that the weight set $\{c^1_{ij}, ..., c^k_{ij}\}$ is sign definite. Since $f_0, ..., f_m$ are all increasing in $\mathbf{z}$ or equivalently in $\vec{c}_{ij}^{\,t} \cdot \vec{X}^*_{ji}$ for every $(i, j) \in \mathcal{G}$ and $t \in \{1, ..., k\}$, it is easy to verify that (see the proof of Theorem 6):

$$\vec{X}^*_{ji} \in -\mathcal{C}_{ij} \tag{5.15}$$

This property is illustrated in Figure 5.1(a). Moreover, the monotonicity of $f_0, ..., f_m$ forces $|\vec{X}^*_{ij}|$ to have the largest possible value, i.e.,

$$|\vec{X}^*_{ji}| = |X^*_{ij}| = \sqrt{X^*_{ii}X^*_{jj}},$$

which makes $X^*\{(i, j\})$ rank 1. This implies that the sign definiteness of the set $\{c^1_{ij}, ..., c^k_{ij}\}$ guarantees the satisfaction of Condition (1) stated above.

So far, it is shown that $\vec{X}^*_{ji}$ belongs to the cone $-\mathcal{C}_{ij}$. Now, to satisfy Condition (2) required for the exactness of the SOCP relaxation, the sum of the angles of the vectors $\vec{X}^*_{ji}$'s must be zero over each cycle in the cycle basis. This trivially happens in two cases:

- If the graph $\mathcal{G}$ is acyclic, then there is no cycle to be concerned about.
- Consider the cycle $\mathcal{O}_r$ for some $r \in \{1, 2, ..., k\}$. If each cone $\mathcal{C}_{ij}$ is one dimensional for every $(i, j) \in \mathcal{O}_r$, then it suffices to have $\sum \measuredangle(-\mathcal{C}_{ij}) = 0$, where the sum is taken over all directed edges (i, j) of the oriented cycle $\vec{\mathcal{O}}_r$ (note that $\measuredangle(-\mathcal{C}_{ij})$ denotes the angle of the 1-d cone $-\mathcal{C}_{ij}$.)

To understand the merit of the above insights, consider the optimization problem (2.1) in the case when the graph $\mathcal{G}$ is bipartite and each complex weight c_{ij}^t has positive real and imaginary parts for every $(i, j) \in \mathcal{G}$ and $t \in \{1, ..., k\}$. Denote the two disjoint vertex sets of the bipartite graph $\mathcal{G}$ as $\mathcal{S}_1$ and $\mathcal{S}_2$, and with no loss of generality, assume that $i \in \mathcal{S}_1$ and $j \in \mathcal{S}_2$ for every $(i, j) \in \mathcal{G}$. Suppose that the constraints of the optimization problem (2.1) are in such a way that the inequality

$$|\measuredangle x_i^* - \measuredangle x_j^*| \leq \frac{\pi}{2}, \quad \forall (i, j) \in \mathcal{G} \tag{5.16}$$

is satisfied for an optimal solution $\mathbf{x}^*$ of this optimization problem. For instance, as will be discussed in the next section, phasor voltages in a power network are forced to satisfy the above condition due to the operational constraints of such networks. Under this circumstance, one can modify the SOCP relaxation by including the extra constraints $\text{Re}\{X_{ij}\} \geq 0$, $\forall (i, j) \in \mathcal{G}$, to account for (5.16). Since $\mathcal{C}_{ij}$ is a subset of a first quadrant in $\mathcal{R}^2$, $\{c_{ij}^1, ..., c_{ij}^k\}$ is a sign definite set and therefore Condition (1) holds. Let X^* denote a solution of the modified SOCP problem. Following the argument leading to (5.15), it can be shown that X_{ji}^* is a negative imaginary number for every $(i, j) \in \mathcal{G}$, meaning that $\vec{X}_{ji}^*$ has the maximum possible angle with respect to all vectors $\vec{c}_{ij}^{\,1}, ..., \vec{c}_{ij}^{\,k}$. Since $\mathcal{G}$ is assumed to be bipartite, Condition (2) holds as a result of this property. Hence, the SOCP, reduced SDP and SDP relaxations are all exact for such graphs $\mathcal{G}$.

The above insight into Conditions (1) and (2) was based on the SOCP relaxation. The same argument can be made about the expanded SOCP relaxation to understand Theorem 10 for weakly cyclic graphs with imaginary weights, for which the regular SOCP relaxation may not be tight.

6. Application in Power Systems. A majority of real-world optimization problems can be regarded as ‘optimization problems with graph structures”, meaning that each of those problems has an underlying graph structure characterizing a physical system. For example, optimization problems in circuits, antenna systems and communication networks fall within this category. Then, the question of interest is: how does the computational complexity of an optimization problem relate to the structure of the system over which the optimization problem is performed? This question will be explored here in the context of electrical power grids. Assume that the graph $\mathcal{G}$ corresponds to an AC power network, where:

- The power network has $|\mathcal{G}|$ nodes.
- For every $(i, j) \in \mathcal{G}$, nodes i and j are connected to each other in the power network via a transmission line with the admittance $g_{ij} + b_{ij}\text{i}$.
- Each node $i \in \mathcal{G}$ of the network is connected to an external device, which exchanges electrical power with the power network.

Figure 5.1(b) exemplifies a sample power network in which two external devices generate power while the remaining ones consume power. As shown in Figure 6.1(a), each line $(i, j) \in \mathcal{G}$ is associated with four power flows:

- p_{ij}: Active power entering the line from node i
- p_{ji}: Active power entering the line from node j
- q_{ij}: Reactive power entering the line from node i
- q_{ji}: Reactive power entering the line from node j

Note that $p_{ij} + p_{ji}$ and $q_{ij} + q_{ji}$ represent the active and reactive losses incurred in the line. Let x_i denote the complex voltage (phasor) for node $i \in \mathcal{G}$. One can write:

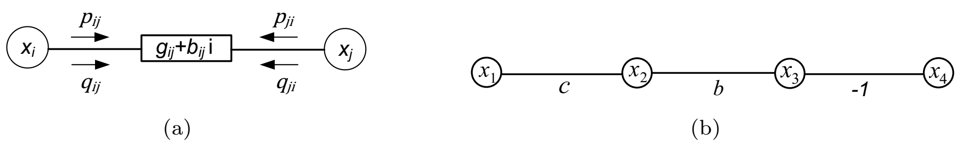


FIG. 6.1. *(a) This figure illustrates that each transmission line has four flows; (b) graph $\mathcal{G}$ corresponding to minimization of $f_0(x_1, x_2)$ given in (7.1).*

$$p_{ij}(\mathbf{x}) = \mathrm{Re}\left\{x_i(x_i - x_j)^H(g_{ij} - b_{ij}\mathrm{i})\right\}, \quad p_{ji}(\mathbf{x}) = \mathrm{Re}\left\{x_j(x_j - x_i)^H(g_{ij} - b_{ij}\mathrm{i})\right\}$$
$$q_{ij}(\mathbf{x}) = \mathrm{Im}\left\{x_i(x_i - x_j)^H(g_{ij} - b_{ij}\mathrm{i})\right\}, \quad q_{ji}(\mathbf{x}) = \mathrm{Im}\left\{x_j(x_j - x_i)^H(g_{ij} - b_{ij}\mathrm{i})\right\}$$

