A Survey of Distributed Optimization and Control Algorithms for Electric Power Systems

Daniel K. Molzahn,* *Member, IEEE*, Florian Dörfler,† *Member, IEEE*, Henrik Sandberg,‡ *Member, IEEE*, Steven H. Low,§ *Fellow, IEEE*, Sambuddha Chakrabarti,¶ *Student Member, IEEE*, Ross Baldick,¶ *Fellow, IEEE*, and Javad Lavaei,** *Member, IEEE*

***Abstract*—Historically, centrally computed algorithms have been the primary means of power system optimization and control. With increasing penetrations of distributed energy resources requiring optimization and control of power systems with many controllable devices, distributed algorithms have been the subject of significant research interest. This paper surveys the literature of distributed algorithms with applications to optimization and control of power systems. In particular, this paper reviews distributed algorithms for offline solution of optimal power flow (OPF) problems as well as online algorithms for real-time solution of OPF, optimal frequency control, optimal voltage control, and optimal wide-area control problems.**

***Index Terms*—Distributed optimization, online optimization, electric power systems**

I. Introduction

CENTRALIZED computation has been the primary way that optimization and control algorithms have been applied to electric power systems. Notably, independent system operators (ISOs) seek a minimum cost generation dispatch for large-scale transmission systems by solving an optimal power flow (OPF) problem. (See [1]–[8] for related literature reviews.) Other control objectives, such as maintaining scheduled power interchanges, are achieved via an Automatic Generation Control (AGC) signal that is sent to the generators that provide regulation services.

These optimization and control problems are formulated using network parameters, such as line impedances, system topology, and flow limits; generator parameters, such as cost functions and output limits; and load parameters, such as an estimate of the expected load demands. The ISO collects all the necessary parameters and performs a central computation to solve the corresponding optimization and control problems.

With increasing penetrations of distributed energy resources (e.g., rooftop PV generation, battery energy storage, plug-in vehicles with vehicle-to-grid capabilities, controllable loads providing demand response resources, etc.), the centralized paradigm most prevalent in current power systems will potentially be augmented with distributed optimization algorithms. Rather than collecting all problem parameters and performing a central calculation, distributed algorithms are computed by many agents that obtain certain problem parameters via communication with a limited set of neighbors. Depending on the specifics of the distributed algorithm and the application of interest, these agents may represent individual buses or large portions of a power system.

Distributed algorithms have several potential advantages over centralized approaches. The computing agents only have to share limited amounts of information with a subset of the other agents. This can improve cybersecurity and reduce the expense of the necessary communication infrastructure. Distributed algorithms also have advantages in robustness with respect to failure of individual agents. Further, with the ability to perform parallel computations, distributed algorithms have the potential to be computationally superior to centralized algorithms, both in terms of solution speed and the maximum problem size that can be addressed. Finally, distributed algorithms also have the potential to respect privacy of data, measurements, cost functions, and constraints, which becomes increasingly important in a distributed generation scenario.

This paper surveys the literature of distributed algorithms with applications to power system optimization and control. This paper first considers distributed optimization algorithms for solving OPF problems in offline applications. Many distributed optimization techniques have been developed concurrently with new representations of the physical models describing power flow physics (i.e., the relationship between the complex voltage phasors and the power injections). The characteristics of a power flow model can have a large impact on the theoretical and practical aspects of an optimization formulation. Accordingly, the offline OPF section of this survey is segmented into sections based on the power flow model considered by each distributed optimization algorithm. This paper then focuses on online algorithms applied to OPF, optimal voltage control, and optimal frequency control problems for real-time purposes.

Note that algorithms related to those reviewed here have found a wide variety of power system applications in distributed optimization and control. See, for instance, surveys on the large and growing literature relevant to distributed optimization of electric vehicle charging schedules [9] and demand response applications [10] as well as work on distributed solution of multi-period formulations for model predictive control problems, e.g., [11], [12]. With an emphasis

*: Argonne National Laboratory, Energy Systems Division, Lemont, IL, USA, dmolzahn@anl.gov
†: Swiss Federal Institute of Technology (ETH), Automatic Control Laboratory, Zürich, Switzerland, dorfler@control.ee.ethz.ch
‡: KTH Royal Institute of Technology, Department of Automatic Control, Stockholm, Sweden, hsan@kth.se
§: California Institute of Technology, Department of Electrical Engineering, Pasadena, CA, USA, slow@caltech.edu
¶: University of Texas at Austin, Department of Electrical and Computer Engineering, Austin, TX, USA, sambuddha.chakrabarti@gmail.com, baldick@ece.utexas.edu. Support from NSF grant ECCS-1406894.
**: University of California, Berkeley, Department of Industrial Engineering and Operations Research, Berkeley, CA, USA, lavaei@berkeley.edu

on algorithmic developments, this paper does not attempt to survey the power systems literature regarding all applications of distributed optimization and control algorithms.

Throughout the paper, we use the following terminology:

1) **Decentralized**: purely local algorithms, i.e., no communication between agents;
2) **Distributed**: algorithms where each agent communicates with its neighbors, but there is not a centralized controller;
3) **Hierarchical**: algorithms where computations are done by agents that communicate with other agents at a higher level in a hierarchical structure, eventually leading to a centralized controller;
4) **Centralized**: Each agent communicates with a centralized controller that performs computations and sends new commands.

