Homotopy Method for Finding the Global Solution of Post-contingency Optimal Power Flow

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***Abstract*—The goal of optimal power flow (OPF) is to find a minimum cost production of committed generating units while satisfying technical constraints of the power system. To ensure robustness of the network, the system must be able to find new operating points within the technical limits in the event of component failures such as line and generator outages. However, finding an optimal, or even a feasible, preventive/corrective action may be difficult due to the innate nonconvexity of the problem. With the goal of finding a global solution to the post-contingency OPF problem of a stressed network, e.g. a network with a line outage, we apply a homotopy method to the problem. By parametrizing the constraint set, we define a series of optimization problems to represent a gradual outage and iteratively solve these problems using local search. Under the condition that the global minimum of the OPF problem for the base-case is attainable, we find theoretical guarantees to ensure that the OPF problem for the contingency scenario will also converge to its global minimum. We show that this convergence is dependent on the geometry of the homotopy path. The effectiveness of the proposed approach is demonstrated on Polish networks.**

I. Introduction

Optimal power flow (OPF) is a fundamental tool for power system network analysis. The goal of OPF is to find a minimum cost production of the committed generating units while satisfying the technical constraints of the power system. To ensure security, additional care must be taken so that the system is able to operate within the technical limits even in the event of component failures (i.e. contingencies). Finding a global optimum for a large-scale OPF problem modeled with AC power flow equations is a difficult task due to the innate nonconvexity of the problem.

The ability to find a feasible and globally optimal solution to the OPF problems is crucial for the reliable and efficient operation of power systems. The difference between a global minimum of the OPF problem and sub-optimal/approximate solutions obtained using the heuristics adopted by the power industry is estimated at billions of dollars each year in the United States. A mere 1% improvement in OPF solution quality would save up to 5 billion dollars annually in the U.S. [1]. Initiated by the work [2], conic optimization has been extensively studied in recent years and proven to be a powerful technique for solving OPF to global or near-global optimality. The paper [2] has indeed shown that a semidefinite programming (SDP) relaxation is able to find a global minimum of OPF for a large class of practical systems, and [3] has discovered that the success of this method is related to the underlying physics of power systems. [4] and [5] have developed different sufficient conditions under which the SDP relaxation of OPF provides zero duality gap. Moreover, [6] has found an upper bound on the the rank of the minimum rank solution of the SDP relaxation of the OPF problem, which is leveraged in [7] to find a near globally optimal solution of OPF via a penalized SDP technique in the case where the SDP relaxation fails to work. These ideas have been refined in many papers to improve the relaxations via branch-and-cut approaches, conic hierarchies, and valid inequalities [8]–[11]. The reader is referred to the survey paper [12] for more details.

Power operators are concerned with security-constrained OPF (SCOPF) instead of an idealistic OPF problem. SCOPF can be regarded as a large number (as high as 10,000) of OPF problems coupled to each other via physical constraints, where the first OPF corresponds to the operating point of the system under the normal condition and the remaining ones are associated with contingencies. It is shown in [7] that SDP relaxations are able to obtain high-quality solutions of SCOPF. However, since SCOPF is a gigantic problem with an enormous number of variables, even simple local search methods are ineffective for real-world systems. The common practice is to approximate SCOPF as a single OPF for the base case subject to many surrogate contingency constraints expressed in terms of the variables of the base case. SDP and many other methods may be used to solve such problems, but that would only find the voltages for the base case. It is essential for the operators to determine the values of the controllable parameters for each contingency in advance. This leads to a high number of decoupled optimization problems. The objective of this paper is to solve each of these problems, named *contingency*-OPF, to global optimality using fast algorithms.

A. Homotopy in Optimization

Homotopy methods have been used to improve the convergence of optimization problems. While convergence to a global minimum with probability one is guaranteed for a convex problem [13], this is not true for nonconvex problems. In order to understand when homotopy can be effective in finding a global solution for nonconvex optimization, we explore a minimization problem of the form: $(P^o)\ \min_x f(x)$ where

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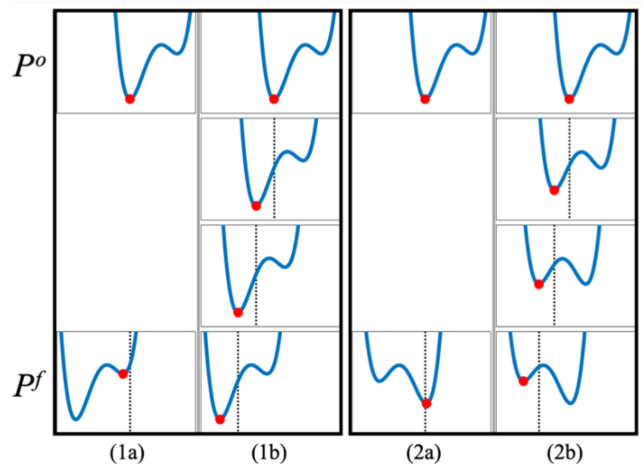

Fig. 1. Evaluating the performance of homotopy on one-dimensional unconstrained minimization problems. The figure compares two different problems (1) and (2), with two different methods (a) and (b). The dotted lines show how the solution from the previous iteration is used in local search algorithms to solve the next problem. The red dots show the solution at each iteration using the position of the dotted lines as the initial point. For the one-shot method (a), the result of P^o is used as the initial point for P^f. For the homotopy method (b), the base problem P^o is gradually transformed to P^f over three iterations, updating the initial point as the solution to the previous problem.

$f : \mathbb{R}^n \to \mathbb{R}$ is a nonconvex function of $x \in \mathbb{R}^n$. Note that the function $f(\cdot)$ can incorporate exact/inexact penalty functions to enforce constraints on x, implying that this formulation is general for both unconstrained and constrained optimization [14]. We will call (P^o) the "base case" problem. We generate a new problem, a deformed version of the base case, which is also a nonconvex minimization problem. Let the deformed problem be: $(P^f)\ \min_x \tilde{f}(x)$. We explore two possible methods for solving the deformed problem that are based on local search:

1) *One-shot method:* Use the solution of P^o as the initial point for any descent numerical algorithm to solve P^f.
2) *Homotopy method:* Generate a (discretized) homotopy map from P^o to P^f. Use the solution of P^o as the initial point, but update it at each step of the homotopy by solving an intermediate problem using local search that is initialized at the solution of the previous step. A linear (un-discretized) homotopy map can be defined as:

$$P(\lambda) = \min_x \left\{ \lambda \tilde{f}(x) + (1-\lambda) f(x) \right\}, \quad 0 \le \lambda \le 1$$

with the property that $P(0) = P^o$ and $P(1) = P^f$.

Depending on $f(x)$ and $\tilde{f}(x)$, homotopy may or may not lead to better results than solving the deformed problem in one shot. In Figure 1, we see an example where homotopy is effective to find the global minimum of a deformed function and another example where homotopy leads to a non-global local minimum whereas solving the problem in one shot leads to the global minimum. Knowing when homotopy will be effective is highly dependent on understanding how the shape of the function changes from the base case to the deformed problem. In the current literature, there is a lack of theoretical results to characterize the performance of homotopy in finding a global optimum.

B. Contributions

In this paper, we develop a homotopy method to improve the quality of the *contingency*-OPF solution. Instead of solving for the solution to a *contingency*-OPF problem via a descent numerical algorithm directly, we generate and solve (using local search) a series of optimization problems wherein we gradually remove a component of the power system. We show that the effectiveness of homotopy to find a global solution of the *contingency*-OPF problem is dependent on the homotopy path, and we introduce new theory to characterize desirable homotopy paths. Note that it is essential to find a global solution because constraint violations in the case of a contingency are very expensive to deal with and a global solution corresponds to the minimum violation.

