

# Promises of Conic Relaxation for Contingency-Constrained Optimal Power Flow Problem

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**Abstract**—This paper is concerned with the security-constrained optimal power flow (SCOPF) problem, where each contingency corresponds to the outage of an arbitrary number of lines and generators. The problem is studied by means of a convex relaxation, named semidefinite program (SDP). The existence of a rank-1 SDP solution guarantees the recovery of a global solution of SCOPF. We prove that the rank of the SDP solution is upper bounded by the treewidth of the power network, which is perceived to be small in practice. We then propose a decomposition method to reduce the computational complexity of the relaxation. In the case where the relaxation is not exact, we develop a graph-theoretic convex program to identify the problematic lines of the network and incorporate the loss over those lines into the objective as a penalization (regularization) term, leading to a penalized SDP problem. We perform several simulations on large-scale benchmark systems and verify that the penalized relaxation is able to find feasible solutions that are at most 1% away from the unknown global minima.

## I. INTRODUCTION

The classical optimal power flow (OPF) problem aims to find a steady-state operating point of a power system that minimizes a desirable cost function, e.g. power loss or generation cost, and satisfies network and physical constraints on loads, powers, voltages and line flows [1]. The OPF problem is not only non-convex but also NP-hard, due to its possible reduction to the (0,1)-quadratic optimization. Started by the work [2] in 1962, many of the existing optimization techniques have been studied for the OPF problem, leading to algorithms based on linear programming, Newton Raphson, quadratic programming, nonlinear programming, Lagrange relaxation, interior point method, artificial intelligence, artificial neural network, fuzzy logic, genetic algorithm, evolutionary programming and particle swarm optimization [3]. Due to the non-convexity of OPF, these algorithms are not robust, lack performance guarantees, and may not find a global optimum.

By exploiting the physical properties of transmission lines, it has been argued in the series of work [4]–[12] that the classical OPF problem corresponding to a practical power system may be convexified and solved efficiently through a semidefinite programming (SDP) relaxation. In particular, the paper [13] shows that the SDP relaxation is exact in two cases under certain technical assumptions: (i) for acyclic networks, (ii) for cyclic networks after relaxing the angle

constraints. However, the SDP relaxation is not always exact for a general mesh network [14]–[17]. To ameliorate the issue, we have recently shown in [18] that: (i) the exactness of the SDP relaxation depends on the formulation of the line capacity constraints, and (ii) the penalization of total reactive loss may enable the recovery of a near-global solution (i.e., a solution that is measurably close to a global minimum) for modest-sized systems (as verified in 7000 simulations).

The major drawback of representing the optimal power flow problem as a semidefinite program is the requirement of defining a square matrix variable, which makes the number of scalar variables of the problem quadratic with respect to the number of network buses. This may yield a very high-dimensional SDP problem for a real-world network. To address this issue, the papers [19]–[21] have leveraged the sparsity of power networks in order to break down the large-scale semidefinite constraint into small-sized constraints. Similarly, the papers [22], [23] and [24] have exploited the general technique proposed in [25] to reduce the complexity of the SDP relaxation of the OPF problem. The simulations performed in those papers suggest that the SDP relaxation would fail to work properly for large-scale systems [21].

Although OPF is a fundamental problem studied extensively in the literature for power systems, a real-work power flow optimization is based on a set of coupled OPFs with a variety of constraints and variables. The latter problem is named security-constrained OPF (SCOPF) [26], [27]. The SCOPF problem is important in practice, since independent system operators tend to design an operating point that satisfies the demand and network constraints not only under normal operation but also under pre-specified contingencies such as line outages. Depending on the network characteristics, one may adopt preventive or corrective approaches for the SCOPF problem. In the preventive formulation of SCOPF, the under-design state of each generator (e.g., production level) is considered identical for the pre- and post-contingency scenarios. This reflects the fact that mechanical facilities may not be able to respond to the changes in the network fast. In the corrective approach, limited changes in certain control parameters are permitted after the network experiences a fault.

In this paper, we study the SCOPF problem—as a general version of OPF—through a convex relaxation. First, we propose an SDP relaxation for this problem. The existence of a rank-1 SDP solution guarantees the recovery of a global solution of SCOPF. We prove that the relaxation has a matrix solution whose rank is at most the treewidth of the pre-contingency network. The treewidth of real-world networks

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is perceived to be small due to their (almost) planarity and sparsity [28]. For example, the treewidth of the graph corresponding to a peak hour setup of a Polish system with over 3000 buses is less than 25. Second, we reduce the computational complexity of the SDP problem using a tree decomposition method to arrive at a decomposed SDP relaxation with a set of small-sized SDP matrices as opposed to a full-scale SDP matrix. We show that the full-scale SDP relaxation has a solution whose rank is upper bounded by the ranks of the small-sized matrices of the decomposed SDP relaxation. By working on the ranks of these small matrices, we propose a technique to identify the problematic lines of the network that may contribute to the failure of the SDP relaxation for SCOPF. This diagnosis method enables us to develop a heuristic method, named penalize SDP relaxation, to find a near-global solution of the problem by penalizing the loss over the problematic lines of the network in the objective of SDP. We test our method on several benchmark systems with as high as 3000 buses and find a solution with a global optimality degree of at least 99% for each case. Note that the proofs of the theorems presented here are available in [29].

**Notations:**  $\mathbb{R}$ ,  $\mathbb{C}$ , and  $\mathbb{H}^n$  denote the sets of real numbers, complex numbers, and  $n \times n$  Hermitian matrices, respectively. The  $m$  by  $n$  rectangular identity matrix whose  $(i, j)$  entry is equal to the Kronecker delta  $\delta_{ij}$  is denoted by  $\mathbf{I}_{m \times n}$ . Notations  $\text{Re}\{\mathbf{W}\}$ ,  $\text{Im}\{\mathbf{W}\}$ , and  $\text{rank}\{\mathbf{W}\}$  denote the real part, imaginary part, and rank of a scalar/matrix  $\mathbf{W}$ , respectively. The notation  $\mathbf{W} \succeq 0$  means that  $\mathbf{W}$  is Hermitian and positive semidefinite. The notation  $\angle x$  denotes the angle of a complex number  $x$ . The notation “ $i$ ” is reserved for the imaginary unit. The superscripts  $(\cdot)^*$  and  $(\cdot)^T$  represent the conjugate transpose and transpose operators respectively. Given a matrix  $\mathbf{W}$ , its  $(l, m)$  entry is denoted as  $W_{lm}$ . The subscript  $(\cdot)_{\text{opt}}$  is used to show the optimal value of an optimization parameter. Given a matrix  $\mathbf{M}$ , its Moore Penrose pseudoinverse is denoted as  $\mathbf{M}^+$ . Given a simple graph  $\mathcal{H}$ , its vertex and edge sets are denoted by  $\mathcal{V}_{\mathcal{H}}$  and  $\mathcal{E}_{\mathcal{H}}$ , respectively. Given two sets  $\mathcal{S}_1$  and  $\mathcal{S}_2$ , the notation  $\mathcal{S}_1 \setminus \mathcal{S}_2$  denotes the set of all elements of  $\mathcal{S}_1$  that do not exist in  $\mathcal{S}_2$ . Given a scalar  $m$  and a real-valued set  $\mathcal{S}$ , define  $\mathcal{S} + m$  as a set obtained by adding  $m$  to every element of  $\mathcal{S}$ . Given a Hermitian matrix  $\mathbf{M}$  and two sets of natural numbers  $\mathcal{S}_1$  and  $\mathcal{S}_2$ , define  $\mathbf{M}\{\mathcal{S}_1, \mathcal{S}_2\}$  as a submatrix of  $\mathbf{M}$  obtained through two operations: (i) removing all rows of  $\mathbf{M}$  whose indices do not belong to  $\mathcal{S}_1$ , and (ii) removing all columns of  $\mathbf{M}$  whose indices do not belong to  $\mathcal{S}_2$ . For instance,  $\mathbf{M}\{\{1, 2\}, \{2, 3\}\}$  is a  $2 \times 2$  matrix with the entries  $M_{12}, M_{13}, M_{22}, M_{23}$ .