This paper is organized as follows. Section II overviews background material: the power flow equations (along with various relaxations and approximations), the OPF problem, and common distributed optimization techniques. Sections III reviews distributed algorithms for offline OPF problems. Section IV summarizes the literature of online algorithms for solving OPF, optimal frequency control, and optimal voltage control problems. Section V concludes the paper.

II. Overview of Background Material

This section overviews the power flow equations, presents the OPF problem, and summarizes several distributed optimization techniques that are used by a variety of algorithms.

A. Power Flow Representations

This section summarizes the power flow equations and some relaxations and approximations which are relevant to existing distributed optimization techniques.

Consider an n-bus electric power system, where $\mathcal{N} := \{1, \ldots, n\}$ denotes the set of buses. Let $\mathcal{L}$ denote the set of lines. The network admittance matrix containing the electrical parameters and topology information is denoted $\mathbf{Y} := \mathbf{G}+\mathbf{j}\mathbf{B}$, where $\mathbf{j} := \sqrt{-1}$. Define $\overline{(\cdot)}$ as the complex conjugate.

For notational brevity and to match the development of many of the distributed optimization approaches that are reviewed in this paper, the power flow equations given here use a balanced single-phase-equivalent network representation. An unbalanced three-phase representation is more appropriate for some applications, such as models of distribution networks. Many of the algorithms surveyed in this paper could be extended to an unbalanced three-phase power flow model.

Each bus has an associated voltage phasor as well as active and reactive power injections. The voltage phasors are denoted $V \in \mathbb{C}^n$, with polar coordinate representation $|V|\,e^{\mathbf{j}\theta} = |V|\angle\theta$, where $|V| > 0 \in \mathbb{R}^n$ and $\theta \in (-180^\circ, 180^\circ]^n$. Each bus $i \in \mathcal{N}$ has active and reactive power injections $P_i + \mathbf{j}Q_i$, $P, Q \in \mathbb{R}^n$.

The power flow equations are

$$P_i + \mathbf{j}Q_i = V_i \sum_{k=1}^{n} \overline{\mathbf{Y}}_{ik}\overline{V}_k. \tag{1a}$$

Squared voltage magnitudes are

$$v_i := V_i\overline{V}_i = |V_i|^2. \tag{1b}$$

Splitting real and imaginary parts of (1) and using polar voltage coordinates yields

$$P_i = |V_i| \sum_{k=1}^{n} |V_k| \left(\mathbf{G}_{ik}\cos(\theta_i - \theta_k) + \mathbf{B}_{ik}\sin(\theta_i - \theta_k)\right) \tag{2a}$$

$$Q_i = |V_i| \sum_{k=1}^{n} |V_k| \left(\mathbf{G}_{ik}\sin(\theta_i - \theta_k) - \mathbf{B}_{ik}\cos(\theta_i - \theta_k)\right) \tag{2b}$$

As an alternative to (2), balanced radial distribution networks can be represented using the *DistFlow* model [13]. Unlike the model (2), the DistFlow model implicitly assumes a *directed* graph with an arbitrary orientation. We will use (i, j) and $i \to j$ interchangeably to denote the directed line from bus i to bus j. Define the active and reactive *sending-end* power flows on the line from bus i to bus k as P_{ik} and Q_{ik}, respectively (note that we abuse notation to use P_i and Q_i to denote nodal injections and P_{ik} and Q_{ik} to denote branch flows). Denote by ℓ_{ik} the squared magnitude of the current flow from bus i to bus k. The DistFlow model is

$$P_{ik} = r_{ik}\ell_{ik} - P_k + \sum_{m:k\to m} P_{km} \tag{3a}$$

$$Q_{ik} = x_{ik}\ell_{ik} - Q_k + \sum_{m:k\to m} Q_{km} \tag{3b}$$

$$v_k = v_i - 2\left(r_{ik}P_{ik} + x_{ik}Q_{ik}\right) + \left(r_{ik}^2 + x_{ik}^2\right)\ell_{ik} \tag{3c}$$

$$\ell_{ik}v_i = P_{ik}^2 + Q_{ik}^2 \tag{3d}$$

for each line $(i, k) \in \mathcal{L}$ with series impedance $r_{ik} + \mathbf{j}x_{ik}$.[1]

The DistFlow model (3) fully represents the power flows for a balanced radial network. However, (3) is a relaxation for mesh network topologies due to the lack of a constraint ensuring consistency in the voltage angles. Indeed, as explained in [14], [15], if a set of non-linear equations, called the cycle condition, is added to (3), the resulting model is equivalent to the models (1) and (2) for general mesh networks, in the sense that there is a bijection between their solution sets [16]–[18]. Hence any power flow analysis or optimization problem can be equivalently posed in any of these models. The cycle condition is vacuous for radial networks.

Use of any of the power flow models (1), (2), or (3) results in non-convex optimization problems that can be difficult to directly handle in distributed optimization algorithms. Therefore, many algorithms have focused on linear approximations and convex relaxations of the power flow equations.

The most commonly used linear approximation is the DC power flow model [19], which is based on several assumptions:

a. Reactive power flows can be neglected.
b. The lines are lossless (i.e., $\mathbf{G} \approx 0$) and shunt elements can be neglected.
c. The voltage magnitudes at all buses are approximately equal, so $|V_i| \approx 1$ at all buses $i \in \mathcal{N}$.