Note that since the flows all depend on $\mathbf{x}$, the argument $\mathbf{x}$ has been added to the above equations (e.g., $p_{ij}(\mathbf{x})$ instead of p_{ij}). The flows $p_{ij}(\mathbf{x})$, $p_{ji}(\mathbf{x})$, $q_{ij}(\mathbf{x})$ and $q_{ji}(\mathbf{x})$ can all be expressed in terms of $|x_i|^2$, $|x_j|^2$ and $\mathrm{Re}\left\{c_{ij}^k x_i x_j^H\right\}$ for $k = 1, 2, 3, 4$, where

$$c_{ij}^1 = -g_{ij} + b_{ij}\mathrm{i}, \quad c_{ij}^2 = -g_{ij} - b_{ij}\mathrm{i}, \quad c_{ij}^3 = b_{ij} + g_{ij}\mathrm{i}, \quad c_{ij}^4 = b_{ij} - g_{ij}\mathrm{i}$$

(note that $\mathrm{Re}\{\alpha x_j x_i^H\} = \mathrm{Re}\{\alpha^H x_i x_j^H\}$ and $\mathrm{Im}\{\alpha x_j x_i^H\} = \mathrm{Re}\{(-\alpha\mathrm{i}) x_i x_j^H\}$ for every value of α). Define

$$\mathbf{p}(\mathbf{x}) = \left\{p_{ij}(\mathbf{x}), p_{ji}(\mathbf{x}) \;\middle|\; \forall (i,j) \in \mathcal{G}\right\}, \quad \mathbf{q}(\mathbf{x}) = \left\{q_{ij}(\mathbf{x}), q_{ji}(\mathbf{x}) \;\middle|\; \forall (i,j) \in \mathcal{G}\right\}$$

Consider the optimization problem

$$\begin{aligned} \min_{\mathbf{x} \in \mathcal{C}^n} \quad & h_0(\mathbf{p}(\mathbf{x}), \mathbf{q}(\mathbf{x}), \mathbf{y}(\mathbf{x})) \\ \text{s.t.} \quad & h_j(\mathbf{p}(\mathbf{x}), \mathbf{q}(\mathbf{x}), \mathbf{y}(\mathbf{x})) \leq 0, \quad j = 1, 2, ..., m \end{aligned} \tag{6.1}$$

for given functions $h_0, ..., h_m$, where $y(\mathbf{x})$ is the vector of $|x_i|^2$'s. Assume for now that the function $h_j(\cdot, \cdot, \cdot)$ accepting three arguments (inputs) is monotonic with respect to its first and second vector arguments. The above optimization problem aims to optimize the flows in a power grid. The constraints of this optimization problem are meant to limit line flows, voltage magnitudes, power delivered to each load, and power supplied by each generator. Observe that $\mathbf{p}(\mathbf{x})$ and $\mathbf{q}(\mathbf{x})$ are both quadratic in $\mathbf{x}$. Since the above optimization problem can be cast as (2.1), the SDP, reduced SDP and SOCP relaxations introduced before can be used to eliminate the effect of quadratic terms. To study under what conditions the relaxations are exact, note that each edge (i, j) of $\mathcal{G}$ has the weight set $\{c_{ij}^1, c_{ij}^2, c_{ij}^3, c_{ij}^4\}$. A customary transmission line is a passive device with nonnegative resistance and inductance, leading to the inequalities $g_{ij} \geq 0$ and $b_{ij} \leq 0$. As a result of this property, the set $\{c_{ij}^1, c_{ij}^2, c_{ij}^3, c_{ij}^4\}$ turns out to be sign definite (see Definition 2). Now, in light of Theorem 11, the relaxations are all exact as long as $\mathcal{G}$ is acyclic.[1]

The optimization of power flows is a fundamental problem, which is solved every 5 to 15 minutes for power grids in practice. This problem, named Optimal Power Flow (OPF), has several variants that are used for different purposes (real-time operation, electricity market, security assessment, etc.).

[1]This result also holds for cyclic power networks with a sufficient number of phase shifters (the graph for a mesh power network with phase shifters can be converted to an acyclic one) [25].

Line flows are restricted in practice to achieve various goals such as avoiding line overheating and guaranteeing the stability of the network [3]. More precisely, it is known that:

i) A thermal limit can be imposed by restricting the line active power flow p_{ij}, the line apparent power flow $|p_{ij} + q_{ij}\mathrm{i}|$ or the line current magnitude. The maximum allowable limits on these parameters can be determined by analyzing the material characteristics of the line. Thermal limits are often imposed on p_{ij}, in practice.

ii) A stability limit may be translated into a constraint on the voltage phase difference across the line, i.e., $|\measuredangle x_i - \measuredangle x_j|$.

These constraints have different implications in the power engineering, but can all be described in terms of $\mathbf{p}(\mathbf{x}), \mathbf{q}(\mathbf{x}), \mathbf{y}(\mathbf{x})$. We will elaborate on this property for the current and angle constraints below:

- *Current constraint:* For every $(i,j) \in \mathcal{G}$, the line current magnitude $|(x_i - x_j)(g_{ij} + b_{ij}\mathrm{i})|$ cannot exceed a maximum number $I_{\max}$. This constraint can be written as:

$$|x_i|^2 + |x_j|^2 - 2\mathrm{Re}\{x_i x_j^H\} \leq \frac{I_{\max}^2}{|g_{ij} + b_{ij}\mathrm{i}|^2} \tag{6.2}$$

- *Angle constraint:* For every $(i,j) \in \mathcal{G}$. the absolute angle difference $|\measuredangle x_i - \measuredangle x_j|$ should not exceed a maximum angle $\theta_{ij}^{\max} \in [0, 90°]$ (due to stability and thermal limits). This constraint can be written as

$$\mathrm{Im}\{x_i x_j^H\} \leq |\tan\theta_{ij}^{\max}| \times \mathrm{Re}\{x_i x_j^H\}$$

or equivalently

$$\begin{aligned} -\tan\theta_{ij}^{\max} \times \mathrm{Re}\{x_i x_j^H\} + \mathrm{Re}\{(+\mathrm{i})\ x_i x_j^H\} \leq 0 \\ -\tan\theta_{ij}^{\max} \times \mathrm{Re}\{x_i x_j^H\} + \mathrm{Re}\{(-\mathrm{i})\ x_i x_j^H\} \leq 0 \end{aligned} \tag{6.3}$$