C. Notations

The symbols $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. $\mathbb{R}^N$ and $\mathbb{C}^N$ denote the spaces of N-dimensional real and complex vectors, respectively. The symbols $(\cdot)^T$ and $(\cdot)^*$ denote the transpose and conjugate transpose of a vector or matrix. Re$\{\cdot\}$ and Im$\{\cdot\}$ denote the real and imaginary part of a given scalar or matrix. The symbol $|\cdot|$ is the absolute value operator if the argument is a scalar, vector, or matrix; otherwise, it is the cardinality of a measurable set. The imaginary unit is denoted by $\mathbf{j} = \sqrt{-1}$.

II. Formulation of the Dispatch Problems

In this section, we present the mathematical formulations for the *base*-OPF with security constraints and the *contingency*-OPF. To begin, let the power network be defined by a graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ with the set of generators $\mathcal{R}$, where $\mathcal{V}$ and $\mathcal{E}$ are the vertex set and the edge set of this graph, respectively. The "classic" optimal power flow problem without contingency considerations is a static optimization problem formulated as

$$\begin{aligned}
\min_{v} \quad & f(v) + \psi(v) \\
\text{subject to} \quad & p_i^g - \sum_{(i,j)\in\mathcal{E}} p_{ij} = P_i^d \quad \forall i \in \mathcal{V} \\
& q_i^g - \sum_{(i,j)\in\mathcal{E}} q_{ij} = Q_i^d \quad \forall i \in \mathcal{V} \\
& p_{ij} = Re\{v_i(v_i - v_j)^* Y_{ij}^*\} \quad \forall (i,j) \in \mathcal{E} \\
& q_{ij} = Im\{v_i(v_i - v_j)^* Y_{ij}^*\} \quad \forall (i,j) \in \mathcal{E}
\end{aligned}$$

where $f(\cdot)$ represents the operating cost (usually a quadratic function of the active power generations) and $\psi(\cdot)$ represents the exact penalty or inexact penalty function that forces the variables to stay within the feasible set defined by:

$$\Psi = \left\{ v \,\middle|\, \begin{array}{ll} P_i^{min} \le p_i^g \le P_i^{max} & \forall i \in \mathcal{R} \\ Q_i^{min} \le q_i^g \le Q_i^{max} & \forall i \in \mathcal{R} \\ V_i^{min} \le |v_i| \le V_i^{max} & \forall i \in \mathcal{V} \\ |p_{ij} + \mathbf{j} q_{ij}| \le S_{ij}^{max} & \forall (i,j) \in \mathcal{E} \end{array} \right\} \tag{1}$$

In this problem, the decision variable v represents the vector of complex voltages of the power system, and v_i is the voltage at the i-th bus. Furthermore, $p_i^g, q_i^g, p_{ij}, q_{ij}, P_i^d$ and Q_i^d are the active/reactive power generation at the i-th bus, active/reactive power flow from bus i to j, and active/reactive power demand at bus i, respectively. $Y_{ij} = G_{ij} + \mathbf{j}B_{ij}$ is the line admittance, whose real and imaginary parts are the line conductance and susceptance, respectively. The constraints model technical limits, such as the power flow equations, bounds on voltage magnitudes, and bounds on power generations and flows. Nonlinearities are introduced to the constraints with the AC power flow equations, and these nonlinearities with the voltage magnitude lower bounds result in the nonconvexity of the problem. In a standardized optimization form, the "classic" OPF problem can be expressed in a compact form as follows (note that $h(\cdot)$ is a vector):

$$\begin{aligned} \min_{v} \quad & f(v) + \psi(v) \\ \text{subject to} \quad & h(v) = 0 \end{aligned} \tag{2}$$

A. Security-constrained optimal power flow

Suppose that there is a set of possible contingencies, namely $\mathcal{K}$, where each contingency corresponds to a line or generator outage. Each contingency $k \in \mathcal{K}$ introduces a new set of voltage variables v^k, and therefore, for a network with N buses and $|\mathcal{K}|$ contingencies, the SCOPF problem will involve optimizing over $N(|\mathcal{K}|+1)$ complex scalar voltage variables. The contingencies also add operational constraints of their own. In addition, there are physical limitations on how the post-contingency network can adapt from the base case, and these limits are added as constraints that are functions of the base case voltages. However, since this extremely high-dimensional problem is cumbersome to solve due to the large number of variables, in practice the contingency constraints are approximated via methods such as LODF and PTDF [15]. In essence, this approximates the contingency voltage v^k as a function of the base case voltage v. Therefore, post-contingency operating constraints for contingency k are approximated by a composite function of the following form:

$$h_k(v) \triangleq c_k(a_k(v))$$

where $a_k(v)$ represents the control actions that are taken in the event of a contingency.

Finally, another important consideration is how SCOPF performs when the problem is infeasible. In other words, the SCOPF modeling should be flexible enough to return a "best possible solution" when all of the physical constraints cannot be met simultaneously. Therefore, we model some operational limits using soft constraints with extra variables that capture the amount of violation. The objective function that is minimized is the sum of active power generation costs in the base case as well as a weighted sum of constraint violation penalties in the base case and contingencies. The standard optimization form is presented below:

$$[\textit{base}\text{-OPF}] \qquad \begin{aligned} \min_{v,\sigma,\sigma_k} \quad & f(v) + \psi(v) + \phi(\sigma) + \sum_{k=1}^{|\mathcal{K}|} \phi_k(\sigma_k) \\ \text{s.t.} \quad & h(v) = \sigma \\ & h_k(v) = \sigma_k \quad \text{for } k = 1, \ldots, |\mathcal{K}| \end{aligned} \tag{3}$$

where $\phi(\cdot)$ and $\phi_k(\cdot)$ represent the penalty functions for the violations. We denote this SCOPF problem as the *base*-OPF, distinguishing it from the *contingency*-OPF presented next.

B. Contingency-OPF

Recall that the SCOPF solves for the base case operating point by taking into account the possible failures in the network. In the process, it approximates the relationship between the contingency operation point v^k and the base case operating point v. However, it does not actually solve for the v^k's. Therefore, for each contingency we need to solve the *contingency*-OPF formulated below to find the best operating point for the specific contingency scenario, given the base solution. This problem resembles the "classic" OPF problem, except that there are additional coupling constraints that tie the problem to the original base case. For instance, the voltage magnitude at a bus must be equal to its base case value unless the reactive capacity of the generators at that bus is exhausted.

We model a contingency, such as a line or generator outage, by changing the constraints from the *base*-OPF. For example, a line outage physically means that power cannot flow over that connection, which can be modeled by setting the resistance of the line to a very high value. In this paper, we focus only on line outages, but an extension to generator outages can be easily done in a similar way. In the event of a line outage, the power is re-routed through a different path and therefore the loss is changed throughout the system. However, the difference in loss is small enough such that there is no need for additional participation from other generators, unlike in the scenario of a generator outage. Therefore, we can fix the active power generation to be equal to the base case values and solve for the v^k's such that the violations for the bus balance equations are spread out as much as possible. This is because a concentrated violation in a few buses can result in serious issues for the power network, whereas small power mismatches can be taken care of by real-time feedback controllers. Taking these into consideration, the *contingency*-OPF under study is given as

$$\begin{aligned} \min_{v,\sigma^p,\sigma^q} \quad & \phi(\sigma^p, \sigma^q) + \psi(v) \\ \text{subject to} \quad & P_i^g - \sum_{(i,j)\in\mathcal{E}} p_{ij} = P_i^d + \sigma_i^P \quad \forall i \in \mathcal{V} \\ & q_i^g - \sum_{(i,j)\in\mathcal{E}} q_{ij} = Q_i^d + \sigma_i^Q \quad \forall i \in \mathcal{V} \\ & p_{ij} = Re\{v_i(v_i - v_j)^* Y_{ij}^*\} \quad \forall (i,j) \in \mathcal{E} \\ & q_{ij} = Im\{v_i(v_i - v_j)^* Y_{ij}^*\} \quad \forall (i,j) \in \mathcal{E} \\ & |v_i| = |v_i|^{base} \quad \forall i \in \mathcal{V} \setminus \mathcal{V}^q \end{aligned}$$