## II. PROBLEM FORMULATION

Consider a power network with the set of buses  $\mathcal{N} := \{1, 2, \dots, n\}$  and the set of flow lines  $\mathcal{L} \subseteq \mathcal{N} \times \mathcal{N}$ . With no loss of generality, each line  $(l, m) \in \mathcal{L}$  of the network is modeled as a series admittance  $y_{lm}$ . Suppose that a known constant-power load with the complex value

$S_{D_k} = P_{D_k} + Q_{D_k}i$  is connected to bus  $k \in \mathcal{N}$ , where  $P_{D_k}, Q_{D_k} \in \mathbb{R}$ . Given a nonnegative integer  $c$ , consider a set of  $c$  contingencies, where each contingency corresponds to an arbitrary number of pre-specified line/generator outages. In this work, we model a line outage by removing the line from the base case and model a generator outage by enforcing its output to be zero. Define  $\mathcal{C} := \{0, 1, \dots, c\}$  as the set of all pre- and post-contingencies, where the base case is treated as contingency 0. Define  $\mathcal{L}^{(0)} = \mathcal{L}$ , and  $\mathcal{L}^{(r)}$  as the set of lines of the network under contingency  $r \in \{1, 2, \dots, c\}$ .

Consider a contingency scenario  $r \in \mathcal{C}$ . Assume that a generator is connected to each bus  $k \in \mathcal{N}$  whose unknown complex output is denoted as  $S_{G_k}^{(r)} = P_{G_k}^{(r)} + Q_{G_k}^{(r)}i$ . Let  $f_k(\cdot)$  be a convex function representing the generation cost for generator  $k$ . The unknown complex voltage at bus  $k \in \mathcal{N}$  is denoted as  $V_k^{(r)}$ . Furthermore, define  $S_{lm}^{(r)} = P_{lm}^{(r)} + Q_{lm}^{(r)}i$  as the unknown complex power transferred from bus  $l \in \mathcal{N}$  to the rest of the network through the line  $(l, m) \in \mathcal{L}^{(r)}$ . Define:

$$\begin{aligned} \mathbf{P}_G^{(r)} &\triangleq [P_{G_1}^{(r)}, \dots, P_{G_n}^{(r)}]^T, & \mathbf{Q}_G^{(r)} &\triangleq [Q_{G_1}^{(r)}, \dots, Q_{G_n}^{(r)}]^T, \\ \mathbf{P}_D &\triangleq [P_{D_1}, \dots, P_{D_n}]^T, & \mathbf{Q}_D &\triangleq [Q_{D_1}, \dots, Q_{D_n}]^T, \\ \mathbf{V}^{(r)} &\triangleq [V_1^{(r)}, \dots, V_n^{(r)}]^T. \end{aligned}$$

Given the known vectors  $\mathbf{P}_D$  and  $\mathbf{Q}_D$ , SCOPF minimizes the generation cost over the unknown parameters  $\mathbf{V}^{(r)}$ ,  $\mathbf{P}_G^{(r)}$  and  $\mathbf{Q}_G^{(r)}$  for  $r = 0, 1, \dots, c$  subject to the power balance equations at all buses and certain network constraints. SCOPF is formalized below.

**SCOPF problem:** Minimize

$$\sum_{k \in \mathcal{N}} f_k \left( P_{G_k}^{(0)} \right) \quad (1)$$

over the variables  $\mathbf{P}_G^{(0)}, \mathbf{P}_G^{(1)}, \dots, \mathbf{P}_G^{(c)} \in \mathbb{R}^n$ ,  $\mathbf{Q}_G^{(0)}, \mathbf{Q}_G^{(1)}, \dots, \mathbf{Q}_G^{(c)} \in \mathbb{R}^n$  and  $\mathbf{V}^{(0)}, \mathbf{V}^{(1)}, \dots, \mathbf{V}^{(c)} \in \mathbb{C}^n$ , subject to

$$P_{G_k}^{(r)} - P_{D_k} = \sum_{l \in \mathcal{N}_k^{(r)}} \text{Re} \left\{ V_k^{(r)} \left( V_k^{(r)} - V_l^{(r)} \right)^* y_{kl}^* \right\} \quad (2a)$$

$$Q_{G_k}^{(r)} - Q_{D_k} = \sum_{l \in \mathcal{N}_k^{(r)}} \text{Im} \left\{ V_k^{(r)} \left( V_k^{(r)} - V_l^{(r)} \right)^* y_{kl}^* \right\} \quad (2b)$$

$$P_{k; \min}^{(r)} \leq P_{G_k}^{(r)} \leq P_{k; \max}^{(r)} \quad (2c)$$

$$Q_{k; \min}^{(r)} \leq Q_{G_k}^{(r)} \leq Q_{k; \max}^{(r)} \quad (2d)$$

$$V_{k; \min}^{(r)} \leq |V_k^{(r)}| \leq V_{k; \max}^{(r)} \quad (2e)$$

$$\left( P_{lm}^{(r)} \right)^2 + \left( Q_{lm}^{(r)} \right)^2 \leq \left( S_{lm; \max}^{(r)} \right)^2 \quad (2f)$$

$$|P_{G_k}^{(r)} - P_{G_k}^{(0)}| \leq \Delta P_{k; \max}^{(r)} \quad (2g)$$

$$|Q_{G_k}^{(r)} - Q_{G_k}^{(0)}| \leq \Delta Q_{k; \max}^{(r)} \quad (2h)$$

$$\left| |V_k^{(r)}|^2 - |V_k^{(0)}|^2 \right| \leq \left( \Delta V_{k; \max}^{(r)} \right)^2 \quad (2i)$$

for every  $k \in \mathcal{N}$ ,  $r \in \mathcal{C}$  and  $(l, m) \in \mathcal{L}^{(r)}$ , where

- $\mathcal{N}^{(r)}(k)$  denotes the set of all neighboring nodes of bus  $k \in \mathcal{N}$  for contingency  $r$ .