Since (6.2) and (6.3) are quadratic in $\mathbf{x}$, they can easily be incorporated into the optimization problem (6.1) and its relaxations. However, the edge set $\{c_{ij}^1, c_{ij}^2, c_{ij}^3, c_{ij}^4\}$ should be extended to $\{c_{ij}^1, c_{ij}^2, c_{ij}^3, c_{ij}^4, -1, \mathrm{i}, -\mathrm{i}\}$ for every $(i,j) \in \mathcal{G}$. It is interesting to note that this set is still sign definite and therefore the conclusion made earlier about the exactness of various relaxations is valid under this generalization.

Another interesting case is the optimization of active power flows for lossless networks. In this case, g_{ij} is equal to zero for every $(i,j) \in \mathcal{G}$. Hence, $p_{ji}(\mathbf{x})$ can be simply replaced by $-p_{ij}(\mathbf{x})$. Motivated by this observation, define the reduced vector of active powers as $\mathbf{p}_r(\mathbf{x}) = \{p_{ij}(\mathbf{x}) \mid \forall (i,j) \in \mathcal{G}\}$, and consider the optimization problem

$$\min_{\mathbf{x} \in \mathcal{C}^n} \quad \bar{h}_0(\mathbf{p}_r(\mathbf{x}), \mathbf{y}(\mathbf{x})) \qquad \text{s.t.} \quad \bar{h}_j(\mathbf{p}_r(\mathbf{x}), \mathbf{y}(\mathbf{x})) \leq 0, \quad j = 1, 2, ..., m$$

for some functions $\bar{h}_0(\cdot,\cdot), ..., \bar{h}_m(\cdot,\cdot)$, which are assumed to be increasing in their first vector argument. Now, each edge (i,j) of the graph $\mathcal{G}$ is accompanied by the singleton weight set $\{b_{ij}\mathrm{i}\}$. Due to Theorem 10, the SDP and reduced SDP relaxations are exact if $\mathcal{G}$ is weakly cyclic. This is the generalization of the result obtained in [31] for optimization over lossless networks.

Remark 4. *For simplicity in the presentation, a transmission line was modeled in this work as a series admittance without taking the capacitive effect of the line into* *account. To make our model more realistic, a shunt element* $g_{ii} + b_{ii}\mathrm{i}$ *should be added to each node* $i \in \mathcal{G}$*. The complex power injected into the network through this shunt element is equal to* $g_{ii}|x_i|^2 - b_{ii}|x_i|^2\mathrm{i}$*. This quadratic expression is only in terms of one variable* x_i *and therefore it does not directly introduce any element to the weight sets of the network. However, the addition of shunt elements may change the weights associated with the line current magnitude or apparent power.*

REMARK 5. *The objective function commonly used for the OPF problem is the total cost of the active power generated by the power plants. The generation cost for a single power plant is a function of its fuel cost (among others), which depends on the type of its consumed fuel (e.g., coal or gas). Based on the simple rule of "the more fuel consumed, the higher amount of electricity generated", the generation cost is usually an increasing function of* $\mathbf{p}(x)$ *and the shape of the function depends on the type of the generator. For simplify, this function is often considered as piece-wise linear for clearing an electricity market and considered as quadratic for solving a real-time dispatch problem.*

The results developed in this section are more general than the existing ones in the literature for the conventional OPF problem [17, 16, 25, 26, 31, 4]. In particular, the conventional OPF problem optimizes the nodal powers, whereas the formulation proposed here can also optimize the electrical power at the level of line flows rather than the aggregated nodal flows. In addition, the traditional OPF problem previously studied in the literature is confined to a simple quadratic formulation, but this work considers a far more generic formation of the problem. Unlike the existing algebraic proofs reported in the aforementioned papers, this section offers a new insight about the role of "passivity of transmission lines" in reducing the complexity of energy optimization problems. Since most distribution power networks are acyclic graphs, this work implies that energy optimization problems are easy to solve at that level. In contrast, transmission networks are made to be cyclic. Although several papers have observed that the SDP relaxation is exact for many instances of OPF over transmission networks, our current results cannot fully explain this observation (except for certain lossless networks) and a more comprehensive study is required. Note that we have observed in multiple simulations in [25] that the cycle condition (3.2) is satisfied for almost all cycles of a transmission network.

7. Examples. In this section, five examples will be provided to illustrate various contributions of this work in certain special cases.

Example 1: The minimization of an unconstrained bivariate quartic polynomial can be carried out via an SDP relaxation obtained from the first-order sum-of-squares technique [22]. In this example, we demonstrate how a computationally cheaper SOCP relaxation (in comparison to the aforementioned SDP relaxation) can be used to solve the minimization of a structured bivariate quartic polynomial subject to an arbitrary number of structured bivariate quartic polynomials. To this end, we first consider the unconstrained case, where the goal is to minimize the polynomial

$$f_0(x_1, x_2) = x_1^4 + ax_2^2 + bx_1^2x_2 + cx_1x_2 \tag{7.1}$$

with the real-valued variables x_1 and x_2, for arbitrary coefficients $a, b, c \in \mathcal{R}$. In order to find the global minimum of this optimization problem, the standard convex optimization technique cannot readily be used due to the possible non-convexity of $f(x_1, x_2)$. To address this issue, the above unconstrained minimization problem will be converted to a constrained quadratic optimization problem. More precisely, the problem of minimizing $f_0(x_1, x_2)$ can be reformulated in terms of x_1, x_2 and two auxiliary variables x_3, x_4 as:

$$\min_{\mathbf{x}\in\mathcal{R}^4} \quad x_3^2 + ax_2^2 + bx_3x_2 + cx_1x_2 \tag{7.2a}$$

$$\text{subject to} \quad x_1^2 - x_3x_4 = 0, \qquad x_4^2 - 1 = 0 \tag{7.2b}$$

where $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^H$. The above optimization problem can be recast as follows:

$$\min_{\mathbf{x}\in\mathcal{R}^4, X\in\mathcal{R}^{4\times 4}} \quad X_{33} + aX_{22} + bX_{23} + cX_{12} \tag{7.3a}$$

$$\text{subject to} \quad X_{11} - X_{34} \leq 0, \qquad X_{44} - 1 = 0 \tag{7.3b}$$