[1]The DistFlow model can be extended to more general line models with shunt admittances, non-zero phase shifts, and off-nominal voltage ratios.

d. Angle differences between connected buses are small such that $\sin(\theta_i - \theta_k) \approx \theta_i - \theta_k$, $\forall (i,k) \in \mathcal{L}$.

Applying these assumptions to (2) yields the DC power flow model:

$$\sum_{(i,k)\in\mathcal{L}} \mathbf{B}_{ik}(\theta_i - \theta_k) = P_i \qquad \forall i \in \mathcal{N} \tag{4}$$

Distribution networks typically violate these assumptions, which motivates the development of alternate linearizations. One approach that is relevant to distributed optimization techniques performs a linearization around the "no-load" voltage profile under the assumptions of negligible shunt impedances and near-nominal voltage magnitudes. The voltage magnitudes can then be approximated as functions of the active and reactive power injection vectors P and Q:

$$|V| = 1 + \mathbf{R}P + \mathbf{X}Q \qquad \forall i \in \mathcal{N} \tag{5}$$

where $\mathbf{Y}^{-1} := \mathbf{R} + \mathbf{j}\mathbf{X}$. See [20]–[22] for further details.

Alternatively, another linear approximation can be formulated by neglecting the losses in the DistFlow model (setting $\ell_{jk} = 0$ in (3)) to obtain the *Linearized DistFlow* model [23]:

$$P_{ik} = -P_k + \sum_{m:k\rightarrow m} P_{km} \tag{6a}$$

$$Q_{ik} = -Q_k + \sum_{m:k\rightarrow m} Q_{km} \tag{6b}$$

$$v_i = v_k + 2\left(r_{ik}P_{ik} + x_{ik}Q_{ik}\right) \tag{6c}$$

for each line $(i,k) \in \mathcal{L}$.

The linearizations (4), (6), and (5) approximate the power flow equations. Alternative approaches form *convex relaxations* of the power flow equations. Convex relaxations enclose the non-convex feasible spaces associated with the power flow equations in a larger space. Convex relaxations bound the optimal objective value for the original non-convex problem and provide sufficient conditions for certifying problem infeasibility. Certain convex relaxations also yield the globally optimal decision variables for some optimization problems.

We next present two convex relaxations of the power flow equations: a semidefinite programming (SDP) relaxation of the model (1) for general networks [24], [25], and a second-order cone programming (SOCP) relaxation of the DistFlow model (3) for radial networks [14], [15], [26]. SOCP relaxations are also proposed in [27], [28] for the model (1). See the tutorial [17], [18] on semidefinite relaxations of OPF for extensive references. See also [29]–[31] for generalizations and convex relaxation approaches as well as [30] for a comparison of various relaxations.

The SDP relaxation is derived by formulating a rank-one matrix $\mathbf{W} = VV^H \in \mathbb{C}^{n\times n}$, where $(\,\cdot\,)^H$ is the complex conjugate transpose operator. The power flow equations (1) are linear in the entries of $\mathbf{W}$. Let $e_i \in \mathbb{R}^n$ denote the i^{th} standard basis vector. Define the matrices

$$\mathbf{H}_i := \frac{\mathbf{Y}^H e_i e_i^T + e_i e_i^T \mathbf{Y}}{2} \tag{7a}$$

$$\tilde{\mathbf{H}}_i := \frac{\mathbf{Y}^H e_i e_i^T - e_i e_i^T \mathbf{Y}}{2\mathbf{j}} \tag{7b}$$

where $(\,\cdot\,)^T$ is the transpose operator. An SDP relaxation of (1) is formed by relaxing the non-convex rank constraint to a positive semidefinite matrix constraint:

$$P_i + \mathbf{j}Q_i = \mathrm{tr}\left(\mathbf{H}_i\mathbf{W}\right) + \mathbf{j}\,\mathrm{tr}\left(\tilde{\mathbf{H}}_i\mathbf{W}\right) \tag{8a}$$

$$|V_i|^2 = \mathrm{tr}\left(e_i e_i^T \mathbf{W}\right) \tag{8b}$$

$$\mathbf{W} \succeq 0 \tag{8c}$$

where $\mathrm{tr}(\,\cdot\,)$ is the matrix trace operator and $\succeq$ indicates positive semidefiniteness. If a solution to an associated optimization problem has $\mathrm{rank}(\mathbf{W}) = 1$, the SDP relaxation is *exact* and yields globally optimal solutions. Specifically, let η denote a unit-length eigenvector of $\mathbf{W}$, rotated such that $\angle\eta_1 = 0$ to set the angle reference at bus 1, with associated non-zero eigenvalue λ. The globally optimal voltage phasors are then $V^* := \sqrt{\lambda}\,\eta$. If $\mathrm{rank}(\mathbf{W}) > 1$, the SDP relaxation does not directly provide globally optimal decision variables, but does yield a bound on the optimal objective value of the non-convex problem.

An SOCP relaxation can be formulated in terms of the DistFlow model (3) variables [14], [15], [26]. With the exception of the quadratic equation (3d), the DistFlow model is linear in the variables $(P_i, Q_i, v_i, \ell_{ij}, P_{ij}, Q_{ij})$. To construct a convex SOCP relaxation of (3), replace the equality constraint (3d) by an inequality:

$$\ell_{ik} v_i \geq P_{ik}^2 + Q_{ik}^2 \qquad \forall (i,k) \in \mathcal{L} \tag{9}$$

The SOCP relaxation considered in this paper is (3a)–(3c), (9), and it applies to single-phase balanced models of radial networks.