- (2a) and (2b) are the power balance equations accounting for the conservation of energy at bus  $k$ .
- (2c), (2d) and (2e) restrict the active power, reactive power and voltage magnitude at bus  $k$  for contingency  $r$ , given the limits  $P_{k;\min}^{(r)}$ ,  $P_{k;\max}^{(r)}$ ,  $Q_{k;\min}^{(r)}$ ,  $Q_{k;\max}^{(r)}$ ,  $V_{k;\min}^{(r)}$  and  $V_{k;\max}^{(r)}$ .
- The constraint (2f) limits the apparent power flow  $|S_{lm}^{(r)}|$  over the line  $(l, m) \in \mathcal{L}^{(r)}$  by  $S_{lm;\max}^{(r)}$ .
- Given the limits  $\Delta P_{k;\max}^{(r)}$ ,  $\Delta Q_{k;\max}^{(r)}$  and  $\Delta V_{k;\max}^{(r)}$ , the constraints (2g), (2h) and (2i) ensure that the potentially controllable parameters  $P_{G_k}^{(r)}$ ,  $Q_{G_k}^{(r)}$  and  $|V_k^{(r)}|$  vary within permissible ranges after a contingency occurs.

Note that a generator outage can be modeled as  $P_{k;\min}^{(r)} = P_{k;\max}^{(r)} = 0$ .

### A. Convex relaxation for SCOPF

The SCOPF problem includes quadratic constraints such as (2a) and (2b). Nevertheless, all constraints of (2) can be expressed linearly in terms of the entries of the matrix variable  $\mathbf{W}$  defined as  $\mathbf{W} \triangleq \mathbf{V}\mathbf{V}^*$ , where  $\mathbf{V}$  denotes a column vector obtained by stacking the voltage vectors  $\mathbf{V}^{(0)}, \mathbf{V}^{(1)}, \dots, \mathbf{V}^{(c)}$ . On the other hand, the variable  $\mathbf{V}$  can be dropped from the optimization problem by equivalently replacing the consistency constraint  $\mathbf{W} = \mathbf{V}\mathbf{V}^*$  with two new constraints: (i)  $\mathbf{W} \succeq 0$ , and (ii)  $\text{rank}\{\mathbf{W}\} = 1$ . Observe that Constraint (ii) is the only non-convex constraint of the reformulated SCOPF problem. Motivated by this fact, the SDP relaxation of SCOPF is defined as the optimization problem reformulated in terms of  $\mathbf{W}$  under the additional constraint  $\mathbf{W} \succeq 0$  without incorporating the rank constraint  $\text{rank}\{\mathbf{W}\} = 1$ . To formalize this relaxation, let  $\mathbf{W}^{(r)} \in \mathbb{H}^n$  denote the submatrix of  $\mathbf{W}$  in the intersection of rows  $rn + 1, \dots, rn + n$  with columns  $rn + 1, \dots, rn + n$ .

**Relaxed SCOPF:** Minimize

$$\sum_{k \in \mathcal{N}} f_k \left( P_{G_k}^{(0)} \right) \quad (3)$$

over the parameters  $\mathbf{P}_G^{(0)}, \mathbf{P}_G^{(1)}, \dots, \mathbf{P}_G^{(c)} \in \mathbb{R}^n$ ,  $\mathbf{Q}_G^{(0)}, \mathbf{Q}_G^{(1)}, \dots, \mathbf{Q}_G^{(c)} \in \mathbb{R}^n$  and  $\mathbf{W} \in \mathbb{H}^{n(c+1)}$ , subject to

$$P_{G_k}^{(r)} - P_{D_k} = \sum_{l \in \mathcal{N}_k^{(r)}} \text{Re} \left\{ \left( W_{kk}^{(r)} - W_{kl}^{(r)} \right) y_{kl}^* \right\} \quad (4a)$$

$$Q_{G_k}^{(r)} - Q_{D_k} = \sum_{l \in \mathcal{N}_k^{(r)}} \text{Im} \left\{ \left( W_{kk}^{(r)} - W_{kl}^{(r)} \right) y_{kl}^* \right\} \quad (4b)$$

$$P_{k;\min}^{(r)} \leq P_{G_k}^{(r)} \leq P_{k;\max}^{(r)} \quad (4c)$$

$$Q_{k;\min}^{(r)} \leq Q_{G_k}^{(r)} \leq Q_{k;\max}^{(r)} \quad (4d)$$

$$\left( V_{k;\min}^{(r)} \right)^2 \leq W_{kk}^{(r)} \leq \left( V_{k;\max}^{(r)} \right)^2 \quad (4e)$$

$$\left( P_{lm}^{(r)} \right)^2 + \left( Q_{lm}^{(r)} \right)^2 \leq \left( S_{lm;\max}^{(r)} \right)^2 \quad (4f)$$

$$|P_{G_k}^{(r)} - P_{G_k}^{(0)}| \leq \Delta P_{k;\max}^{(r)} \quad (4g)$$

$$|Q_{G_k}^{(r)} - Q_{G_k}^{(0)}| \leq \Delta Q_{k;\max}^{(r)} \quad (4h)$$

$$\left| W_{kk}^{(r)} - W_{kk}^{(0)} \right| \leq \left( \Delta V_{k;\max}^{(r)} \right)^2 \quad (4i)$$

$$\mathbf{W} \succeq 0 \quad (4j)$$

for every  $k \in \mathcal{N}$ ,  $r \in \mathcal{C}$  and  $(l, m) \in \mathcal{L}^{(r)}$ .