and subject to the additional constraint $X = \mathbf{x}\mathbf{x}^H$. Note that $X_{11} - X_{34} \leq 0$ should have been $X_{11} - X_{34} = 0$, but this modification does not change the solution. To eliminate the non-convexity induced by the constraint $X = \mathbf{x}\mathbf{x}^H$, one can use an SOCP relaxation obtained by replacing the constraint $X = \mathbf{x}\mathbf{x}^H$ with the convex constraints $X = X^H$, $X\{(1,2)\} \succeq 0$, $X\{(2,3)\} \succeq 0$ and $X\{(3,4)\} \succeq 0$. To understand the exactness of this relaxation, the weighted graph $\mathcal{G}$ capturing the structure of the optimization problem (7.2) should be constructed. This graph is depicted in Figure 6.1(b). Due to Corollary 1, since $\mathcal{G}$ is acyclic, the SOCP relaxation is exact for all values of a, b, c. Note that this does not imply that every solution X of the SOCP relaxation has rank 1. However, there is a simple systematic procedure for recovering a rank-1 solution from an arbitrary optimal solution of this relaxation.

Now, consider the constrained optimization case where a set of constraints

$$f_j(x_1, x_2) = x_1^4 + a_jx_2^2 + b_jx_1^2x_2 + c_jx_1x_2 \leq d_j \quad j = 1, ..., m$$

is added to the optimization problem (7.1) for given coefficients a_j, b_j, c_j, d_j. In this case, the graph $\mathcal{G}$ depicted in Figure 6.1(b) needs to be modified by replacing its edge sets $\{b\}$ and $\{c\}$ with $\{b, b_1, ..., b_m\}$ and $\{c, c_1, ..., c_m\}$, respectively. Due to Corollary 1, the SOCP and SDP relaxations corresponding to the new optimization problem are exact as long as the sets $\{c, c_1, ..., c_m\}$ and $\{b, b_1, ..., b_m\}$ are both sign definite. Moreover, in light of Theorem 4, if these sets are not sign definite, then the SDP relaxation will still have a low rank (rank 1 or 2) solution.

Example 2: Consider the optimization problem

$$\min_{\mathbf{x}\in\mathcal{C}^7} \quad \mathbf{x}^H M\mathbf{x} \qquad \text{s.t.} \quad |x_i| = 1, \quad i = 1, 2, ..., 7 \tag{7.4}$$

where M is a given Hermitian matrix. Assume that the weighted graph $\mathcal{G}$ depicted in Figure 2.1(c) captures the structure of this optimization problem, meaning that (i) $M_{ij} = 0$ for every pair $(i, j) \in \{1, 2, ...7\}$ such that $(i, j) \notin \mathcal{G}$, $(j, i) \notin \mathcal{G}$ and $i \neq j$, (ii) M_{ij} is equal to the edge weight c_{ij} for every $(i, j) \in \mathcal{G}$. The SDP relaxation of this optimization problem is as follows:

$$\min_{X\in\mathcal{C}^{7\times 7}} \quad \text{trace}\{MX\} \qquad \text{s.t.} \qquad X_{11} = \cdots = X_{77} = 1, \qquad X = X^H \succeq 0$$

Define $\mathcal{O}_1$ and $\mathcal{O}_2$ as the cycles induced by the vertex sets $\{1, 2, 3\}$ and $\{1, 4, 5\}$, respectively. Now, the reduced SDP and SOCP relaxations can be obtained by replacing the constraint $X = X^H \succeq 0$ in the above optimization problem with certain small-sized constraints based on $\mathcal{O}_1$ and $\mathcal{O}_2$, as mentioned before. In light of Theorem 11, the following statements hold:

- The SDP, reduced SDP and SOCP relaxations are all exact in the case where $c_{12}, c_{13}, c_{14}, c_{15}, c_{23}, c_{45}$ are real numbers satisfying the inequalities $c_{12}c_{13}c_{23} \leq 0$ and $c_{14}c_{15}c_{45} \leq 0$.
- The SDP, reduced SDP and SOCP relaxations are all exact in the case where each of the sets $\{c_{12}, c_{13}, c_{23}\}$ and $\{c_{14}, c_{15}, c_{45}\}$ has at least one zero element.
- The SDP and reduced SDP are exact in the case where $c_{12}, c_{13}, c_{14}, c_{15}, c_{23}, c_{45}$ are imaginary numbers. Note that the SOCP relaxation may not be tight. To illustrate this fact, assume that the weights of the graph $\mathcal{G}$ are all equal to +i and that the diagonal entries of the matrix M are zero. In this case, the SDP relaxation is known to be tight, but the optimal objective values of the SOCP and SDP relaxations are equal to two different numbers -16 and -14.3923. Hence, the SOCP relaxation cannot be exact.

The above results demonstrate how the combined effect of the graph topology and the edge weights makes various relaxations exact for the quadratic optimization problem (7.4).

Example 3: Consider the optimization problem

$$\min_{\mathbf{x} \in \mathcal{C}^n} \quad \mathbf{x}^H M \mathbf{x} \qquad \text{s.t.} \quad |x_j| = 1, \quad j = 1, 2, ..., m \tag{7.5}$$

where M is a symmetric real-valued matrix. It has been proven in [33] that this problem is NP-hard even in the case when M is restricted to be positive semidefinite. Consider the graph $\mathcal{G}$ associated with the matrix M. As an application of Theorem 8, the SDP and reduced SDP relaxations are exact for this optimization problem and therefore this problem is polynomial-time solvable with an arbitrary accuracy, provided that $\mathcal{G}$ is bipartite and weakly cyclic. To understand how well the SDP relaxation works, we pick $\mathcal{G}$ as a cycle with 4 vertices. Consider a randomly generated matrix M:

$$M = \begin{bmatrix} 0 & -0.0961 & 0 & -0.1245 \\ -0.0961 & 0 & -0.1370 & 0 \\ 0 & -0.1370 & 0 & 0.7650 \\ -0.1245 & 0 & 0.7650 & 0 \end{bmatrix}$$

After solving the SDP relaxation numerically, an optimal solution X^* is obtained as

$$X^* = \begin{bmatrix} 1.0000 & 0.1767 & -0.5516 & 0.6505 \\ 0.1767 & 1.0000 & 0.7235 & -0.6327 \\ -0.5516 & 0.7235 & 1.0000 & -0.9923 \\ 0.6505 & -0.6327 & -0.9923 & 1.0000 \end{bmatrix}$$

This matrix has rank-2 and thus it seems as if the SDP relaxation is not exact. However, the fact is that this relaxation has a hidden rank-1 solution. To recover that solution, one can write X^* as the sum of two rank-1 matrices, i.e., $X^* = (\mathbf{u}_1)(\mathbf{u}_1)^H + (\mathbf{u}_2)(\mathbf{u}_2)^H$ for two real vectors $\mathbf{u}_1$ and $\mathbf{u}_1$. It is straightforward to inspect that the complex-valued rank-1 matrix $(\mathbf{u}_1 + \mathbf{u}_2\text{i})(\mathbf{u}_1 + \mathbf{u}_2\text{i})^H$ is another solution of the SDP relaxation. Thus, $\mathbf{x}^* = \mathbf{u}_1 + \mathbf{u}_2\text{i}$ is an optimal solution of the optimization problem (7.5).