where $\psi(\cdot)$ represents a exact/inexact penalty function that forces the variables to stay within the feasible set defined by Ψ in (1). The set $\mathcal{V}^q$ is the set of buses that hit their upper or lower reactive power generation bounds in the base case, and $|v_i|^{base}, \forall i \in \mathcal{V}$ is the voltage magnitude of bus i in the base case. Note that active power generation is now a fixed parameter obtained from a solution of the *base*-OPF and therefore has been denoted by capital P^g. Denoting $x = [v, \sigma^p, \sigma^q]$ as the combined variable, *contingency*-OPF in a standard optimization form would be:

$$[\textit{contingency}\text{-OPF}] \qquad \begin{aligned} \min_{x} \quad & f(x) \\ \text{subject to} \quad & h(x) = 0 \end{aligned} \tag{4}$$

Note that $f(\cdot)$ is the not the same as the objective function used in (2) or (3) but a comprehensive objective function that includes both the generation cost functions and the violation penalty functions. Similarly, $h(\cdot)$ is the not the same as the constraint functions used in (2) or (3).

If the optimal objective value of the *contingency*-OPF is zero, it means that the solution of the base case could be modified to stay feasible in case of the contingency. However, the focus of the paper is on hard instances with a nonzero optimal cost, meaning that some of the constraints must be violated to accommodate the outage. In these cases, since taking corrective actions to deal with nodal power violations is expensive, it is essential to find a global solution.

III. Background on Homotopy Methods

Homotopy and continuation methods have long been used in mathematics and engineering to solve systems of nonlinear algebraic equations [16]. Continuation methods in mathematics describe the continuous transformation of an easy problem into the given hard problem [17]. The benefit of homotopy methods compared to other iterative methods is that homotopy methods may yield global rather than local convergence. These methods are most useful for problems where convergence to a global solution is heavily dependent on a good initial point, which can be hard to obtain.

The development of probability-one homotopy methods in the 1970s created a globally-convergent framework for solving nonlinear systems of equations [18]. For these probability-one methods, almost all choices of parameter for the homotopy map yield no singular points in the Jacobian and thereby global convergence. While homotopy methods have been shown to be accurate and robust, they are computationally expensive and should be reserved for highly nonlinear problems for which a good initial point is hard to find [17].

More recently, these probability-one homotopy methods have been applied to solving optimization problems. The applications include optimal control ([19], [20]) and statistical learning [21]. Typically the homotopy methods in optimization focus on cases when the Karush-Kuhn-Tucker (KKT) optimality conditions are parametrized ([17], [22]) or when the objective is ([13], [23]). Our method is similar to the homotopy optimization method described in [13], wherein a series of local minimization problems are solved, rather than tracing a path of zeros to the KKT conditions. However, we will focus on a more generalized theoretical analysis of homotopy, allowing for a homotopy map on the set of constraints.

Homotopy methods have also been applied in the field of power systems, primarily to solve the power flow (PF) problem for cases that do not converge. The continuation power flow (CPF) problem is used to find a set of solutions of the power flow problem, starting at some base load and ending at an operating point near the voltage stability limit [24]. The power flow Jacobian is singular at the voltage stability limit, which results in convergence issues for solving PF; however, the CPF formulation allows the problem to stay well-conditioned at all possible loading conditions. Homotopy methods are also used to solve the PF problem when the convergence of the problem is dependent on a good initial point, which may be hard to find. It has been shown that standard iterative methods for power flow, such as Newton-Raphson, may diverge due to a poor initial point [25]. Homotopy methods have been shown to improve convergence of the PF problem ([26], [27]) as well as compute all possible solutions to the PF problem [28].

A. Classical Homotopy Framework

Let the given hard problem be to find x such that $f(x) = 0$ and an easy (fictitious) version of the problem be to find x such that $s(x) = 0$. Assume that we choose $s(x)$ such that $s(x) = 0$ has a unique solution x_0. We introduce the continuation parameter λ to generate a homotopy map. A linear homotopy map may be defined as:

$$H(\lambda, x) = \lambda f(x) + (1 - \lambda)s(x), \quad 0 \leq \lambda \leq 1$$

We solve $H(\lambda, x) = 0$ as we continuously vary λ from 0 to 1. Starting from $\lambda = 0$, we can solve the easy problem $s(x) = 0$ to find the unique solution x_0. The goal is to track the solutions of the problems so that at $\lambda = 1$, we find a solution $\bar{x}$ where $f(\bar{x}) = 0$.

However, the continuous variation of λ is not implemented in practice, i.e. we discretize λ and solve the series of problems $H(\lambda^i, x) = 0$, where $\lambda^i = \lambda^{i-1} + \Delta\lambda$, for $i = 1, ..., I$. Modern homotopy methods allow the discretization of λ, as long as $\Delta\lambda$ is sufficiently small [17]. If there are singularities or divergence issues, the homotopy method will not converge to a solution of $f(x) = 0$. However, we can construct a homotopy map so that with probability-one, the Jacobian matrix of the homotopy map has full rank [17]. For almost all choices of the homotopy map, there exists a homotopy path that converges to the global solution with probability-one, in a sense of the Lebesgue measure.

In the following section, we present a homotopy method that parametrizes the constraint set of *contingency*-OPF to model a line or a generator outage, which is analogous to the "deformed problem" discussed in the one-dimensional example above. In Section IV-A, we develop a theory to characterize cases when homotopy will lead to a global solution of the deformed problem.

IV. Methods

A. Homotopy Method for the Contingency-OPF

In order to solve the contingency-OPF problem, we introduce a homotopy method that gradually changes certain parameters of the problem, rather than physically changing the structure of the network. For instance, a transmission line outage can be modeled by physically removing the line from the network, by limiting the apparent power flow over the line, or by assigning a high resistance and reactance value to the line such that effectively no power flows through it. We use the third method to construct a homotopy map for a line outage. In other words, we solve a series of contingency-OPF problems, each with a slightly higher impedance value than the previous problem which uses the solution of the previous problem as an initial point.

Let $\ell \in \mathcal{E}$ be a line that connects bus i and j. Now, consider a contingency scenario in which the line ℓ is out.

The active and reactive power over line ℓ can be expressed by the following power flow equations:

$$p_{ij} = Re\{v_i(v_i - v_j)^* Y_{ij}^*\} \tag{5}$$
$$q_{ij} = Im\{v_i(v_i - v_j)^* Y_{ij}^*\} \tag{6}$$

We introduce the homotopy parameter $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$ to create the following homotopy map:

$$Y_{ij}(\lambda) = G_{ij}^0 \lambda_1 + \mathbf{j} B_{ij}^0 \lambda_2 \tag{7}$$

where G_{ij}^0 and B_{ij}^0 represent the initial admittance of line ℓ. By varying λ from $\lambda^o = (1, 1)$ to $\lambda^f = (0, 0)$, the homotopy map allows us to trace a gradual line outage event, rather than an abrupt jump in the system. The series of homotopy problems, $H(\lambda)$, parametrized by λ can be written in the standard form:

$$\begin{bmatrix} \textit{homotopy}\text{-OPF} \\ H(\lambda) \end{bmatrix} \qquad \begin{array}{ll} \min\limits_{x} & f(x, \lambda) \\ \text{subject to} & h(x, \lambda) = 0 \end{array} \tag{8}$$

A generator outage can also be modeled in a similar way by gradually removing it and adjusting the participation of other generators to compensate for the loss in power.