The relaxed SCOPF is alternatively referred to as **SDP relaxation** henceforth. Let  $f_{\text{opt}}$  and  $f_{r\text{-opt}}$  denote the optimal objective values of the SCOPF and relaxed SCOPF. As shown in [6] and [30], the relaxed SCOPF is equivalent to the dual of the dual of the SCOPF problem and therefore it provides a lower bound  $f_{r\text{-opt}}$  on the globally minimum solution  $f_{\text{opt}}$  of the original problem (2). Hence,  $f_{\text{opt}} - f_{r\text{-opt}}$  represents the duality gap for the non-convex SCOPF problem, which is not necessarily zero or even small [14], [15]. Zero duality gap is a favorable property because it guarantees the existence of a rank-1 solution  $\mathbf{W}_{\text{opt}}$  for the SDP relaxation from which an optimal vector of voltage phasors  $\mathbf{V}_{\text{opt}}$  can be constructed. It has been shown in [30] that whenever the duality gap of the classical OPF problem is zero for a specific power network, the SCOPF problem also possesses zero duality gap, leading to the presence of a rank-1 solution for the relaxed SCOPF. As investigated in [31] and [13], the duality gap for the OPF problem (without any contingency scenarios) is highly correlated with the topology of the network. In addition, the gap heavily depends on the mathematical formulation of the line capacity constraints [18]. Since a power network has a sparse graph in general, the relaxed SCOPF problem may have infinitely many solutions. As an extreme case, the duality gap could be zero and yet there exist a set of rank-1 and higher-rank solutions for the relaxed problem. To alleviate this issue, the paper [6] suggests adding a small resistance ( $10^{-5}$  per unit) to every ideal transformer with zero resistance.

Using a graph-theoretic approach combined with the SDP relaxation, we aim to study three problems:

- Since the dimension of the matrix variable  $\mathbf{W}$  is prohibitive for a large-scale network, how can the computational complexity of the relaxed SCOPF be reduced?
- What is the rank of an optimal solution  $\mathbf{W}_{\text{opt}}$  of the relaxed SCOPF and how does it relate to the topology of the power grid?
- If the rank of  $\mathbf{W}_{\text{opt}}$  is not 1, how can a near-global solution be recovered for the non-convex SCOPF problem?

## III. LOW-RANK SDP SOLUTION

In this section, the objective is twofold. First, the computational complexity of the relaxed SCOPF will be reduced. Second, the rank of its lowest-rank solution will be studied.

### A. Reduction of computational complexity

*Definition 1 (Treewidth):* Given a graph  $\mathcal{H} = (\mathcal{V}_{\mathcal{H}}, \mathcal{E}_{\mathcal{H}})$ , a tree  $\mathcal{T}$  is called a tree decomposition of  $\mathcal{H}$  if it satisfies the following properties:

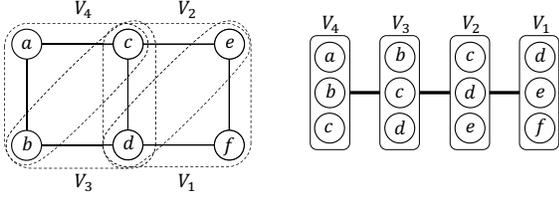


Fig. 1: A minimal tree decomposition for a ladder

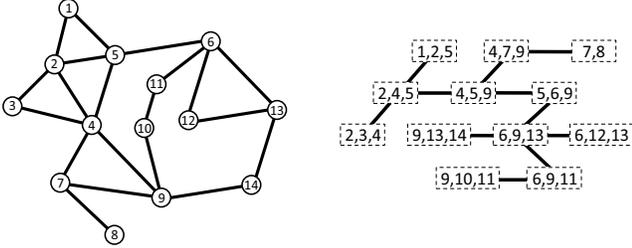


Fig. 2: The IEEE 14-bus test case and its minimal tree decomposition

- 1) Every node of  $\mathcal{T}$  corresponds to and is identified by a subset of  $\mathcal{V}_{\mathcal{H}}$ . Alternatively, each node of  $\mathcal{T}$  is regarded as a group of vertices of  $\mathcal{H}$ .
- 2) Every vertex of  $\mathcal{H}$  is a member of at least one node of  $\mathcal{T}$ .
- 3)  $\mathcal{T}_k$  is a connected graph for  $k \in \mathcal{V}_{\mathcal{H}}$ , where  $\mathcal{T}_k$  denotes the subgraph of  $\mathcal{T}$  induced by all nodes of  $\mathcal{T}$  containing the vertex  $k$  of  $\mathcal{H}$ .
- 4) If  $(i, j) \in \mathcal{E}_{\mathcal{H}}$ , then the subgraphs  $\mathcal{T}_i$  and  $\mathcal{T}_j$  have at least one node in common.

Each node of  $\mathcal{T}$  is a bag (collection) of vertices of  $\mathcal{H}$  and hence it is referred to as **bag**. The width of  $\mathcal{T}$  is the cardinality of its biggest bag minus one. The treewidth of  $\mathcal{H}$  is the minimum width over all possible tree decompositions of  $\mathcal{H}$  and is denoted by  $\text{tw}(\mathcal{H})$ .

Note that the treewidth of a tree is equal to 1. Figure 1 shows a graph  $\mathcal{H}$  with 6 vertices named  $a, b, c, d, e, f$ , together with its minimal tree decomposition  $\mathcal{T}$ . Every node of  $\mathcal{T}$  is a set containing three members of  $\mathcal{V}_{\mathcal{H}}$ . The width of this decomposition is therefore equal to 2. The graph of the IEEE 14-bus system and its minimal tree decomposition are depicted in Figure 2. As shown in Table I, the treewidth of IEEE systems and various setups of Polish systems with as high as 3000 buses is at most 24. This empirical evidence signifies that real-world power grids may have a small treewidth, which is leveraged in this work to solve the SCOPF problem.

*Definition 2 (Sparsity graph):* The sparsity graph of the relaxed SCOPF problem is defined as a graph with  $n(c+1)$  vertices such that  $(i, j)$  is an edge of the graph if  $i \neq j$ ,  $i, j \in \{1, 2, \dots, n(c+1)\}$  and  $W_{ij}$  participates in either of the constraints (4a) and (4b).

Consider a tree decomposition of the power network in the pre-contingency case and denote its bags (nodes) as  $\mathcal{J}_1^{(0)}, \mathcal{J}_2^{(0)}, \dots, \mathcal{J}_p^{(0)} \subseteq \mathcal{N}$ .

*Theorem 1:* The following statements hold:

- i) The sparsity graph of the relaxed SCOPF problem has

a tree decomposition with  $p(c+1)$  bags given by the set:

$$\left\{ \mathcal{J}_m^{(0)} + nr \mid m = 1, \dots, p \text{ and } r = 0, \dots, c \right\} \quad (5)$$

- ii) The optimal objective value of the relaxed SCOPF problem does not change if its constraint  $\mathbf{W} \succeq 0$  is replaced by

$$\mathbf{W} \left\{ \mathcal{J}_m^{(0)} + nr, \mathcal{J}_m^{(0)} + nr \right\} \succeq 0 \quad (6)$$

or equivalently

$$\mathbf{W}^{(r)} \left\{ \mathcal{J}_m^{(0)}, \mathcal{J}_m^{(0)} \right\} \succeq 0 \quad (7)$$

for every  $r \in \mathcal{C}$  and  $m \in \{1, \dots, p\}$ .