As stated before, the SDP and reduced SDP relaxations are exact in this example. To evaluate the SOCP relaxation, it can be shown that the optimal objective values of the SDP and SOCP relaxations are equal to -1.9124 and -2.2452, respectively. The discrepancy between these two numbers implies that the SOCP relaxation is not exact.

Example 4: Consider the optimization problem

$$\min_{\mathbf{x}\in\mathcal{C}^n} \quad \mathbf{x}^H M_0 \mathbf{x} \qquad \text{s.t.} \;\; \mathbf{x}^H M_j \mathbf{x} \leq 0, \quad j = 1, 2, ..., m$$

where $M_0, ..., M_m$ are symmetric real matrices, while $\mathbf{x}$ is an unknown complex vector. Similar to what was done in Example 1, a generalized weighted graph $\mathcal{G}$ can be constructed for this optimization problem. Regardless of the edge weights, as long as the graph $\mathcal{G}$ is acyclic, the SDP, reduced SDP and SOCP relaxations are all tight (see Theorem 6). As a result, this class of optimization problems is polynomial-time solvable with an arbitrary accuracy.

Example 5: As a generalization of linear programs, consider the non-convex optimization problem

$$\begin{aligned} \min_{\mathbf{x}\in\mathcal{R}^n} \quad & \sum_{i=1}^{k} a_{0i} e^{\mathbf{x}^H M_{0i} \mathbf{x}} + \sum_{i=k+1}^{l} \mathbf{x}^H M_{0i} \mathbf{x} + \mathbf{b}_0^T \mathbf{x} \\ \text{s.t.} \quad & \sum_{i=1}^{k} a_{ji} e^{\mathbf{x}^H M_{ji} \mathbf{x}} + \sum_{i=k+1}^{l} \mathbf{x}^H M_{ji} \mathbf{x} + \mathbf{b}_j^T \mathbf{x} \leq 0, \quad j = 1, 2, ..., m \end{aligned}$$

where a_{ij}'s are scalars, $\mathbf{b}_j$'s are $n \times 1$ vectors , and M_{ij}'s are $n \times n$ symmetric matrices. This problem involves linear terms, quadratic terms, and exponential terms with quadratic exponents. Using the technique stated in Section 2.2, the above optimization problem can be reformulated in terms of the rank-1 matrix $\tilde{\mathbf{x}}\tilde{\mathbf{x}}^H$ where $\tilde{\mathbf{x}} = \begin{bmatrix} 1 & \mathbf{x}^H \end{bmatrix}^H$, from which an SDP relaxation can subsequently be obtained by replacing the matrix $\tilde{\mathbf{x}}\tilde{\mathbf{x}}^H$ with a new matrix variable $\tilde{X}$ under the constraint $\tilde{X}_{11} = 1$. By mapping the structure of the optimization into a generalized weighted graph and noticing that e^x is an increasing function in x, it can be concluded that the SDP relaxation is exact if the following conditions are all satisfied:

- a_{ji} is nonnegative for every $j \in \{0, ..., m\}$ and $i \in \{1, ..., k\}$.
- $\mathbf{b}_j$ is a non-positive vector for every $j \in \{0, ..., m\}$.
- M_{ji} has only non-positive off-diagonal entries for every $j \in \{0, ..., m\}$ and $i \in \{1, ..., l\}$.

8. Conclusions. This work deals with three conic relaxations for a broad class of nonlinear real/complex optimization problems, where the argument of each objective and constraint function is quadratic (as opposed to linear) in the optimization variable. Several types of optimization problems, including polynomial optimization, can be cast as the problem under study. To explore the exactness of the proposed relaxations, the structure of the optimization problem is mapped into a generalized weighted graph with a weight set assigned to each edge. In the case of real-valued optimization problems, it is shown that the relaxations are exact if a set of conditions are satisfied, which depend on some weak properties of the underlying generalized weighted graph. A similar result is derived in the complex-valued case after introducing the notion of "sign-definite complex weight sets", under the assumption that the graph is acyclic. The complex case is further studied for general graphs, and it is shown that if the graph can be decomposed as the union of edge-disjoint subgraphs, each satisfying one of the four derived structural properties, then two relaxations are exact.

In the past five years, several papers have reported that a specific SDP relaxation would be exact for many instances of power optimization problems. The present paper provides an insight into this observation and partly relates the exactness of the relaxation to the passivity of transmission lines. This paper also proves that the relaxation is exact for power networks with acyclic graphs. A deeper understanding of the SDP relaxation for mesh power networks would shed light on the true complexity of practical energy optimization problems. This is left for future work. An optimal power flow problem (OPF) is often formulated in terms of nodal voltages and this amounts to $O(n)$ variables. To convexify the problem through an SDP relaxation, the problem size is increased to $O(n^2)$. Given that a large-scale OPF problem must be solved every 5 to 15 minutes in practice, it would be challenging to handle this increase in the size of the problem. As far as the computational complexity is concerned, the reduced SDP and SOCP relaxations are far more appealing for implementation. For instance, the number of important variables associated with an SOCP relaxation is $O(n)$ as opposed to $O(n^2)$, under a reasonably practical assumption that the graph of the power network is planar.

9. Appendix. In what follows, Theorem 10 will be proved.

Proof of Part (i): Let $\mathbf{x}^*$ denote an arbitrary solution of (2.1). For every $\mathcal{G}_s \in \Omega$, define $\alpha(\mathcal{G}_s)$ as:

- If $\mathcal{G}_s \in \Omega \backslash (\mathcal{O}_1 \cup \cdots \cup \mathcal{O}_p)$, then we set $\alpha(\mathcal{G}_s)$ equal to any arbitrary complex number with norm 1.
- If $(\mathcal{G}_s) = \mathcal{O}_r$ for some $r \in \{1, ..., p\}$, then we set $\alpha(\mathcal{G}_s)$ equal to $e^{-(\measuredangle x^*_{\mu_r} \mathrm{i})}$.