B. Connecting the contingency-OPF with the base-OPF

Starting with a solution to the *base*-OPF, we aim to iteratively solve a series of *homotopy-OPF* to eventually arrive at the *contingency*-OPF. In order to proceed, we assume that the *base*-OPF has a unique global solution that is available (known). The availability of a global solution is a reasonable assumption because a good initial point is usually provided for the *base*-OPF, and also because more time is allocated to solving it compared to a large set of *contingency*-OPF problems for different outages, allowing the use of various convex-relaxation techniques.

If the violation cost at the base case is non-zero, the global solution will be unique with overwhelming probability. Furthermore, even if the violation cost at the base case is zero, it will immediately become non-zero during the next homotopy iteration if removing that line introduces inflexibilities that the network cannot accommodate. In fact, these near-infeasible problems where a contingency will make the system "stressed" are the cases where homotopy can be useful and is the focus of this paper.

C. Implementation of Homotopy-OPF

This section describes how the solution to the *base*-OPF can be used to find the solution to the *contingency*-OPF via a homotopy method. First, a series of *homotopy*-OPF problems is formulated as a parametrization of the *contingency*-OPF problem (as in Section IV-A). To define a physically implementable homotopy map, we must discretize the homotopy parameter λ. Then, the first *homotopy*-OPF problem is initialized as the solution to the *base*-OPF problem. The series of *homotopy*-OPF problems is solved with local search methods, and the initial point is updated at each iteration of homotopy to be the solution of the last *homotopy*-OPF problem. See Algorithm 1 for the complete details.

To develop intuition on when a homotopy method may or may not lead to the global solution, we consider the basin of attraction of the global solution to the *contingency*-OPF problem. The "basin of attraction" of a local solution is the set of initial points that lead to the solution using an iterative search method. Note that the size of the basin of attraction is dependent on both the problem geometry and the choice of search method.

Algorithm 1 Homotopy-OPF Algorithm for Line Outages

Given:
1. Power network $\mathcal{G}(\mathcal{V}, \mathcal{E})$ and generators $\mathcal{R}$
2. Set of contingencies $\mathcal{K}$ with line outages $l_k \in \mathcal{E}$
3. Homotopy scheme: I iterations of homotopy with parameter $\lambda^i \in \mathbb{R}^2$ at each $i \in \{0, ..., I\}$ such that $\lambda^0 = (1, 1)$ and $\lambda^I = (0, 0)$

Initialize: Solve *base*-OPF problem given by Equation (3) to find a base case solution (v_0, σ, σ_k) and obtain $p_1^g, \sigma^p, \sigma^q$ through calculation.

Formulate the *contingency*-OPF problem in Equation (4):
1. Fix real power generation to base case solution: $P_i^g = p_i^g \quad \forall i \in \mathcal{R}$
2. Find $\mathcal{V}^q$, the set of buses that hit their upper or lower reactive power generation bounds in the base case

for $k \in \mathcal{K}$ **do**
  Define (i,j) as the *from* and *to* buses of line l_k
  Define G_{ij}^0, B_{ij}^0 as the initial admittance values
  Let $(\tilde{v}, \tilde{\sigma}^p, \tilde{\sigma}^q) \leftarrow (v_0, \sigma^p, \sigma^q)$
  for $i \in \{1, ..., I\}$ **do**
    $Y_{ij} \leftarrow G_{ij}^0 \lambda_1^i + \mathbf{j} B_{ij}^0 \lambda_2^i$
    Use local search to solve *contingency*-OPF with updated Y_{ij} using initial point $(\tilde{v}, \tilde{\sigma}^p, \tilde{\sigma}^q)$ and obtain a solution (v, σ^p, σ^q)
    Update $(\tilde{v}, \tilde{\sigma}^p, \tilde{\sigma}^q) \leftarrow (v, \sigma^p, \sigma^q)$
  end for
  Return (v, σ^p, σ^q) and violation cost $\phi(\sigma^p, \sigma^q)$
end for

Because we employ a local search method at each step of homotopy, each point along the homotopy path is a local (or saddle), if not global, minimum of the function as defined for that iteration. At each iteration of homotopy, the problem is initialized as the solution to the previous problem with the hope that the previous point will be in the basin of attraction of the new solution. The homotopy method will find the global solution of the final problem if at some point along the homotopy path, the solution to the intermediate problem enters and stays within the basin of attraction of the global solution to the final problem. Because of this, homotopy is only useful for problems where the initial point, i.e. the solution to *base*-OPF, is not in the basin of attraction of the global minimum of the final problem, *contingency*-OPF. If the initial point is within the basin of attraction of the global solution to the final problem, then the intermediate problems are unnecessary.

Proposition 1. *If the global solution along the homotopy path is unique, then a sufficiently small step-size $\Delta\lambda$ will ensure that the solution to each intermediate problem is a global solution.*

Under the uniqueness assumption mentioned above, the solution to some intermediate problem will enter and remain within the basin of attraction of the global solution to the final problem, and we will obtain the global minimum of the final problem. In the next section, we find some conditions under which the global solution along the path is unique.

V. Analysis of Homotopy Paths

While probability-one homotopy methods almost surely guarantee algorithm convergence, they do not necessarily result in convergence to a global minimum [13]. In Section I-A, we offered two examples of nonconvex optimization: one in which the homotopy method resulted in the global minimum and another in which the homotopy method resulted in a local minimum (see Figure 1). In this section, we describe a theoretical framework that describes when homotopy can be used to obtain a global minimum. We apply this framework to analyze the performance of *homotopy*-OPF in finding the global solution of the *contingency*-OPF. The results developed in this section have implications for homotopy methods in a broad range of optimization problems.

Remark 1. *To simplify the presentation, we make the assumption that homotopy-OPF has a unique global solution at the initial point of the path. The "uniqueness" of the global solution (in this assumption and Theorem 1 to be presented next) can be replaced by the "connectivity" of the set of all global solutions (this allows having infinitely many possible solutions for post-contingency OPF with a zero violation cost).*

A. Characterization of desirable homotopy path

The path we take to change the homotopy parameter λ defines how the constraints and objective function of $H(\lambda)$ will change, and this in turn affects the series of global solutions obtained throughout the homotopy process. Therefore, choosing a good homotopy path is directly correlated with the success of the method. Note that even though Algorithm 1 works on a discretized homotopy path, its analysis requires working on the continuous path. In Figure 5 of Section VIII-A, we have presented an example in which homotopy fails to find the global solution of the final problem. The major cause of this breakdown is the emergence of two global solutions in the (continuous) homotopy path, which is followed by a change in the relative positions of the global solution and next best local solution. In order to better characterize this, consider the KKT conditions for the *homotopy*-OPF problem defined in (8):

$$\begin{aligned} \nabla f(x,\lambda) + \nu^T \nabla h(x,\lambda) &= 0 \\ h(x,\lambda) &= 0 \end{aligned} \tag{9}$$

where ν is the vector of dual variables. We assume that constraint qualification holds for the problem $H(\lambda)$ defined in (8) for all $\lambda \in \Lambda$, which implies the KKT conditions are satisfied for every local minimum. The set Λ can be defined as a box region containing the relevant homotopy path. For a given λ, let $\mathcal{X}(\lambda)$ be the set of all x that satisfy the KKT conditions in equation (9). Note that the goal is not to solve the KKT conditions directly but is to merely use them as a necessary condition for all local solutions. Before proceeding to the main theorem of this work, below we make one basic assumption on the KKT conditions and define a concept called the "dividing midpoint zone" (DMZ).