Define **decomposed relaxed SCOPF** as a convex optimization obtained from the relaxed SCOPF by replacing  $\mathbf{W} \succeq 0$  with the constraints  $\mathbf{W}^{(r)} \left\{ \mathcal{J}_m^{(0)}, \mathcal{J}_m^{(0)} \right\} \succeq 0$  for every  $r \in \mathcal{C}$  and  $m \in \{1, \dots, p\}$ . Theorem 1 reduces the computational cost of the SDP relaxation dramatically for a large-scale system with a relatively small treewidth. Note that many entries of the matrix variable  $\mathbf{W}$  may not appear in the objective or constraints of the decomposed relaxed SCOPF, and those redundant entries can be eliminated. For example, the relaxed OPF for a Polish system has about 9,000,000 scalar variables, while the decomposed relaxed OPF has only about 100,000 parameters. As will be illustrated later, this enables us to solve a large-scale problem efficiently.

### B. Existence of low-rank solution

Let  $\mathbf{W}_{\text{ref}} \in \mathbb{H}^{n(c+1)}$  denote an arbitrary solution of the relaxed SCOPF or decomposed relaxed SCOPF. Note that if  $\mathbf{W}_{\text{ref}}$  corresponds to the decomposed problem, its redundant entries may not have been found by the numerical algorithm and are regarded as “missing”. The following question arises: is it possible to fine-tune the entries of  $\mathbf{W}_{\text{ref}}$  or design its missing entries to arrive at a different, but lower rank, solution of the (decomposed) relaxed problem? To address this problem, we propose an iterative algorithm that transforms  $\mathbf{W}_{\text{ref}}$  into a solution  $\mathbf{W}_{\text{opt}}$  whose rank is upper bounded by the treewidth of the network. To introduce our algorithm, consider a tree decomposition  $\mathcal{T}_{\mathcal{C}}$  of the sparsity graph of the relaxed SCOPF problem with the bags specified in (5). For simplicity, we name the bags as  $\mathcal{J}_1, \mathcal{J}_2, \dots, \mathcal{J}_{p(c+1)}$ .

### Matrix completion algorithm:

- 1) Set  $\mathcal{T}' := \mathcal{T}_{\mathcal{C}}$  and  $\mathbf{W} := \mathbf{W}_{\text{ref}}$ .
- 2) If  $\mathcal{T}'$  has a single node, then consider  $\mathbf{W}_{\text{opt}}$  as  $\mathbf{W}$  and terminate; otherwise continue to the next step.
- 3) Choose a pair of bags  $\mathcal{J}_x, \mathcal{J}_y$  of  $\mathcal{T}'$  such that  $\mathcal{J}_x$  is a leaf of  $\mathcal{T}'$  and  $\mathcal{J}_y$  is its unique neighbor.
- 4) Define

$$\mathbf{A} \triangleq \mathbf{W} \{ \mathcal{J}_x \cap \mathcal{J}_y, \mathcal{J}_x \cap \mathcal{J}_y \} \quad (8a)$$

$$\mathbf{B}_x \triangleq \mathbf{W} \{ \mathcal{J}_x \setminus \mathcal{J}_y, \mathcal{J}_x \cap \mathcal{J}_y \} \quad (8b)$$

$$\mathbf{B}_y \triangleq \mathbf{W} \{ \mathcal{J}_y \setminus \mathcal{J}_x, \mathcal{J}_x \cap \mathcal{J}_y \} \quad (8c)$$

$$\mathbf{X} \triangleq \mathbf{W}\{\mathcal{J}_x \setminus \mathcal{J}_y, \mathcal{J}_x \setminus \mathcal{J}_y\} \in \mathbb{C}^{d_x \times d_x} \quad (8d)$$

$$\mathbf{Y} \triangleq \mathbf{W}\{\mathcal{J}_y \setminus \mathcal{J}_x, \mathcal{J}_y \setminus \mathcal{J}_x\} \in \mathbb{C}^{d_y \times d_y} \quad (8e)$$

$$\mathbf{S}_x \triangleq \mathbf{X} - \mathbf{B}_x \mathbf{A}^+ \mathbf{B}_x^* = \mathbf{Q}_x \mathbf{\Lambda}_x \mathbf{Q}_x^* \quad (8f)$$

$$\mathbf{S}_y \triangleq \mathbf{Y} - \mathbf{B}_y \mathbf{A}^+ \mathbf{B}_y^* = \mathbf{Q}_y \mathbf{\Lambda}_y \mathbf{Q}_y^* \quad (8g)$$

where  $\mathbf{Q}_x \mathbf{\Lambda}_x \mathbf{Q}_x^*$  and  $\mathbf{Q}_y \mathbf{\Lambda}_y \mathbf{Q}_y^*$  denote the eigenvalue decompositions of  $\mathbf{S}_x$  and  $\mathbf{S}_y$  with the diagonals of  $\mathbf{\Lambda}_x$  and  $\mathbf{\Lambda}_y$  arranged in descending order. Update a part of  $\mathbf{W}$  as follows:

$$\begin{aligned} \mathbf{W}\{\mathcal{J}_y \setminus \mathcal{J}_x, \mathcal{J}_x \setminus \mathcal{J}_y\} &:= \mathbf{B}_y \mathbf{A}^+ \mathbf{B}_x^* \\ &+ \mathbf{Q}_y \sqrt{\mathbf{\Lambda}_y} \mathbf{I}_{d_y \times d_x} \sqrt{\mathbf{\Lambda}_x} \mathbf{Q}_x^* \end{aligned}$$

and update  $\mathbf{W}\{\mathcal{J}_x \setminus \mathcal{J}_y, \mathcal{J}_y \setminus \mathcal{J}_x\}$  as well to preserve the symmetry of  $\mathbf{W}$ .

- 5) Update  $\mathcal{T}'$  by merging  $\mathcal{J}_x$  into  $\mathcal{J}_y$ , i.e., replace  $\mathcal{J}_y$  with  $\mathcal{J}_x \cup \mathcal{J}_y$  and then remove  $\mathcal{J}_x$  from  $\mathcal{T}'$ .
- 6) Go back to step 2.

*Theorem 2:* Consider an arbitrary solution  $\mathbf{W}_{\text{ref}}$  of the (decomposed) relaxed SCOPF problem. The output of the matrix completion algorithm, denoted as  $\mathbf{W}_{\text{opt}}$ , is a solution of the (decomposed) relaxed SCOPF problem whose rank is smaller than or equal to:

$$\max \left\{ \text{rank} \left\{ \mathbf{W}_{\text{ref}}^{(r)} \{ \mathcal{J}_m^{(0)}, \mathcal{J}_m^{(0)} \} \right\} \mid 1 \leq m \leq p, r \in \mathcal{C} \right\}.$$

Note that Theorem 2 is valid for not only relaxed SCOPF but also decomposed relaxed SCOPF. The following two results are the by-products of the above theorem.