Distribution systems are mostly radial and unbalanced. The power flow model (1) can be generalized to an unbalanced network (radial or mesh topologies). By considering its single-phase equivalent circuit, the SDP relaxation is extended in [32], [33] to this generalized model. For radial networks, the DistFlow model (3) is extended in [33] to unbalanced networks and the SOCP relaxation (9) is extended to an SDP relaxation using a chordal decomposition.

B. Optimal Power Flow Formulation

The OPF problem optimizes system performance according to a specified objective function. Typical objective functions C are based on generation cost (i.e., $C := \sum_{i\in\mathcal{G}} c_{2,i}P_{Gi}^2 + c_{1,i}P_{Gi} + c_{0,i}$, where $c_{2,i} \geq 0$, $c_{1,i}$, and $c_{0,i}$ are scalar coefficients associated with the generator at bus i, P_{Gi} is the generation at bus i, and $\mathcal{G}$ is the set of generator buses), losses (i.e., $C := \sum_{i\in\mathcal{N}} P_{Gi}$), proximity to a desired voltage profile (i.e., $C := \sum_{i\in\mathcal{N}} \left(|V_i|^2 - |V_i^\bullet|^2\right)^2$, where $|V_i^\bullet|$ denotes a desired voltage profile), or some combination of these.

Engineering constraints in the OPF problem limit the power injections and voltage magnitudes, and the power flow equations must be satisfied. Flow limits (typically based on apparent power, active power, or current magnitude) are also generally enforced. The specific line flow formulation depends on the power flow model and type of flow. Denote $f_{ik}(V_i, V_k)$ as the appropriate flow function for line $(i,k) \in \mathcal{L}$, with the specific function descriptions excluded for brevity.

The OPF problem considered in this paper is[2]

$$\min \quad C \tag{10a}$$

subject to

$$P_i^{min} \leq P_i \leq P_i^{max} \qquad \forall i \in \mathcal{N} \tag{10b}$$

$$Q_i^{min} \leq Q_i \leq Q_i^{max} \qquad \forall i \in \mathcal{N} \tag{10c}$$

$$\left(V_i^{min}\right)^2 \leq |V_i|^2 \leq \left(V_i^{max}\right)^2 \qquad \forall i \in \mathcal{N} \tag{10d}$$

$$f_{ik}\left(V_i, V_k\right) \leq I_{i,k}^{max} \qquad \forall \left(i,k\right) \in \mathcal{L} \tag{10e}$$

$$\text{A power flow model} \tag{10f}$$

where "max" and "min" denote specified upper and lower limits on the corresponding quantities and the power flow model (10f) may be

- a non-convex formulation (1), (2), or (3);
- the DC power flow formulation (4), in which case the reactive power and voltage magnitude constraints (10c) and (10d) are ignored;
- the linear power flow representation (5) from [20], [21];
- the linearized DistFlow model (6);
- the SDP relaxation (8);
- the SOCP relaxation (3a)–(3c) and (9).

An advantage of solving OPF problems via a relaxation is the ability to certify a solution as being globally optimal: if an optimal solution of a relaxation satisfies an easily checkable condition (e.g., if the optimal matrix for the SDP relaxation is of rank 1 or if the optimal solution of the SOCP relaxation attains equality in (9)), then a globally optimal solution to the original non-convex OPF problem can be recovered. We say in this case that the relaxation is exact. The SOCP relaxation is much simpler computationally than the SDP relaxation, but SDP relaxation is tighter for general networks. For single-phase models of radial networks, however, they have the same tightness, i.e., given any OPF instance, its SOCP relaxation is exact if and only if its SDP relaxation is exact [16], [17].

Semidefinite relaxations of OPF, however, are generally inexact [34]–[38]. This is not surprising as OPF has been shown in [25], [39], [40] to be NP-hard in general. When it is not exact, the solution of a relaxation does not satisfy Kirchhoff's laws, but it does provide a lower bound on objective value of the non-convex OPF problem. For radial networks, a set of sufficient conditions have been derived under which SOCP (and hence SDP) relaxations of OPF are always exact, e.g., [28], [41]–[47] for power flow models (1) and (2), and [14], [15], [26], [48]–[50] for the DistFlow model (3); see [18] for other references. These sufficient conditions may not be satisfied in practical networks.

C. *Summary of Distributed Optimization Techniques*

This section next summarizes several distributed optimization techniques. Adopting from the exposition in [51], the first set of distributed optimization techniques are based on augmented Lagrangian decomposition. These include Dual Decomposition, the Alternating Direction Method of Multipliers with Proximal Message Passing, Analytical Target Cascading, and the Auxiliary Problem Principle. The second set of techniques are based on decentralized solution of the Karush-Kuhn-Tucker (KKT) necessary conditions for local optimality [52]. These include Optimality Condition Decomposition and Consensus+Innovation. Two other approaches, Gradient Dynamics and Dynamic Programming with Message Passing, are discussed in Section III-C6.

[2]There exist a variety of generalizations and extensions of the OPF problem, many of which are often used in practical applications. See, e.g., [7], [8].