Assumption 1. *The cardinality of set $\mathcal{X}(\lambda)$ as a function of λ is constant for all $\lambda \in \Lambda$.*

Assumption 1 is essential to guarantee that a local solution is not suddenly created or disappeared along the homotopy path, in which case we cannot trace it back to the local solutions of the original problem to track it. Using techniques in algebraic geometry, one can study the satisfaction of Assumption 1 [29].

Definition 1. *At λ^o, we order all the elements in $\mathcal{X}(\lambda^o)$ in a way such that $f(x^{(1)},\lambda^o) < f(x^{(2)},\lambda^o) \le ... \le f(x^{(|\mathcal{X}|)},\lambda^o)$. Furthermore, let a be the midpoint objective value of the first and second best KKT points. In other words,*

$$a = \frac{f(x^{(1)},\lambda^o) + f(x^{(2)},\lambda^o)}{2}$$

Define $\mathcal{S}$ to be the set of all λ for which there exists a KKT point with the objective value equal to a:

$$\mathcal{S} = \{\lambda \in \Lambda \ | \quad f(x,\lambda) = a \text{ for some } x \in \mathcal{X}(\lambda)\} \tag{10}$$

Here we define a to be the "dividing midpoint" between $f(x^{(1)},\lambda^o)$ and $f(x^{(2)},\lambda^o)$. In practice, a wide range of values that are slightly above or below the point a, within the DMZ, would lead to the same implications. The optimal choice of a depends on the knowledge of how the shape of the curve changes with respect to λ. We are now ready to state the first theorem.

Theorem 1. *Let $\rho(\lambda) = 0$ be a homotopy path of λ with two end-points λ^o and λ^f. In other words, the set of λ's satisfying $\rho(\lambda) = 0$ can be parametrized by $t \in [0,T]$ such that $\lambda(0) = \lambda^o$ and $\lambda(T) = \lambda^f$. If $\rho(\lambda) = 0$ does not intersect with the set $\mathcal{S}$, then the homotopy problem (8) has a unique global minimum for all values of λ along the path $\rho(\lambda) = 0$.*

proof. The proof is provided in the Appendix.

According to Theorem 1, the success of homotopy in finding the global optimum depends on the geometry of the set $\mathcal{S}$. To illustrate this, suppose that the set $\mathcal{S}$ is described by the blue area in Figure 2. We wish to design a homotopy method that starts from $\lambda^o = (1,1)$ and ends at $\lambda^f = (0,0)$. However, this is not possible without crossing the set $\mathcal{S}$ because it fully encompasses the final λ^f and blocks any path from entering.

Directly analyzing the geometry of the set $\mathcal{S}$ is not an easy job. Therefore, we introduce a method to certify whether a path is a successful homotopy path or not. The following theorem offers a dual certificate.

Theorem 2. *Let $\rho(\lambda) = 0$ define the homotopy path used to solve the homotopy-OPF problem (8). Consider the following feasibility problem and denote it by (P) :*

$$\begin{aligned} (P) \quad p(x^*,\lambda^*,\mu^*) = \min_{x,\lambda,\mu} \quad & 0 \\ s.t. \quad & \nabla f(x,\lambda) + \mu^T \nabla h(x,\lambda) = 0 && (11) \\ & h(x,\lambda) = 0 && (12) \\ & f(x,\lambda) = a && (13) \end{aligned}$$

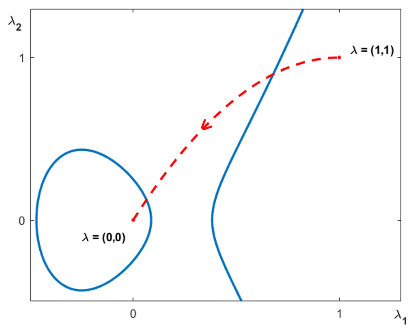

Fig. 2. An example of the set $\mathcal{S}$ (shown in blue) where the homotopy path (shown in red) cannot reach the origin without passing through a point in $\mathcal{S}$.

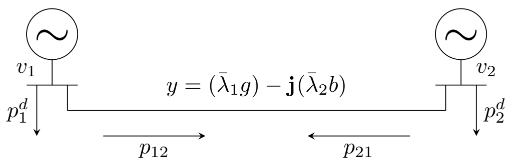

Fig. 3. A two-bus network

$$\rho(\lambda) = 0 \tag{14}$$

Let the corresponding dual problem be denoted by (D)*, written as* $\max_{\omega_1,\omega_2,\omega_3,\omega_4} d(\omega_1,\omega_2,\omega_3,\omega_4)$ *where* $\omega_1,\omega_2,\omega_3,\omega_4$ *are the dual variables for the constraints (11), (12), (13) and (14). If there exists a quadruplet* $(\omega_1,\omega_2,\omega_3,\omega_4)$ *such that* $d(\omega_1,\omega_2,\omega_3,\omega_4) > 0$*, then the homotopy-OPF problem (8) attains a unique global minimum along the path* $\rho(\lambda) = 0$*.*

proof. The proof is provided in the Appendix.

Note that the dual problem is convex and finding it is easy for certain problems, for example in the case where *homotopy*-OPF is cast as a non-convex quadratically-constrained quadratic program. In essence, finding a dual feasible point $(\omega_1,\omega_2,\omega_3,\omega_4)$ for which $d(\omega_1,\omega_2,\omega_3,\omega_4) > 0$ provides a certificate that guarantees that the homotopy path $\rho(\lambda)$ will never intersect with set $\mathcal{S}$. Then, by Theorem 1, we can conclude that the homotopy method will have a unique global minimum along its path and therefore Algorithm 1 is able to solve contingency-OPF to global optimality using iterative local search due to Proposition 1.

B. Geometry of the homotopy path: Two-bus example

Consider a simple two-bus example as shown in Figure 3. Each bus has a corresponding voltage magnitude and voltage angle associated with it. The voltage magnitude of bus i is denoted by $|v_i|$ and the voltage angle of bus i is denoted by θ_i. The line connecting the two buses have admittance $y = (\bar{\lambda}_1 g) - \mathbf{j}(\bar{\lambda}_2 b)$. The active power injection and demand at bus i are denoted by p_i^{inj} and $P_i^d > 0$, respectively. Furthermore, there is a lower bound Q^{min} on reactive power injection at both buses. Assume the following:

1) $|v_1| = |v_2| = 1$
2) $-\Delta \le \theta_1 - \theta_2 \le \Delta$
3) $0 < Q^{min} < q(\Delta)$

where $\Delta = \tan^{-1}(\lambda_2 b/\lambda_1 g)$ and $q(\cdot)$ denotes the reactive power injection as a function of the soley the angle difference, which is due to the fact that voltage magnitudes are fixed. Note that the second constraint on angle difference is reasonable for the secure operation of power systems and is also used in [5] in order to restrict the two-bus active power injection reagion to be the Pareto front of the original feasible region. In mathematical terms, suppose that the corresponding OPF problem takes the following form:

$$\begin{aligned} \min_{p_1^{inj},p_2^{inj}} \quad & (p_1^{inj} + P_1^d)^2 + c(p_2^{inj} + P_2^d)^2 \\ \text{subject to} \quad & h(p_1^{inj}, p_2^{inj}) = 0 \end{aligned} \tag{15}$$

The feasible set of the two-bus injection region is the Pareto front of an ellipse, which is partially removed due to the reactive power constraints (the details can be found in [5]). The following lemma characterizes the set of homotopy parameters for which there are at least two global solutions.