*Corollary 1:* If the (decomposed) relaxed SCOPF problem is feasible, then it has a solution  $\mathbf{W}_{\text{opt}}$  whose rank is upper bounded by the treewidth of the power network in the pre-contingency case.

*Corollary 2:* The non-convex SCOPF problem has the same globally optimal value as that of the (decomposed) relaxed SCOPF under the additional constraints

$$\text{rank} \{ \mathbf{W}^{(r)} \{ \mathcal{J}_m^{(0)}, \mathcal{J}_m^{(0)} \} \} = 1 \quad (9)$$

for every  $r \in \mathcal{C}$  and  $m \in \{1, \dots, p\}$ .

#### IV. RECOVERY OF A NEAR-GLOBAL SOLUTION

We explored the properties of the decomposed relaxed SCOPF in the preceding section. In this part, we aim to address two problems: (i) how to find a tree decomposition of the power network in order to be able to formulate the decomposed problem, (ii) how to recover a near-global solution of the SCOPF problem through an SDP relaxation.

##### A. Tree decomposition algorithm

In what follows, we describe an effective algorithm for finding a tree decomposition that is used in all of the simulations offered in the next section. This algorithm combines the greedy degree and greedy fill-in algorithms presented in [32] in order to obtain a tree decomposition for a graph with a low maximum clique order. The algorithm described here designs a graph  $\mathcal{T}$  with a set of nodes and no edges. Since the decomposed relaxed SCOPF only needs the bags of  $\mathcal{T}$ , it is unnecessary to find the edges of the tree decomposition.

Consider a graph  $\mathcal{H} = (\mathcal{V}_{\mathcal{H}}, \mathcal{E}_{\mathcal{H}})$  together with an arbitrary vertex  $u$  of this graph.  $\delta_{\mathcal{H}}(u)$  denotes the degree of  $u \in \mathcal{V}_{\mathcal{H}}$ . The fill-in of  $u$ , is defined as the number of edges whose addition to the subgraph formed by the neighbors of  $u$  makes the resulting subgraph a clique (complete subgraph) which is denoted by  $\phi_{\mathcal{H}}(u)$ . The vertex  $u$  is called simplicial if  $\phi_{\mathcal{H}}(u) = 0$  (i.e., if the neighbors of  $u$  are all connected to one another).

##### Greedy decomposition algorithm:

- 1) Consider  $\alpha$  as an arbitrary constant and define  $\mathcal{H}' = \mathcal{H}$ .
- 2) If  $\mathcal{H}'$  has a single vertex, then define  $\mathcal{T}$  as the single node  $\mathcal{V}_{\mathcal{H}}$  and terminate; otherwise continue.
- 3) Choose a vertex  $u$  in  $\mathcal{H}'$  according to the following rules:
  - If  $\mathcal{H}'$  has a simplicial node, then set  $u$  as that vertex.
  - Otherwise, set  $u$  as a (non-unique) vertex of  $\mathcal{H}'$  that minimizes the function  $\phi_{\mathcal{H}'}(u) + \alpha \times \delta_{\mathcal{H}'}(u)$ .
- 4) Define  $\mathcal{U}$  as the set of all neighboring vertices of  $u$  in  $\mathcal{H}'$ . Add the bag  $\mathcal{U} \cup \{u\}$  to  $\mathcal{T}$ , and then update the graph  $\mathcal{H}'$  by first connecting all vertices in  $\mathcal{U}$  to each other and then removing  $u$ . Jump to Step 2.

Based on [32], it is straightforward to show that a set of edges can be added to the nodes of  $\mathcal{T}$  to make it a tree decomposition for  $\mathcal{H}$ .

##### B. Penalization method

Consider the case where the (decomposed) relaxed SCOPF has no rank-1 solution. Suppose that it is possible to design a convex function  $g(\mathbf{W}^{(0)}, \dots, \mathbf{W}^{(c)})$  such that the SDP relaxation admits a rank-1 solution if the objective of the relaxed SCOPF is replaced by this function. Then, penalizing the objective of the relaxed SCOPF with  $\varepsilon \times g(\cdot)$  may lead to an approximate rank-1 SDP solution and subsequently a near-global SCOPF solution for an appropriate choice of the penalty coefficient  $\varepsilon$ . Therefore, the main challenge is to seek a penalization function  $g(\cdot)$ . The recent literature of compressed sensing suggests a penalty term consisting of a weighted sum of the diagonal entries of  $\mathbf{W}$  [33]. However, this idea fails to work for SCOPF since all feasible solutions of the SDP relaxation have similar diagonal values due to tight voltage control in practice. We propose a different penalty function in this paper. Consider a positive semidefinite matrix  $\mathbf{X}$  with constant (fixed) diagonal entries  $X_{11}, \dots, X_{nn}$  and variable off-diagonal entries. If we maximize a weighted sum of the off-diagonal entries of  $\mathbf{X}$  with positive weights, then it turns out that  $X_{lm} = \sqrt{X_{ll} X_{mm}}$  for all  $l, m \in \{1, \dots, n\}$ , in which case  $\mathbf{X}$  becomes rank-1. Motivated by this fact, we employ the idea of elevating the off-diagonal entries of  $\mathbf{W}$  to obtain a low-rank solution. For a lossless network, sum of the total reactive power generation increases the weighted sum of the real parts of the off-diagonal entries of  $\mathbf{W}$ . Also, penalizing the apparent power loss over the series impedance of the lines of the network plays a similar role for a lossy network.

**Penalized SDP relaxation:** This optimization is obtained from the (decomposed) relaxed SCOPF problem by replacing

its objective function with

$$\sum_{k \in \mathcal{N}} f_k \left( P_{G_k}^{(0)} \right) + \varepsilon_b \sum_{\substack{k \in \mathcal{N} \\ r \in \mathcal{C}}} Q_{G_k}^{(r)} + \varepsilon_l \sum_{(r,l,m) \in \mathcal{L}^{\text{prob}}} L_{lm}^{(r)} \quad (10)$$

for given positive numbers  $\varepsilon_b, \varepsilon_l$  and a set of triples  $\mathcal{L}^{\text{prob}} \subseteq \{(r, l, m) \mid r \in \mathcal{C}, (l, m) \in \mathcal{L}^{(r)}\}$ , where  $L_{lm}^{(r)}$  represents the apparent power loss over the series impedance of line  $(l, m)$  for contingency  $r$ .