Given its widespread use, the Alternating Direction Method of Multipliers (ADMM) is given a more detailed overview, while other decomposition techniques have a more summary treatment. These techniques have a broad conceptual similarity in that each considers distributed agents that pass information among one another and perform local computations to solve the overall problem. However, the details of the mathematical structure (which information is shared, how the algorithms ensure consistency between different subproblems, the specific computations performed by each agent, etc.) lead to differences in practical performance and theoretical properties. See, e.g., [51], [53] or the references below for more detailed discussions of these techniques. See also [51], [54], [55] for numerical comparisons between different distributed optimization techniques in the context of power system optimization problems, including empirical analyses of convergence rates.

1) Precursors to ADMM and Literature Survey: Keeping in view our aim at providing a detailed survey of ADMM, we first give a brief literature overview for ADMM in which we describe the algorithm's evolution. This is followed by two sections where we detail the Dual Decomposition algorithm and the ADMM algorithm.

The Alternating Direction Method of Multipliers first originated in the 1970s with the works of Mercier-Gabay [56], Glowinski-Marocco [57], etc. Gabay and Eckstein-Bertsekas first offered the convergence properties of the ADMM algorithm in their works [58] and [59], respectively. In that same work, Gabay also showed that there exists a more generalized method called the Douglas-Rachford method of splitting monotone operators [60], [61], of which ADMM is a special case. ADMM came into being as a result of the amalgamation of two previously proposed algorithms: Dual Decomposition (which is, in turn, based on the Dual Ascent algorithm) and the Method of Multipliers for solving augmented Lagrangian problems in a distributed manner (which is also similar in flavor to the Gauss-Siedel iterative method). ADMM combines the robustness of the augmented Lagrangian and the method of multipliers with the distributed computational capability of dual decomposition. Hestenes in [62] and Powell in [63] first proposed the augmented Lagrangian and the method of multipliers in the 1960s. Dual Decomposition also made its appearance in the 1960s in the works of Everett [64], Dantzig-Wolfe [65], Benders [66], and Dantzig [67].

2) Dual Decomposition: The Lagrangian functions of optimization problems that have a separable structure can be exploited using dual decomposition techniques [53], [64], [68]. Consider an optimization problem of the form

$$\min_{x} \quad \sum_{i=1}^{N} f_i(x_i) \tag{11a}$$

$$\text{subject to} \quad \sum_{i=1}^{N} \mathbf{A}_i x_i = b \tag{11b}$$

where, for $i = 1, \ldots, N$, $f_i(\cdot)$ is a cost function, $x_i \in \mathbb{R}^{n_i}$ is the length n_i vector of decision variables associated with the function f_i, $\mathbf{A}_i \in \mathbb{R}^{m \times n_i}$ is a specified matrix, and $b \in \mathbb{R}^m$ is a specified vector. The Lagrangian for (11) is

$$L(x, y) := \sum_{i=1}^{N} L_i(x_i, y) \tag{12}$$

where $L_i(x_i, y) := f_i(x_i) + y^T \mathbf{A}_i x_i - (1/N) y^T b$ and $y \in \mathbb{R}^m$ is the vector of dual variables. A decomposable mathematical structure in this form can often be constructed by duplicating variables shared by multiple functions f_i along with additional equality constraints that ensure consistency among the duplicated variables.

Dual decomposition methods use an iterative method called "dual ascent":

$$x_i^{k+1} := \operatorname*{argmin}_{x_i} L_i\left(x_i, y^k\right) \tag{13a}$$

$$y^{k+1} := y^k + \alpha^k \left(\sum_{i=1}^{N} \left(\mathbf{A}_i x_i^{k+1} \right) - b \right) \tag{13b}$$

where k is the iteration counter and $\alpha^k > 0$ is the specified step size at iteration k. Observe that each update of (13a) can be performed independently, which enables a decentralized implementation of this step. (The dual variable update step (13b) requires a central coordinator.) Note that the convergence of dual decomposition techniques is generally not guaranteed, even for convex problems, and depends on the step size α^k and problem characteristics.

3) Alternating Direction Method of Multipliers: Many distributed optimization approaches are based on the ADMM algorithm or its variants. Similar to dual decomposition, ADMM has minimization and dual variable update steps, but it uses an *augmented* Lagrangian function. This section provides an overview; see [68] for a detailed tutorial.

ADMM is applicable to optimization problems of the form

$$\min_{x,z} \; f(x) + g(z) \tag{14a}$$

$$\text{subject to} \quad \mathbf{A}x + \mathbf{B}z = c \tag{14b}$$

where x and z are decision variables, $\mathbf{A}$ and $\mathbf{B}$ are specified matrices, c is a specified vector, and $f(x)$ and $g(z)$ are specified functions. The ADMM algorithm is based on the augmented Lagrangian for (14):

$$L_\rho := f(x) + g(z) + y^T (\mathbf{A}x + \mathbf{B}z - c) + \frac{\rho}{2} ||\mathbf{A}x + \mathbf{B}z - c||_2^2 \tag{15}$$

where $\rho > 0$ is a specified penalty parameter and $||\cdot||_2$ is the two-norm. Observe that (15) is the Lagrangian of (14) augmented with a weighted squared norm of the constraint residual. The ADMM algorithm iteratively minimizes the augmented Lagrangian by performing the following updates:

$$x^{k+1} := \operatorname*{argmin}_{x} \; L_\rho\left(x, z^k, y^k\right) \tag{16a}$$

$$z^{k+1} := \operatorname*{argmin}_{z} \; L_\rho\left(x^{k+1}, z, y^k\right) \tag{16b}$$

$$y^{k+1} := y^k + \rho\left(\mathbf{A}x^{k+1} + \mathbf{B}z^{k+1} - c\right) \tag{16c}$$

where superscripts indicate the iteration index and y is the dual variable. Since the x and z updates in (16a) and (16b) are independent, they can be performed in a decentralized fashion.