Lemma 1. *Denote* $\alpha = cos^{-1}(\frac{-Q^{min}+b\lambda_2}{|y|})$*, and define two polynomial functions of* $\lambda = (\lambda_1, \lambda_2)$ *as follows:*

$$w_1(\lambda_1,\lambda_2) = \frac{2\lambda_2 b}{|y|}\big(\lambda_2 b \cdot \sin\alpha + \alpha \cdot \lambda_1 g\big)$$
$$w_2(\lambda_1,\lambda_2) = 2\lambda_1 g - \frac{2\lambda_1 g}{|y|}\big(-\lambda_1 g \cdot \sin\alpha + \alpha \cdot \lambda_2 b\big)$$

Define also the set $\widetilde{\mathcal{S}}$ *as:*

$$\begin{aligned}\widetilde{\mathcal{S}} = \{\lambda \in \mathbb{R}^2 \mid (1-c)\cdot w_1(\lambda_1,\lambda_2)\cdot w_2(\lambda_1,\lambda_2) \\ + 2P_1^d \cdot w_1(\lambda_1,\lambda_2) - 2cP_2^d \cdot w_1(\lambda_1,\lambda_2) = 0\}\end{aligned}$$

If $(\bar{\lambda}_1, \bar{\lambda}_2) \in \widetilde{\mathcal{S}}$*, then the two-bus OPF problem has two global solutions.*

proof. The proof is provided in the Appendix.

We can view this set $\widetilde{\mathcal{S}}$ as an equivalent if not a subset of $\mathcal{S}$. This is a set of measure zero in general and as long as the homotopy path does not intersect with this set, Algorithm 1 will work (see Appendix for more details).

VI. Simulations

In order to implement the contingency-OPF using the MATPOWER format, we introduce virtual generators that allow for violations of real and reactive power balances at all nodes after an outage occurs. These violations are penalized in a modified objective function. The benefit of this formulation is that there always exists a feasible solution to *contingency*-OPF. By adding power generation flexibility with virtual generators, we aim to find a feasible point (equivalent to a zero objective value) or an infeasible point for the network but with the minimum violations (such solutions could yet be implemented via corrective actions taken by real-time feedback controllers).

Three different homotopy schemes are tested and compared to the performance of local search without homotopy:

- Scheme 1: Continuously decrease in λ from $(1,1) \to (0,0)$
- Scheme 2: Decrease λ_1 from $1 \to 0$, then λ_2 from $1 \to 0$
- Scheme 3: Decrease λ_2 from $1 \to 0$, then λ_1 from $1 \to 0$

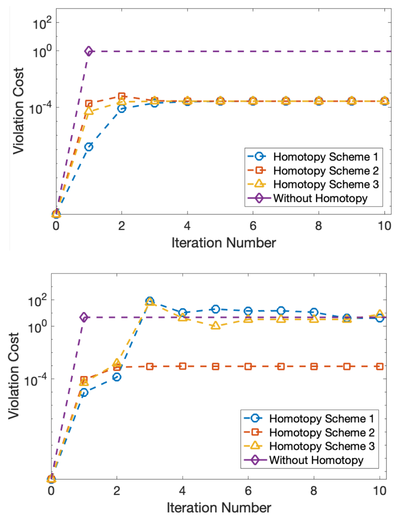

Fig. 4. Performance of proposed homotopy method on the 3375-bus Polish network with single line outages. Note that this test case is `case3375wp` with real and reactive power demand scaled up by 10% of their original values. In the top figure, the ID of the line out is 719, and in the bottom figure, the ID of the line out is 1031. For more simulations, see Appendix.

We run the homotopy simulations with the MATPOWER Interior Point Solver (MIPS). By testing different line outages on Polish networks, we were able to identify cases where the using homotopy to solve *contingency*-OPF resulted in a significantly better performance. These are hard contingency instances where an outage has a real impact on the network.

Figure 4 shows a case for which the homotopy method results in an objective value that is significantly lower than that obtained with the one-shot method. For the first line outage (top figure), all three homotopy schemes result in a violation cost for *contingency*-OPF over 10^3 times less than the one obtained without deploying homotopy. For the second line outage (bottom figure), only one homotopy scheme results in a violation cost lower than the one obtained without homotopy. Note that each of the homotopy schemes for this 3375-bus network took less than 30 seconds to solve on a standard laptop. The simulations are aligned with our message delivered in section V stating that choosing the correct homotopy path is significant for the success of the method.

VII. Conclusions

This paper studies the *contingency*-OPF problem, which is used to find an optimal operating point in the case of a line or generator outage. Unlike OPF which is a single optimization problem, there are many contingency-OPF problems, which should all be solved in a short period of time. Recognizing that the *contingency*-OPF problem is a more challenging version of the classical OPF problem, we introduce a new homotopy method to find the best solution of the *contingency*-OPF problem. By solving the *contingency*-OPF problem over a series of optimization problems using simple local search methods, we can ensure convergence to a global solution under certain conditions. We show on Polish networks that the homotopy method can result in a lower value of the objective, and we introduce theoretical notions to understand when homotopy works on a given problem.

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VIII. Appendix

A. Additional Figures

In this section, we report some additional figures that did not make to the main body of the paper due to page limitations. Figure 5 explains the intuition behind the definition of set $\mathcal{S}$ and also provides insight into the proof of Theorem 1. Figure 6 is referenced in the proof of Lemma 1. Figures 7 through 9 provide additional results of the proposed homotopy method's performance on Polish networks with line outages.

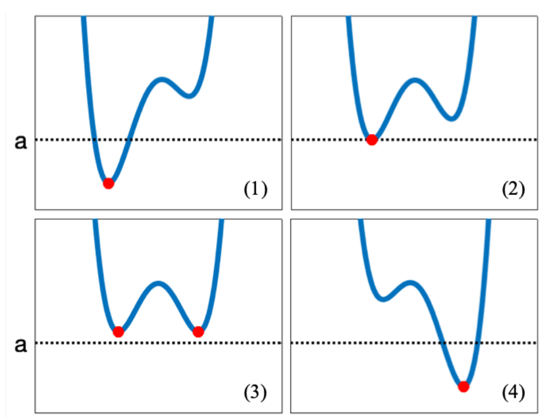

Fig. 5. Red dots denote the global min of the functions. Homotopy may not be effective for cases where the global minimum of the base case becomes a local minimum for the deformed problem. For any problem where a global minimum for the initial problem is transformed into a non-global local solution for the final problem, by continuity, there must exist a point along the deformation path where the problem has two global minima. In this example, continuously changing from the initial curve (1) to the final curve (4) requires passing through curves (3) with two global minima. Point a is defined in Definition 1.

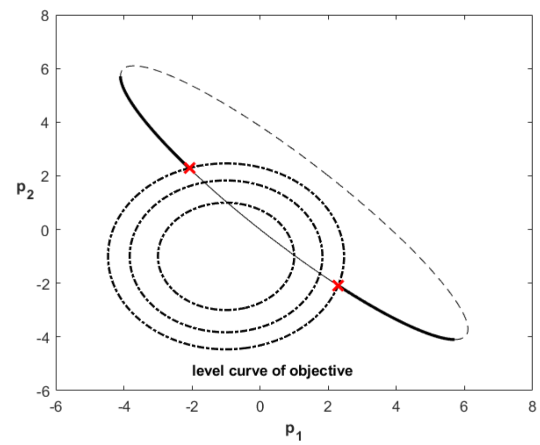

Fig. 6. An example of two-bus network for which there are two global solutions to the OPF.