For every $i \in \mathcal{G}$, define q_i^o as $|x_i^*|^2$. In addition, for every $\mathcal{G}_s \in \Omega$ and $(i,j) \in \mathcal{G}_s$, define:

$$\begin{aligned} U_{ii}^o(\mathcal{G}_s) &= \mathrm{Re}\{x_i^*\alpha(\mathcal{G}_s)\}^2, \quad U_{jj}^o(\mathcal{G}_s) = \mathrm{Re}\{x_j^*\alpha(\mathcal{G}_s)\}^2 \\ W_{ii}^o(\mathcal{G}_s) &= \mathrm{Im}\{x_i^*\alpha(\mathcal{G}_s)\}^2, \quad W_{jj}^o(\mathcal{G}_s) = \mathrm{Im}\{x_j^*\alpha(\mathcal{G}_s)\}^2 \\ V_{ij}^o(\mathcal{G}_s) &= \mathrm{Re}\{x_i^*\alpha(\mathcal{G}_s)\}\mathrm{Im}\{x_j^*\alpha(\mathcal{G}_s)\}, \quad V_{ji}^o(\mathcal{G}_s) = \mathrm{Re}\{x_j^*\alpha(\mathcal{G}_s)\}\mathrm{Im}\{x_i^*\alpha(\mathcal{G}_s)\} \end{aligned} \tag{9.1}$$

Consider those entries of $U^o(\mathcal{G}_s), V^o(\mathcal{G}_s), W^o(\mathcal{G}_s)$ that are not specified above as arbitrary. The first goal is to show that $(\mathbf{q}, U, V, W) = (\mathbf{q}^o, U^o, V^o, W^o)$ is a feasible solution of the expanded SOCP problem. To this end, it is straightforward to verify that (5.10d) and (5.10e) are satisfied. Moreover, for every $\mathcal{G}_s \in \Omega$ and $(i,j) \in \mathcal{G}_s$, one can write:

$$q_i^o = |x_i^*|^2 = |x_i^*\alpha(\mathcal{G}_s)|^2 = U_{ii}^o(\mathcal{G}_s) + W_{ii}^o(\mathcal{G}_s) \tag{9.2}$$

Besides,

$$W_{\mu_r\mu_r}^o(\mathcal{O}_r) = \mathrm{Im}\{x_{\mu_r}^*\alpha(\mathcal{O}_r)\}^2 = \mathrm{Im}\{x_{\mu_r}^* e^{-(\measuredangle x_{\mu_r}^* \mathrm{i})}\}^2 = 0$$

for every $r \in \{1, 2, ..., p\}$. Hence, $(\mathbf{q}^o, U^o, V^o, W^o)$ satisfies (5.10c)-(5.10f). On the other hand, for every $(i,j) \in \mathcal{G}$, there is a unique subgraph $\mathcal{G}_s \in \Omega$ such that $(i,j) \in \mathcal{G}_s$ (because $\mathcal{G}$ is weakly cyclic by assumption). Now, since the edge weights are imaginary numbers, one can write:

$$\begin{aligned} \mathrm{Re}\big\{c_{ij}^t (x_i^*)(x_j^*)^H)\big\} &= -\mathrm{Im}\{c_{ij}^t\} \times \mathrm{Im}\left\{(x_i^*\alpha(\mathcal{G}_s))(x_j^*\alpha(\mathcal{G}_s))^H)\right\} \\ &= \mathrm{Im}\{c_{ij}^t\}(V_{ij}^o(\mathcal{G}_s) - V_{ji}^o(\mathcal{G}_s)) \end{aligned} \tag{9.3}$$

for every $t \in \{1, ..., k\}$. It follows from (9.2) and (9.3) that

$$\mathbf{q}^o = l_1\left((\mathbf{x}^*)(\mathbf{x}^*)^H\right), \quad \bar{l}(V^o) = l_2\left((\mathbf{x}^*)(\mathbf{x}^*)^H\right) \tag{9.4}$$

Therefore,

$$0 \geq f_j\left(l_1\left((\mathbf{x}^*)(\mathbf{x}^*)^H\right), l_2\left((\mathbf{x}^*)(\mathbf{x}^*)^H\right)\right) = f_j(\mathbf{q}^o, \bar{l}(V^o)), \qquad j = 1, 2, ..., m$$

This means that $(\mathbf{q}, U, V, W) = (\mathbf{q}^o, U^o, V^o, W^o)$ is a feasible solution of the expanded SOCP problem. Similarly,

$$f^* = f_0\left(l_1\left((\mathbf{x}^*)(\mathbf{x}^*)^H\right), l_2\left((\mathbf{x}^*)(\mathbf{x}^*)^H\right)\right) = f_0(\mathbf{q}^o, \bar{l}(V^o)) \geq f^*_{\text{e-SOCP}}$$

Proof of Part (ii): Given an arbitrary solution $\mathbf{x}^*$ of the optimization problem (2.1), consider $(\mathbf{q}^o, U^o, V^o, W^o)$ defined in (9.1). As shown in the proof of Part (i), this is a feasible solution of the expanded SOCP relaxation. Furthermore, observe that

$$\begin{bmatrix} U^o_{ii}(\mathcal{G}_s) & V^o_{ij}(\mathcal{G}_s) \\ V^o_{ij}(\mathcal{G}_s) & W^o_{jj}(\mathcal{G}_s) \end{bmatrix} = \begin{bmatrix} \text{Re}\{x^*_i\alpha(\mathcal{G}_s)\} \\ \text{Im}\{x^*_j\alpha(\mathcal{G}_s)\} \end{bmatrix} \begin{bmatrix} \text{Re}\{x^*_i\alpha(\mathcal{G}_s)\} & \text{Im}\{x^*_j\alpha(\mathcal{G}_s)\} \end{bmatrix},$$
$$\begin{bmatrix} U^o_{jj}(\mathcal{G}_s) & V^o_{ji}(\mathcal{G}_s) \\ V^o_{ji}(\mathcal{G}_s) & W^o_{ii}(\mathcal{G}_s) \end{bmatrix} = \begin{bmatrix} \text{Re}\{x^*_j\alpha(\mathcal{G}_s)\} \\ \text{Im}\{x^*_i\alpha(\mathcal{G}_s)\} \end{bmatrix} \begin{bmatrix} \text{Re}\{x^*_j\alpha(\mathcal{G}_s)\} & \text{Im}\{x^*_i\alpha(\mathcal{G}_s)\} \end{bmatrix}$$