If the functions $f(x)$ and $g(z)$ are convex, the constraint residual under ADMM (16) is guaranteed to converge to zero and the objective value to the minimum of (14). Typically, the iterations converge quickly to moderate accuracy but can be slow to converge to high accuracy. The convergence rate depends on the choice of ρ, and different strategies have been proposed for adaptatively choosing this parameter [68]. (The literature also describes optimal strategies for choosing ADMM parameters for certain problems [69].) The ADMM algorithm can be applied to non-convex problems, but there is no guarantee of convergence.

The flexibility afforded by the choice of the functions $f(x)$ and $g(z)$ allows for consideration of optimization problems with non-linear constraints. Consider, for instance, the optimization problem $\min_x f(x)$ s.t. $g_i(x) \geq 0,\ i = 1, \ldots, m$. ADMM can be applied to this problem using the reformulation $\min_{x,z} f(x) + h(z)$ s.t. $x = z$, where $h(z)$ is the indicator function $h(z) := \begin{cases} 0 & g_i(z) \geq 0,\ i = 1, \ldots, m \\ \infty & \text{otherwise} \end{cases}$. The variations among the ADMM algorithms considered in this survey are often related to different choices for the decomposition between $f(x)$ and $g(x)$.

As described above, ADMM algorithms require a central coordinator to manage the dual variable update step (16c). However, a modification known as Proximal Message Passing (PMP) facilitates a distributed algorithm. At each iteration of the proximal message passing algorithm, each agent evaluates a "prox" function:

$$\text{prox}_{f_i,\rho}(v) := \operatorname*{argmin}_{w_i} \left(f_i(w_i) + (\rho/2) ||w_i - v_i||_2^2 \right). \tag{17}$$

The vector w_i contains both the decision variables (which themselves are chosen based on the power flow model) and the dual variables for agent i.[3] The vector v_i contains the average values of the variables in w_i for all neighboring nodes. The function $f_i(w_i)$ is the local objective for a specific agent with respect to the decision variables in w_i. The scalar ρ is a tuning parameter. Thus, the prox function optimizes an agent's local objective $f_i(w_i)$ while minimizing the weighted mismatch to the primal and dual variables from the agent's neighbors. The agents pass the results of the prox algorithm (i.e., their local copy in the variable w_i) to their neighbors such that each agent can compute the average value v_i to execute the next iteration. The algorithm converges when the agents agree on common

[3]In the notation of (16), the vector w_i in (17) for each agent contains local copies of both the primal variables (x or z in (14), depending on the agent) and the dual variables y.

values for w_i. The Proximal Message Passing algorithm is a special case of ADMM and thus inherits the convergence guarantees for convex problems. See [70] for further details.

4) Analytical Target Cascading: Analytical Target Cascading (ATC) is a hierarchial, iterative approach for distributed solution of an optimization problem. The optimization problem is split into subproblems which are related by a tree structure. Parent and children subproblems in this tree share optimization variables, with the coupling modeled using penalty functions that are modified at each iteration. If all subproblems are convex, the algorithm is guaranteed to converge to the solution. Note that ATC algorithms require a central coordinator to manage the distributed computations. See, e.g., [71], [72] for further details.

5) Auxiliary Problem Principle: Similar to the previous techniques, the Auxiliary Problem Principle (APP) technique decomposes an optimization problem into subproblems with shared variables [73]. Each subproblem corresponds to a region of the system with shared variables at the tie-lines connecting to neighboring regions. An augmented Lagrangian approach is again used to ensure consistency between the subproblems for neighboring regions. The key difference for APP techniques is that the cross-terms in the two-norm expression employed in the augmented Lagrangian (15) are linearized rather than modeled directly as in ADMM and ATC techniques. This decouples the subproblems such that no central coordinator is required for APP techniques. Convergence is guaranteed if all subproblems are convex.

6) Optimality Condition Decomposition: Rather than duplicating shared variables as in the previous techniques, the Optimality Condition Decomposition (OCD) technique assigns each primal and dual variable to a specific subproblem [74]. Each agent considers a subproblem under the condition that only its assigned variables are allowed to change (i.e., all variables that are assigned to other subproblems are fixed to their previous values). The couplings for the variables assigned to other subproblems are modeled using linear penalties that are added to the objective. The coefficients for these linear penalties are defined by the Lagrange multipliers resulting from other subproblems. At each iteration, each agent applies one step of a Newton-Raphson method to the KKT conditions for its subproblem and then shares the resulting primal and dual values with its neighboring agents. Thus, the OCD technique is effectively an approach for distributed solution of the KKT conditions for an optimization problem. Note that OCD techniques do not require a central coordinator. A sufficient condition for convergence holds when the coupling between subproblems is relatively weak (i.e., there is a small number of sparsely connected subproblems) [74]. A modified OCD algorithm using "correction terms" improves the convergence rate at the cost of a some additional communication between agents [75], [76].