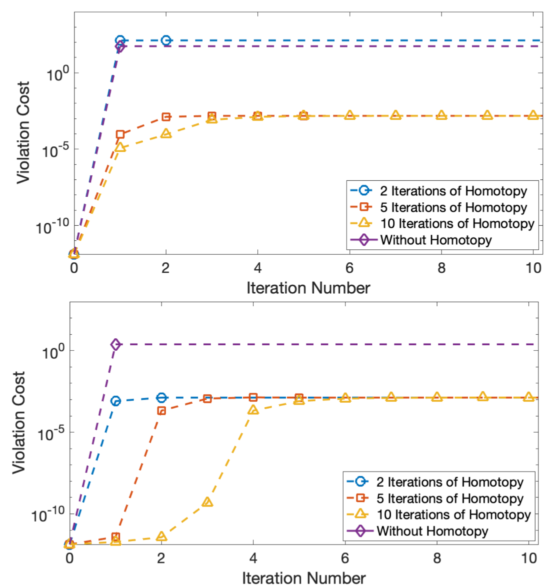

Fig. 7. Performance of proposed homotopy method on the 3012-bus Polish network (`case3012wp` with real and reactive power demand scaled up by 8%) with single line outages. Homotopy scheme 1, as described in Section VI, is tested with a varying number of homotopy iterations. In the top figure (line out ID: 332), we see a case where the one-shot and 2-iteration homotopy methods result in much higher objective values than the 5 and 10-iteration homotopy methods. In the bottom figure (line out ID: 1604), we see a case where the 2, 5, and 10-iteration homotopy methods result in an objective value much lower than that obtained by the one-shot method. For this scenario, by introducing even a 2-iteration homotopy scheme we outperform the one-shot method.

B. Proof of Theorem 1

Let the *homotopy*-OPF problem at λ^o, $H(\lambda^o)$, have a set of KKT points $\mathcal{X}(\lambda^o) = \{x^{(1)}, \dots, x^{(|\mathcal{X}|)}\}$ that are ordered in a way such that $f(x^{(1)}, \lambda^o) < f(x^{(2)}, \lambda^o) \leq \dots \leq f(x^{(|\mathcal{X}|)}, \lambda^o)$. The first strict inequality between the global minimum and the next best local minimum implies that there is a unique global minimum $x^{(1)}$, which is the assumption made in Remark 1. Therefore, by definition, $\lambda^o \notin \mathcal{S}$. Let's prove the theorem by proving its contrapositive. Suppose that there exists a $\tau \in [0, T]$ for which the *homotopy*-OPF problem $H(\lambda(\tau))$ has two global solutions, $x_\tau^{(1)}$ and $x_\tau^{(2)}$. To show that the contrapositive is true, we have to show that the path described by $\rho(\lambda) = 0$ intersects with the set $\mathcal{S}$. There are two scenarios that can happen:

(i) When $f(x_\tau^{(1)}, \lambda(\tau)) = f(x_\tau^{(2)}, \lambda(\tau)) \geq a$
(ii) When $f(x_\tau^{(1)}, \lambda(\tau)) = f(x_\tau^{(2)}, \lambda(\tau)) < a$

Note that a is defined in Definition 1. For scenario (i), since $f(x^{(1)}, \lambda^o) < a$, $\exists t \in [0, \tau]$ such that $f(x_t^{(1)}, \lambda(t)) = a$ where $\rho(\lambda(t)) = 0$. This is due to Assumption 1 and the fact that the

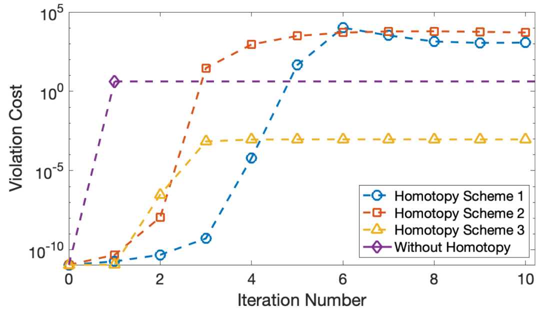

Fig. 8. Performance of proposed homotopy method on the 3120-bus Polish network (`case3120sp` with original real and reactive power demand) with a single line outage (line out ID: 1602). Homotopy schemes 1 through 3, as described in Section VI, are tested with 10 iterations. For this example, homotopy schemes 1 and 2 result in an objective value higher than that obtained by the one-shot method, while the third homotopy scheme outperforms the one-shot method. This example shows that convergence to a global minimum is dependent on the choice of homotopy path.

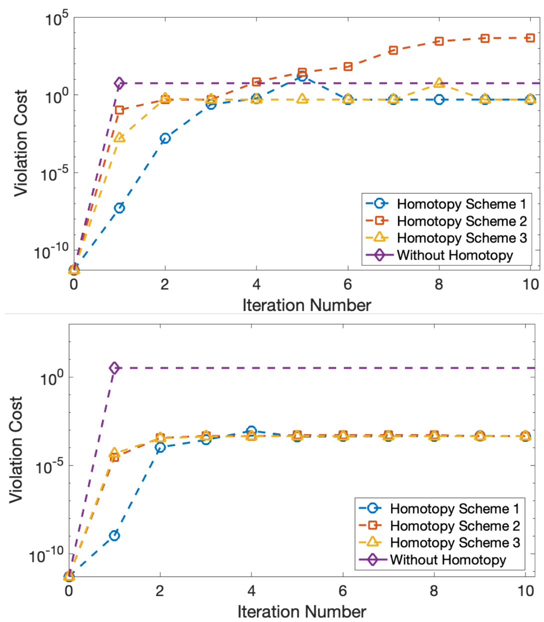

Fig. 9. Performance of proposed homotopy method on the 3120-bus Polish network (`case3120sp` with real and reactive power demand scaled up by 10%) with a multiple line outages. Homotopy schemes 1 through 3, as described in Section VI, are tested with 10 iterations. By introducing multiple line outages, we make the *contingency*-OPF problem more difficult to solve, which makes it a good candidate for the proposed homotopy method. In the top figure, the IDs of the outed lines are 31 and 32, and in the bottom figure, the IDs of the outed lines are 438, 439, and 3150.

KKT points change continuously with respect to the parameter λ. Similarly, for scenario (ii), since $f(x^{(2)}, \lambda^o) > a$, $\exists t \in [0, \tau]$ such that $f(x_t^{(2)}, \lambda(t)) = a$ where $\rho(\lambda(t)) = 0$. In both scenarios, the path described by $\rho(\lambda) = 0$ intersects with the set $\mathcal{S}$, which proves the contrapositive and completes the proof.

C. Proof of Theorem 2

Due to Theorem 1, it is sufficient to show the equivalence between the following two statements:

(i) The path described by $\rho(\lambda) = 0$ does not intersect with the set $\mathcal{S}$.

(ii) There exists a dual variable quadruplet $(\omega_1, \omega_2, \omega_3, \omega_4)$ such that $d(\omega_1, \omega_2, \omega_3, \omega_4) > 0$.

By definition of the set $\mathcal{S}$, statement (i) is equivalent to saying that the following set of equations do not have a solution:

$$\begin{aligned} \nabla f(x, \lambda) + \mu^T \nabla h(x, \lambda) &= 0 \\ h(x, \lambda) &= 0 \\ f(x, \lambda) &= a \\ \rho(\lambda) &= 0 \end{aligned}$$

In other words, the following feasibility problem is infeasible:

$$\begin{aligned} (P) \quad & \min_{x,\lambda,\mu} \quad 0 \\ \text{s.t. } & \nabla f(x, \lambda) + \mu^T \nabla h(x, \lambda) = 0 \\ & h(x, \lambda) = 0 \\ & f(x, \lambda) = a \\ & \rho(\lambda) = 0 \end{aligned}$$

By duality theory, if the dual problem (D) is unbounded, then the primal problem (P) must be infeasible. However, since the primal objective value is zero and the dual problem should provide a lower bound to the primal, finding a dual certificate that gives a positive dual objective value is sufficient in proving that the primal problem is infeasible. This completes the proof.