This implies that the above matrices have rank 1, which completes the proof of the "only if" part. To prove the "if" part, let $(\mathbf{q}^*, U^*, V^*W^*)$ be a solution of the expanded SOCP relaxation satisfying the rank condition stated in Part (ii). Therefore, for every $\mathcal{G}_s \in \Omega$ and $(i,j) \in \mathcal{G}_s$, one can decompose the 2×2 matrices in (5.10d) and (5.10e) at the point $(\mathbf{q}, U, V, W) = (\mathbf{q}^*, U^*, V^*W^*)$ as

$$\begin{bmatrix} U^*_{ii}(\mathcal{G}_s) & V^*_{ij}(\mathcal{G}_s) \\ V^*_{ij}(\mathcal{G}_s) & W^*_{jj}(\mathcal{G}_s) \end{bmatrix} = \begin{bmatrix} u^*_i(\mathcal{G}_s) \\ w^*_j(\mathcal{G}_s) \end{bmatrix} \begin{bmatrix} u^*_i(\mathcal{G}_s) \\ w^*_j(\mathcal{G}_s) \end{bmatrix}^H,$$
$$\begin{bmatrix} U^*_{jj}(\mathcal{G}_s) & V^*_{ji}(\mathcal{G}_s) \\ V^*_{ji}(\mathcal{G}_s) & W^*_{ii}(\mathcal{G}_s) \end{bmatrix} = \begin{bmatrix} u^*_j(\mathcal{G}_s) \\ w^*_i(\mathcal{G}_s) \end{bmatrix} \begin{bmatrix} u^*_j(\mathcal{G}_s) \\ w^*_i(\mathcal{G}_s) \end{bmatrix}^H$$

for some real numbers $u^*_i(\mathcal{G}_s), u^*_j(\mathcal{G}_s), w^*_i(\mathcal{G}_s), w^*_j(\mathcal{G}_s)$. Following the proof of Part (i) and by making a comparison with (9.5), it suffices to show the existence of a vector $\mathbf{x}^*$ and a complex set $\{\sigma(\mathcal{G}_s) \mid \forall \mathcal{G}_s \in \Omega\}$ satisfying the relations:

$$u^*_i(\mathcal{G}_s) + w^*_i(\mathcal{G}_s)\text{i} = x^*_i\alpha(\mathcal{G}_s), \quad \forall \mathcal{G}_s \in \Omega,\ i \in \mathcal{G}_s \tag{9.6a}$$
$$|\sigma(\mathcal{G}_s)| = 1, \qquad \forall \mathcal{G}_s \in \Omega \backslash (\mathcal{O}_1 \cup \cdots \cup \mathcal{O}_p) \tag{9.6b}$$
$$\sigma(\mathcal{O}_r) = e^{-(\measuredangle x^*_{\mu_r}\text{i})}, \qquad \forall r \in \{1, ..., p\} \tag{9.6c}$$

It can be verified that

$$q^*_i = |u^*_i(\mathcal{G}_s) + w^*_i(\mathcal{G}_s)\text{i}|^2, \quad \forall \mathcal{G}_s \in \Omega,\ i \in \mathcal{G}_s$$

Hence, the equations in (9.6) consistently find $|x^*_i|$ as $|x^*_i|^2 = \sqrt{q^*_i}$ for every $i \in \mathcal{G}$. Now, it remains to find the phase of x^*_i. To this end, (9.6) can be equivalently expressed as:

- If $\mathcal{G}_s = \mathcal{O}_r$ for some $r \in \{1, 2, ..., p\}$, then

$$\measuredangle x^*_i - \measuredangle x^*_{\mu_r} = \tan^{-1}\frac{w^*_i(\mathcal{O}_r)}{u^*_i(\mathcal{O}_r)} \tag{9.7}$$

- If $\mathcal{G}_s \in \Omega \backslash (\mathcal{O}_1 \cup \cdots \cup \mathcal{O}_p)$, then

$$\measuredangle x_i^* + \measuredangle \sigma(\mathcal{G}_s) = \tan^{-1} \frac{w_i^*(\mathcal{G}_s)}{u_i^*(\mathcal{G}_s)} \tag{9.8}$$

Note that if the index i in (9.7) is chosen as μ_r, then the left side of this equation becomes zero. Equation (5.10f) guarantees that the right side of (9.7) is also zero in this case. The goal is to show that (9.7) and (9.8) have a solution $\{\measuredangle x_1^*, ..., \measuredangle x_n^*\}$. For this purpose, we order the subgraphs in the set Ω in such a way that every two consecutive subgraphs in the ordered set share a vertex. Denote the ordered set as $\{\mathcal{G}_1, ..., \mathcal{G}_{|\Omega|}\}$. Since the graph $\mathcal{G}$ is weakly cyclic, $\mathcal{G}_1 \cup \cdots \cup \mathcal{G}_s$ and the subgraph $\mathcal{G}_{s+1}$ share exactly one vertex for every $r \in \{1, 2, ..., |\Omega| - 1\}$. Hence, the following algorithm can be used to find $\{\measuredangle x_1^*, ..., \measuredangle x_n^*\}$:

Step 1: Set $s = 1$ and $\measuredangle x_i^* = 0$ for an arbitrary vertex i of the subgraph $\mathcal{G}_1$.

Step 2: So far, the elements of $\mathbf{x}$ corresponding to all vertices of $\mathcal{G}_1 \cup \cdots \cup \mathcal{G}_{s-1}$ and only one vertex of $\mathcal{G}_s$ have been found. Let j denote the index of the only vertex of $\mathcal{G}_s$ for which x_j^* has been obtained. Now, depending on whether or not $\mathcal{G}_s$ belongs to $\Omega \backslash (\mathcal{O}_1 \cup \cdots \cup \mathcal{O}_p)$, (9.7) or (9.8) can be uniquely solved to find all entries of $\mathbf{x}^*$ corresponding to the vertices of $\mathcal{G}_s$.

Step 3: Increment s and jump to Step 2 unless $s = |\Omega|$.

Proof of Part (iii): Given an arbitrary feasible point $(\mathbf{q}, U, V, W)$ of the expanded SOCP relaxation, consider the entries of X in the SOCP relaxation (2.5) as:

- For every $i \in \mathcal{G}$, set X_{ii} equal to q_i.
- For every $(i, j) \in \mathcal{G}$, find the unique subgraph $\mathcal{G}_s \in \Omega$ such that $(i, j) \in \mathcal{G}_s$, and set $X_{ij} = X_{ji}^H = V_{ji}(\mathcal{G}_s) - V_{ij}(\mathcal{G}_s)$.
- Choose the remaining entries of X arbitrarily.