Let  $\mathbf{W}_{\text{opt}}$  and  $\mathbf{W}_\varepsilon$  denote arbitrary solutions of the SDP and penalized SDP relaxations, respectively. Assume that  $\mathbf{W}_{\text{opt}}$  does not have rank 1, whereas  $\mathbf{W}_\varepsilon$  has rank 1. By decomposing  $\mathbf{W}_\varepsilon$  as  $\mathbf{V}_\varepsilon \mathbf{V}_\varepsilon^*$ , a feasible solution  $\mathbf{V}_\varepsilon$  of the SCOPF can be obtained. In addition, the optimal value  $f_{\text{opt}}$  of the SCOPF problem is lower bounded by the optimal value  $f_{r\text{-opt}}$  of the SDP relaxation and upper bounded by  $f_\varepsilon$ , where  $f_\varepsilon$  is defined as the total generation cost associated with the operating point  $\mathbf{V}_\varepsilon$ . Define *global optimality degree* as  $100 - \frac{f_\varepsilon - f_{r\text{-opt}}}{f_\varepsilon} \times 100$ . This shows the closeness of the feasible solution  $\mathbf{V}_\varepsilon$  to the unknown globally optimal solution (in percentage). For example, if the global optimality degree is 99%, then the cost associated with the global solution of SCOPF can be at most 1% better than the cost corresponding to the feasible point. In summary, if the penalized SDP relaxation has a rank-1 solution, then a feasible solution of SCOPF together with its global optimality degree can be computed.

The success of the penalized SDP relaxation is in part related to the choice of  $\mathcal{L}^{\text{prob}}$ . Sometimes, a good choice is to consider this set as the collection of all lines of the system in pre- and post-contingency cases. In what follows, we propose an effective heuristic method for designing  $\mathcal{L}^{\text{prob}}$ . Consider a bag  $\mathcal{J}_i$  of the tree decomposition  $\mathcal{T}_\mathcal{C}$  together with a matrix  $\mathbf{W}$ .  $\mathcal{J}_i$  is called a **problematic bag** associated with  $\mathbf{W}$  if  $\mathbf{W}\{\mathcal{J}_i, \mathcal{J}_i\}$  does not have rank 1. Any line of the pre- or post-contingency network corresponding to an off-diagonal entry of  $\mathbf{W}\{\mathcal{J}_i, \mathcal{J}_i\}$  is called a *problematic line* associated with  $\mathbf{W}$ . It follows from Theorem 2 and Corollary 2 that the SDP relaxation is exact if there is no problematic bags/lines associated with the solution of the decomposed relaxed SCOPF.

**Problematic line selection algorithm:** Consider the penalized SDP relaxation with  $\varepsilon_l = 0$ . Using a bisection method, find a value for  $\varepsilon_b$  such that the number of problematic lines associated with the penalized SDP solution is small (minimum). Consider  $\mathcal{L}^{\text{prob}}$  as the set of problematic lines.

Assume that the penalized SDP relaxation results in a rank-1 solution or a low-rank matrix solution with a dominant nonzero eigenvalue. The next algorithm can be used to find an approximate feasible solution of SCOPF.

**Recovery algorithm:** Given a low-rank solution  $\mathbf{W}_{\text{opt}}$  of the penalized SDP relaxation, we recover an approximate solution  $\mathbf{V}$  for the SCOPF according to the following procedure:

- 1) Set the voltage magnitude  $V_i$  equal to the square root of the  $(i, i)$  entry of  $\mathbf{W}_{\text{opt}}$  for  $i = 1, \dots, n(c+1)$ .

- 2) Find the phases of the entries of  $\mathbf{V}$  through a convex program by minimizing

$$\sum_{r \in \mathcal{C}} \sum_{(l,m) \in \mathcal{L}^{(r)}} \left| \angle(\mathbf{W}_{\text{opt}})_{lm}^{(r)} - \angle \mathbf{V}_l^{(r)} + \angle \mathbf{V}_m^{(r)} \right| \quad (11)$$

over the variables  $\angle \mathbf{V}^{(0)}, \angle \mathbf{V}^{(1)}, \dots, \angle \mathbf{V}^{(c)} \in [-\pi, \pi]^n$ , subject to

$$\angle V_1^{(r)} = 0, \quad r = 0, 1, \dots, c \quad (12)$$

Note that the above recovery algorithm retrieves a globally optimal solution of the SCOPF problem in the case where  $\text{rank}\{\mathbf{W}_{\text{opt}}\} = 1$ . Under that circumstance, we have  $\angle(\mathbf{W}_{\text{opt}})_{lm}^{(r)} - \angle \mathbf{V}_l^{(r)} + \angle \mathbf{V}_m^{(r)} = 0$ . If the rank of  $\mathbf{W}_{\text{opt}}$  is not 1 but this matrix has a dominant nonzero eigenvalue, the above recovery method aims to find a vector  $\mathbf{V}$  for which the corresponding line angle differences are as closely as possible to those suggested by the matrix  $\mathbf{W}_{\text{opt}}$ .

## V. SIMULATIONS RESULTS

In what follows, we offer several simulations for OPF and SCOPF problems. We have written a custom OPF Solver to perform these simulations [34].

**3-bus system:** As our first example, consider the 3-bus system presented in [14]. The SDP relaxation may not result in a rank-1 solution for this system if a certain line is under stress (i.e. the capacity constraint of the line is binding at optimality). To address this issue, we use the penalized SDP relaxation with the objective function (10), where the parameter  $\varepsilon_b$  is set to be zero and the line under stress is chosen for penalization. The resulting optimal cost is reported in Figure 3(a) for different values of  $\varepsilon_l$ . It can be seen that there exists a relatively large interval for  $\varepsilon_l$  that makes the penalized SDP relaxation possess a rank-1 solution with a fixed cost. This cost overlaps with the globally optimal cost of the OPF problem. Hence, our method is able to bridge the duality gap reported in [14].

**IEEE and Polish systems:** As our second example, we evaluate the penalization method for the OPF problem performed over benchmark systems. The results are reported in Table I. For each system, the following numbers are reported:

- TW: an upper bound on the treewidth of the system
- # prob. bags: number of problematic bags for the SDP relaxation with  $\varepsilon_b = \varepsilon_l = 0$
- Lower bound: lower bound on the globally minimum cost of OPF
- Upper Bound: upper bound on the globally optimal cost of OPF, corresponding to the recovered solution
- Opt. degree: global optimality degree (in percentage)
- Com. time: the total computation time (in seconds) including those consumed towards tree decomposition, solving the SDP relaxation, and recovering a solution (the simulations were run on a desktop computer with an Intel Core i7 quad-core 3.4 GHz CPU and 16 GB RAM).