7) Consensus+Innovation: The Consensus+Innovation (C+I) technique [77], [78] is similar to the OCD technique in that both perform a distributed solution of the KKT conditions. However, rather than assigning each variable to a certain subproblem as in the OCD technique, the C+I technique uses an iterative algorithm that allows all variables in a subproblem to vary. A limit point of the iterative algorithm satisfies the KKT conditions. For convex problems, any limit point of this iterative algorithm is therefore an optimal solution [78]. Since each step of the iterative algorithm can be performed using only local and neighboring information, computations in the C+I technique can be performed in a distributed fashion without the need for a central coordinator. Unlike OCD techniques, the C+I technique is applicable at any level of partitioning: an individual agent could potentially represent a single bus or a large region of the network. Various modifications of C+I speed convergence via additional communication links [79] and facilitate consideration of communication delays [80].

III. Distributed Algorithms for Optimal Power Flow Problems

The OPF problem (10) minimizes the total system cost subject to engineering limits and the physical constraints dictated by the power flow equations. This section surveys the application of distributed optimization techniques to the OPF problem for offline applications. The survey is organized by the type of power flow representation (linear, convex non-linear, and non-convex) and optimization technique.

A. Distributed Algorithms for Linear Approximations of the OPF Problem

This section reviews distributed algorithms developed for optimization problems that employ the DC power flow model (4) for transmission systems and two power flow linearizations applicable to distribution systems, (5) and (6).

1) Distributed Optimization with a DC Power Flow Model: Following the exposition in [51], this section summarizes distributed optimization approaches for DC OPF problems categorized by the associated solution technique discussed in Section II-C. See [51] for an extensive review with detailed mathematical descriptions for many relevant algorithms and formulations.

a) Applications of Dual Decomposition to DC OPF Problems: Early work [81] in distributed approaches for solving DC OPF problems employs a dual decomposition technique that adds fictitious buses at the interconnections between independently coordinated areas. Note that the approach in [81] augments the DC power flow model (4) with an approximation of the line losses. Other work that applies dual decomposition techniques includes [82], which incorporates discrete decision variables. The approach in [82] uses so-called "ordinal optimization" techniques that aim to achieve "good enough" choices for the discrete variables while using a dual decomposition for the continuous variables. Recent publications [83], [84] study the integration of demand response resources, including privacy considerations and multiple time periods. Other recent work [85] applies the dual decomposition approach to the DC OPF problem (with a quadratic line loss approximation) in an electricity market context.

b) Applications of ADMM to DC OPF Problems: ADMM techniques have recently been applied to a variety of power system optimization problems. Reference [86] presents

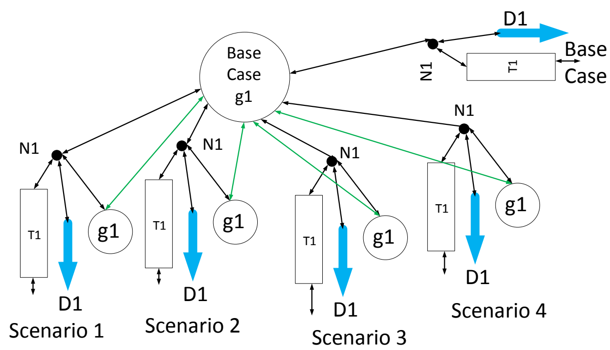

Fig. 1. Depiction of the proximal message passing variant of the ADMM algorithm proposed in [87] for the Security-Constrained DC OPF problem. Each generator "g", transmission line "T", load "D", and bus "N" has an associated computing agent for the base case and each contingency scenario.

a mathematical treatment of ADMM in the context of DC OPF problems, including the consideration of asynchronous updates (i.e., only some of subproblems are updated at each iteration of the ADMM algorithm).

The proximal message passing variant of ADMM (see Section II-C3) eliminates the need for a central coordinator to perform the dual update step, thus enabling a distributed implementation. In [70], ADMM with proximal message passing is applied to DC OPF problems, including a multi-period formulation with many possible device types (HVDC lines, storage devices, controllable loads, etc.). Each component (generator, transmission line, load, and bus) has an associated computing agent. At every iteration, the computing agents solve (in parallel) prox functions (17) to update the variables associated with each component. For generators, $f_i(\cdot)$ in the prox function consists of the generation cost, while v_i is computed by averaging the neighboring components' values for the power generation and voltage phase angles. For lines, $f_i(\cdot)$ is an indicator function signaling satisfaction of the relationship between the power flow and phase angle difference across the line, such that the prox function (17) can be computed analytically. The prox functions for loads consist of update equations for active power and phase angle that can also be evaluated analytically. Computing agents for the buses update the Lagrange multipliers for nodal power balance and nodal phase angle consistency. The iterations proceed until the agents agree on all values for the variables shared by multiple components.