By adopting the argument leading to (9.4), it can be shown that

$$f_j(l_1(X), l_2(X)) = f_j(\mathbf{q}, \bar{l}(V)), \qquad j = 0, 1, ..., m \tag{9.9}$$

Thus, it only remains to prove that the defined X is a feasible point of the SOCP relaxation (2.5). Given an edge $(i, j) \in \mathcal{G}$, let $\mathcal{G}_s \in \Omega$ be the subgraph containing this edge. One can write:

$$X\{(i,j)\} = \begin{bmatrix} U_{ii}(\mathcal{G}_s) & -V_{ij}(\mathcal{G}_s)\mathrm{i} \\ V_{ij}(\mathcal{G}_s)\mathrm{i} & W_{jj}(\mathcal{G}_s) \end{bmatrix} + \begin{bmatrix} W_{ii}(\mathcal{G}_s) & V_{ji}(\mathcal{G}_s)\mathrm{i} \\ -V_{ji}(\mathcal{G}_s)\mathrm{i} & U_{jj}(\mathcal{G}_s) \end{bmatrix}$$

Since $X\{(i, j)\}$ has been expressed as the sum of two positive semidefinite matrices, it must be a positive semidefinite matrix. This implies that X is a feasible point of the SOCP relaxation.

Proof of Part (iv): Let X denote an arbitrary feasible point of the reduced SDP relaxation. Given a subgraph $\mathcal{G}_s \in \Omega$, the matrix $X\{\mathcal{G}_s\}$ can be decomposed as $D\{\mathcal{G}_s\}D\{\mathcal{G}_s\}^H$, where $D\{\mathcal{G}_s\}$ is a matrix in $\mathcal{C}^{|\mathcal{G}_s| \times |\mathcal{G}_s|}$ whose last row is entirely real valued. Such a decomposition can be obtained using the eigen-decomposition method. Now, consider the matrix variable $\begin{bmatrix} U(\mathcal{G}_s) & V(\mathcal{G}_s) \\ V(\mathcal{G}_s)^H & W(\mathcal{G}_s) \end{bmatrix}$ in the expanded SOCP relaxation as

$$\begin{bmatrix} \mathrm{Re}\{D(\mathcal{G}_s)\}\mathrm{Re}\{D(\mathcal{G}_s)\}^H & \mathrm{Re}\{D(\mathcal{G}_s)\}\mathrm{Im}\{D(\mathcal{G}_s)\}^H \\ \mathrm{Im}\{D(\mathcal{G}_s)\}\mathrm{Re}\{D(\mathcal{G}_s)\}^H & \mathrm{Im}\{D(\mathcal{G}_s)\}\mathrm{Im}\{D(\mathcal{G}_s)\}^H \end{bmatrix}$$

Moreover, consider q_i as X_{ii} for every $i \in \mathcal{G}$. It is straightforward to show that (9.9) holds for this choice of $(\mathbf{q}, U, V, W)$, and that $(\mathbf{q}, U, V, W)$ is a feasible point of the expanded SOCP relaxation. This completes the proof.

Proof of Part (v): Consider the optimization problem

$$\min_{\mathbf{u},\mathbf{w}} \quad f_0(\mathbf{q}, \bar{l}(V)) \tag{9.10a}$$

subject to:

$$f_j(\mathbf{q}, \bar{l}(V)) \leq 0, \qquad j = 1, ..., m \tag{9.10b}$$

$$U_{ii}(\mathcal{G}_s) + W_{ii}(\mathcal{G}_s) = q_i, \quad \forall \mathcal{G}_s \in \Omega,\ i \in \mathcal{G}_s \tag{9.10c}$$

$$U_{ii} = u_i(\mathcal{G}_s)^2, \qquad \forall \mathcal{G}_s \in \Omega,\ i \in \mathcal{G}_s \tag{9.10d}$$

$$W_{ii} = w_i(\mathcal{G}_s)^2, \qquad \forall \mathcal{G}_s \in \Omega,\ i \in \mathcal{G}_s \tag{9.10e}$$

$$V_{ij} = u_i(\mathcal{G}_s) w_j(\mathcal{G}_s), \qquad \forall \mathcal{G}_s \in \Omega,\ (i,j) \in \mathcal{G}_s \tag{9.10f}$$

$$V_{ji} = u_j(\mathcal{G}_s) w_i(\mathcal{G}_s), \qquad \forall \mathcal{G}_s \in \Omega,\ (i,j) \in \mathcal{G}_s \tag{9.10g}$$

$$W_{\mu_r \mu_r}(\mathcal{O}_r) = 0, \qquad r = 1, 2, ..., p \tag{9.10h}$$

where $\mathbf{u} = \{u_i(\mathcal{G}_s) \mid \forall \mathcal{G}_s \in \Omega,\ i \in \mathcal{G}_s\}$ and $\mathbf{w} = \{w_i(\mathcal{G}_s) \mid \forall \mathcal{G}_s \in \Omega,\ i \in \mathcal{G}_s\}$. Note that U, V, W are considered as implicit (dependent) variables in the optimization problem (9.10), because they can be readily expressed in terms of $\mathbf{u}$ and $\mathbf{w}$. The optimization problem (9.10) is real-valued and can be cast in the form of (2.1). Therefore, one can find its SDP, reduced SDP and SOCP relaxations. It is easy to verify that the SOCP relaxation for this optimization problem is indeed the expanded SOCP relaxation (5.10). Assume that this relaxation is tight for (5.10). Then, it follows from Theorem 1 that the expanded SOCP relaxation has a solution for which the matrices in (5.10d) and (5.10e) have rank 1. In this case, the proof of Part (v) is an immediate consequence of Parts (ii)-(iv). Therefore, it suffices to show that the relaxation (5.10) is tight for the optimization problem (9.10). To this end, according to Corollary 1, it is enough to show that the graph capturing the structure of the optimization problem (9.10) is acyclic. To construct this graph, notice that not every quadratic term in the matrix $\begin{bmatrix} \mathbf{u} \\ \mathbf{w} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{w} \end{bmatrix}^H$ appears in the constraints of the optimization problem (9.10). The ones creating an edge in the graph of this optimization problem are given by the set $\{u_i(\mathcal{G}_s) w_j(\mathcal{G}_s), u_j(\mathcal{G}_s) w_i(\mathcal{G}_s) \mid \forall \mathcal{G}_s \in \Omega,\ (i,j) \in \mathcal{G}_s\}$. This graph is cyclic. However, since $w_{\mu_r}(\mathcal{O}_r)$ is equal to zero for $r = 1, ..., p$, all vertices associated with $w_{\mu_r}(\mathcal{O}_r)$'s can be removed from the graph. Now, the remaining graph becomes acyclic (given that $\mathcal{G}$ is weakly cyclic). This completes the proof. ■

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