Test cases	$\alpha$	TW	# prob. bags	$\epsilon_b$	$\epsilon_l$	Lower bound	Upper bound	Opt. degree	Com. time
Chow's 9 bus	0	2	2	10	0	5296.68	5296.68	%100	$\leq 5$
IEEE 14 bus	0	2	0	0	0	8081.53	8081.53	%100	$\leq 5$
IEEE 24 bus	0	4	0	0	0	63352.20	63352.20	%100	$\leq 5$
IEEE 30 bus	0	3	1	0.1	0	576.89	576.89	%100	$\leq 5$
NE 39 bus	0	3	1	10	0	41862.08	41864.40	%99.994	$\leq 5$
IEEE 57 bus	0	5	0	0	0	41737.78	41737.78	%100	$\leq 5$
IEEE 118 bus	0	4	61	10	0	129654.61	129660.81	%99.995	$\leq 5$
IEEE 300 bus	0	6	7	0.1	100	719711.63	719725.10	%99.998	13.9
Polish 2383wp	0	23	651	3500	3000	1861510.42	1874322.65	%99.316	529
Polish 2736sp	0	23	1	1500	0	1307882.29	1308270.20	%99.970	701
Polish 2737sop	0	23	3	1000	0	777626.26	777664.02	%99.995	675
Polish 2746wop	0	23	1	1000	0	1208273.91	1208453.93	%99.985	801
Polish 2746wp	0	24	1	1000	0	1631772.83	1632384.87	%99.962	699
Polish 3012wp	1	24	605	0	10000	2587740.98	2608918.45	%99.188	814
Polish 3120sp	-1.5	24	20	0	10000	2140765.92	2160800.42	%99.073	910

TABLE I: Performance of the penalization method for several benchmark systems.

Note that the permissible feasibility violation for the recovered solution was set to  $10^{-6}$  for all cases reported in Table I, except for Polish 3012wp and Polish 3120sp for which the violation level was set to  $1.5 \times 10^{-5}$ . For Polish 3012wp and Polish 3120sp, we penalized the apparent power loss over all lines of each system. For IEEE 300 bus system, we penalized apparent power loss over two lines: (i) line 38 between buses 9053 and 9533, (ii) line 402 between buses 7023 and 23. These lines are problematic for the penalized SDP problem in the case  $\epsilon_b = 0.1$  and  $\epsilon_l = 0$ . Similarly, the apparent power loss penalization for Polish 2383wp system was performed over problematic lines for the case  $\epsilon_b = 3500$  and  $\epsilon_l = 0$ , leading to the following 9 lines: line 100 between buses 35 and 34, line 101 between buses 34 and 51, line 102 between buses 183 and 34, line 103 between buses 183 and 35, line 104 between buses 617 and 35, line 130 between buses 51 and 50, line 134 between buses 727 and 51, line 819 between buses 546 and 545, and line 821 between buses 727 and 545. It can be observed that the penalized SDP relaxation was able to find feasible solutions with global optimality degrees above 99% for many benchmark examples.

Contingency number	Line number	Initial node	Terminal node
1	1	1	2
2	2	1	39
3	3	2	3
4	4	2	25
5	6	3	4
6	7	3	18
7	15	7	8
8	20	10	32
9	40	25	26
10	45	28	29

TABLE II: List of contingencies for New England system.

**New England 39 bus system:** Consider the New England system under 10 contingency scenarios, each representing the outage of one line as described in Table II. The corrective active power production for each generator in case of contingency is set to be within 2 MW away from the base

Contingency number	Line number	Initial node	Terminal node
1	266	19	231
	388	234	236
	400	7130	130

TABLE III: List of lines outages of the contingency scenario considered for IEEE 300 bus system.

case production level. We solve the penalized SDP relaxation by setting  $\epsilon_b$  to zero and minimizing the apparent power loss over all lines of the network. The result is depicted in Figures 3(b) for different values of the coefficient  $\epsilon_l$ . It can be seen that a near-global solution for the SCOPF problem is associated with the cost 45141.70. This SCOPF cost is 7% higher than the optimal cost of the OPF problem with no contingency.

**IEEE 300 bus system:** Consider the 300 bus system with one contingency scenario associated with the simultaneous outage of three highly congested lines of the base OPF. These lines are listed in table III. The corrective active power production for each generator in case of contingency is set to be within 1 MW away from the base case production level. As before, we solve the penalized SDP relaxation by setting  $\epsilon_b$  to zero and minimizing the apparent power loss over all lines of the network. The result is depicted in Figures 3(c). It can be seen that a near-global solution for the SCOPF problem is associated with the cost 740493.80 which is 3% different from that of the OPF problem.

## VI. CONCLUSIONS

This paper studies the security-constrained optimal power flow (SCOPF) problem by means of a semidefinite programming (SDP) relaxation. The existence of a rank-1 solution guarantees that this convex program will find a globally optimal solution of the SCOPF problem. First, we prove that the SDP relaxation has a solution whose rank is at most equal to the treewidth of the power network, which is expected to be very small for real-world systems. Second, we propose a decomposition method to reduce the computational

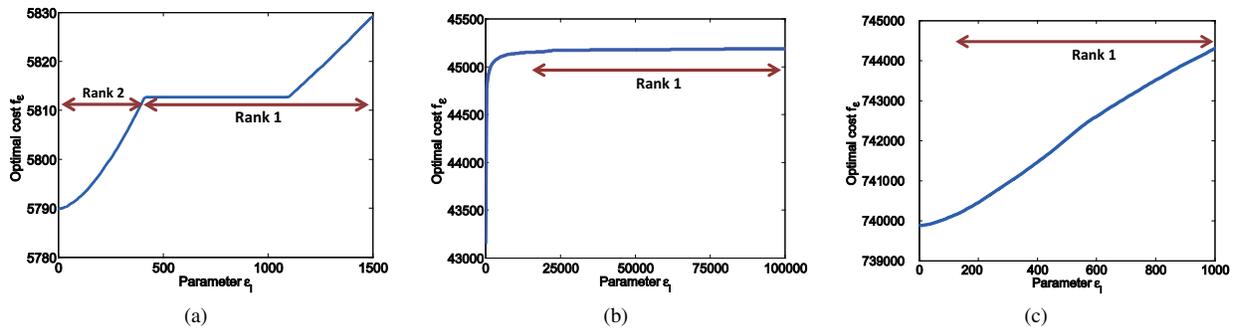


Fig. 3: (a) The 3-bus system presented in [14]; (b) contingency analysis of New England system; (c) contingency analysis of IEEE 300 bus system.

complexity of the SDP relaxation. In the case where the SDP relaxation fails to work, we develop a graph-theoretic method to identify the problematic lines of the network that make SCOPF difficult to solve. By penalizing the loss over those lines in the SDP relaxation, we develop a rank-enforcing SDP relaxation. We test our relaxation method on several benchmark examples and demonstrate its ability in finding feasible solutions of SCOPF that are at least 99% globally optimal.

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