D. Proof of Lemma 1

Let us start with the equation for the reactive power injections. Let θ_1 and θ_2 denote the voltage phasor angles at bus 1 and 2, respectively. Then after denoting $\theta = \theta_1 - \theta_2$, we have the following:

$$\begin{aligned} q_1^{inj} &= \lambda_2 b - \lambda_1 g \cdot \sin\theta - \lambda_2 b \cdot \cos\theta \\ q_2^{inj} &= \lambda_2 b + \lambda_1 g \cdot \sin\theta - \lambda_2 b \cdot \cos\theta \end{aligned}$$

A lower bound of $Q^{\min}$ on q_1^{inj} results in the following calculations:

$$\begin{aligned} & Q^{\min} \leq \lambda_2 b - \lambda_1 g \cdot \sin\theta - \lambda_2 b \cdot \cos\theta \\ \iff & -Q^{\min} + \lambda_2 b \geq \lambda_1 g \cdot \sin\theta + \lambda_2 b \cdot \cos\theta \\ & \qquad = \sqrt{(\lambda_1 g)^2 + (\lambda_2 b)^2} \cdot \cos(\theta - \Delta) \\ & \qquad \text{where } \Delta = \tan^{-1}\left(\frac{\lambda_1 g}{\lambda_2 b}\right) \\ \iff & \cos(\theta - \Delta) \leq \frac{-Q^{\min} + b \cdot \lambda_2}{\sqrt{(\lambda_1 g)^2 + (\lambda_2 b)^2}} \\ \iff & \theta \geq \cos^{-1}\left(\frac{-Q^{\min} + b \cdot \lambda_2}{\sqrt{(\lambda_1 g)^2 + (\lambda_2 b)^2}}\right) + \Delta \\ & \text{or} \quad \theta \leq -\cos^{-1}\left(\frac{-Q^{\min} + b \cdot \lambda_2}{\sqrt{(\lambda_1 g)^2 + (\lambda_2 b)^2}}\right) + \Delta \end{aligned} \tag{16}$$

From the lower bound on q_2^{inj}, we can perform a similar derivation and arrive at the following inequality:

$$\theta \geq \cos^{-1}\left(\frac{-Q^{\min} + b \cdot \lambda_2}{\sqrt{(\lambda_1 g)^2 + (\lambda_2 b)^2}}\right) - \Delta$$

$$\text{or} \quad \theta \leq -\cos^{-1}\left(\frac{-Q^{\min} + b \cdot \lambda_2}{\sqrt{(\lambda_1 g)^2 + (\lambda_2 b)^2}}\right) - \Delta. \tag{17}$$

Therefore, combining inequalities (16) and (17), we derive the following inequality:

$$\theta \geq \cos^{-1}\left(\frac{-Q^{\min} + b \cdot \lambda_2}{\sqrt{(\lambda_1 g)^2 + (\lambda_2 b)^2}}\right) + \Delta$$
$$\text{or } \theta \leq -\cos^{-1}\left(\frac{-Q^{\min} + b \cdot \lambda_2}{\sqrt{(\lambda_1 g)^2 + (\lambda_2 b)^2}}\right) - \Delta. \tag{18}$$

Furthermore, we assume that

$$-\tan^{-1}\left(\frac{\lambda_2 b}{\lambda_1 g}\right) \leq \theta \leq \tan^{-1}\left(\frac{\lambda_2 b}{\lambda_1 g}\right)$$

which is equivalent to the following using basic trigonometry:

$$-\left(\frac{\pi}{2} - \Delta\right) \leq \theta \leq \left(\frac{\pi}{2} - \Delta\right) \tag{19}$$

Combining the two inequalities (18) and (19), and using the definition of α, we get the final constraint on θ:

$$\alpha + \Delta \leq \theta \leq \left(\frac{\pi}{2} - \Delta\right) \text{ or}$$
$$-\left(\frac{\pi}{2} - \Delta\right) \leq \theta \leq -\alpha - \Delta. \tag{20}$$

This feasible region of θ is reflected in the feasible region of the active power injections, as shown in the bolded part of the ellipse in Figure 6. As illustrated in the figure, the two red points are active power injections, corresponding to $\theta = \alpha + \Delta$ and $\theta = -\alpha - \Delta$. Let the first red point, (P_1^{inj}, P_2^{inj}), be generated by $\theta = \alpha + \Delta$. Then, the following is true for P_1^{inj}:

$$\begin{aligned}
P_1^{inj} &= \lambda_1 g + \lambda_2 b \cdot \sin\theta - \lambda_1 g \cdot \cos\theta \\
&= \lambda_1 g + \lambda_2 b \cdot \sin(\alpha + \Delta) - \lambda_1 g \cdot \cos(\alpha + \Delta) \\
&= \lambda_1 g + \lambda_2 b \cdot (\sin\alpha \cdot \cos\Delta + \alpha \sin\Delta) \\
&\qquad - \lambda_1 g \cdot (\alpha \cos\Delta - \sin\alpha \cdot \sin\Delta) \\
&= \lambda_1 g + \frac{\lambda_2 b}{|y|}(\lambda_2 b \cdot \sin\alpha + \alpha \cdot \lambda_1 g) \\
&\qquad - \frac{\lambda_1 g}{|y|}(\alpha \cdot \lambda_2 b - \lambda_1 g \cdot \sin\alpha)
\end{aligned}$$

Similarly, if we let the second red point $(\bar{P}_1^{inj}, \bar{P}_2^{inj})$, be generated by $\theta = -\alpha - \Delta$, we have

$$\begin{aligned}
\bar{P}_1^{inj} = \lambda_1 g &- \frac{\lambda_2 b}{|y|}(\lambda_2 b \cdot \sin\alpha + \alpha \cdot \lambda_1 g) \\
&- \frac{\lambda_1 g}{|y|}(\alpha \cdot \lambda_2 b - \lambda_1 g \cdot \sin\alpha)
\end{aligned}$$

Also, note that due to symmetry, $P_2^{inj} = \bar{P}_1^{inj}$ and $\bar{P}_2^{inj} = P_1^{inj}$. Let's define the following two functions:

$$\begin{aligned}
w_1(\lambda_1, \lambda_2) &= P_1^{inj} - \bar{P}_1^{inj} = \frac{2\lambda_2 b}{|y|}\big(\lambda_2 b \cdot \sin\alpha + \alpha \cdot \lambda_1 g\big) \\
w_2(\lambda_1, \lambda_2) &= P_1^{inj} + \bar{P}_1^{inj} \\
&= 2\lambda_1 g - \frac{2\lambda_1 g}{|y|}\big(-\lambda_1 g \cdot \sin\alpha + \alpha \cdot \lambda_2 b\big).
\end{aligned}$$

If the two points (P_1^{inj}, P_2^{inj}) and $(\bar{P}_1^{inj}, \bar{P}_2^{inj})$ are both globally optimal, their objective values must be equal. In other words,

$$\begin{aligned}
&(P_1^{inj} + P_1^d)^2 + c(P_2^{inj} + P_2^d)^2 \\
&\qquad = (\bar{P}_1^{inj} + P_1^d)^2 + c(\bar{P}_2^{inj} + P_2^d)^2 \\
\iff &(1-c)\{(P_1^{inj})^2 - (\bar{P}_1^{inj})^2\} + 2P_1^d(P_1^{inj} - \bar{P}_1^{inj}) \\
&\qquad - 2cP_2^d(P_1^{inj} - \bar{P}_1^{inj}) = 0 \\
\iff &(1-c) \cdot w_1(\lambda_1, \lambda_2) \cdot w_2(\lambda_1, \lambda_2) + 2P_1^d \cdot w_1(\lambda_1, \lambda_2) \\
&\qquad - 2cP_2^d \cdot w_1(\lambda_1, \lambda_2) = 0
\end{aligned}$$

This completes the proof.