Extension of the proximal message passing variant of ADMM is proposed for security-constrained DC OPF problems in [87]. As depicted in Fig. 1, this extension requires that each component (generator, transmission line, load, and bus) has a computing agent for the base case and each contingency scenario.

c) Applications of ATC to DC Unit Commitment Problems: Studies of ATC techniques with DC power flow models have been conducted in the context of security-constrained unit commitment problems [88], [89]. The approach in [88] has one central coordinator with multiple lower-level agents, each associated with a region of the transmission network. The approach in [89] models a transmission system with multiple connected distribution systems, decomposed at the boundary substations. Note that a DC power flow model is used for the distribution systems, which is generally not appropriate. However, the general decomposition approach could conceivably be applied using a more realistic power flow model for the distribution systems.

d) Applications of APP to DC Unit Commitment Problems: An APP technique is applied in the context of the unit commitment problem in [90] using a two-level generalized Benders' decomposition approach. The top level determines a generator schedule by solving a conventional unit commitment problem. Multi-period DC OPF subproblems, each decomposed regionally using the APP technique, provide cuts for the master problem. This improves computational tractability and protects private utility data. Reference [91] also uses the APP technique to solve a two-stage stochastic unit commitment problem which considers wind uncertainty with geographically distributed reserves.

e) Applications of OCD to DC OPF Problems: DC OPF problems were among the first applications of OCD techniques. Rather than adding fictitious border buses, [92] uses OCD to decompose the DC OPF problem at the tie lines to neighboring regions. Reference [92] also demonstrates the capabilities of OCD techniques using a 583-bus model of the Balkan system. Demonstration on a network of computers is presented in [93], which includes some modifications that require a central coordinator to check for convergence.

A so-called Heterogeneous Decomposition (HGD) algorithm related to OCD techniques is used in [94] to jointly model transmission and distribution systems, decomposed at the boundary substations. The transmission system sends Locational Marginal Prices (LMPs) at the boundary substations to the distribution system, while the distribution systems pass power consumptions back to the transmission system. The approach in [94] uses a DC power flow model for both transmission and distribution systems, with the consideration of possible modifications to account for voltage constraints.

Improvements in the convergence speed of OCD techniques can be achieved by computing linear sensitivities for the dual variables passed to each subproblem [95]. A similar approach is applied in [96], which extends [94] by computing the sensitivities of the LMPs to the load injections at the boundary substations.

f) Applications of C+I to DC OPF Problems: The C+I decomposition technique has solely been applied to DC OPF problems [77], [78]. At each iteration, the buses send their phase angle, power generation, and dual variables for the power balance and line flow constraints to their neighbors. Each bus then uses these shared variables to analytically compute an update for the next iteration. The C+I technique is guaranteed to converge to the DC OPF solution. Improvements made to the C+I technique include faster convergence rates via communicating with buses beyond immediate neighbors [79], the consideration of asynchronous updates [80], and incorporation of security constraints [97].

2) Distributed Optimization with Linearized Power Flow Models for Distribution Networks: While generally well suited for transmission systems, the DC power flow model is typically inappropriate for distribution systems. The other power flow linearizations discussed in Section II-A (i.e., (5) and (6)) are better models for distribution networks. This section next surveys the literature of distributed optimization algorithms that use these power flow models.

a) Applications of Dual Decomposition to Linear Power Flow Models: Reference [98] uses the linearized DistFlow model (6) in concert with dual decomposition. Specifically, [98] considers a distributed two-level stochastic optimization problem, with the first level representing the decisions for a microgrid and the second level representing the decisions for the distribution network operator. The microgrids are coupled by penalty functions that are iteratively determined by the distribution network operator.

b) Applications of ADMM to Linear Power Flow Models: The linearized DistFlow model (6) is also used as the basis of the work in [99]–[101]. The approach in [99] minimizes power losses in a distribution system subject to limits on voltage magnitudes and inverter reactive power capabilities. ADMM is found to outperform a dual decomposition method for this problem. The approach in [100] uses ADMM in concert with stochastic programming in order to consider uncertainty in distribution systems, with decomposition over each bus and each scenario in the stochastic program. Using a regret minimization approach, [101] also considers uncertainty.

As an alternative to the linearized DistFlow equations, the approach in [102] uses the power flow linearization (5) from [20], [21] to optimize distribution systems with large penetrations of solar PV. The ability to regulate voltage magnitudes is validated using test cases with realistic solar generation data.

The power flow model employed in [103] is based on a linearization of the DistFlow model (3) about a specified operating point. The approach in [103] provides an optimal reactive power dispatch for voltage regulation in unbalanced radial distribution systems.

B. Distributed Algorithms for Non-Linear Convex Approximations of the OPF Problem

Convex relaxations based on SDP and SOCP have shown promise for a variety of power system optimization problems. This section reviews distributed approaches for solving these relaxations.

1) Distributed Optimization with the SDP Relaxation: As formulated in Section II-A, the positive semidefinite constraint (8c) in the SDP relaxation couples the variables associated with all buses. There exists an equivalent, sparsity-exploiting reformulation of this constraint that results in a mathematical structure that more closely represents the network topology [104]–[106]. Specifically, the positive semidefinite constraint on the $n \times n$ matrix $\mathbf{W}$ in (8c) can be decomposed into positive semidefinite constraints on certain submatrices of $\mathbf{W}$. (The submatrices are determined by the maximal cliques of a chordal extension of the network graph. See [107] for further details.) This helps facilitate the application of various decomposition techniques. This section reviews applications of dual decomposition and ADMM techniques used in the context of SDP relaxations of the power flow equations.

a) Applications of Dual Decomposition to the SDP Relaxation: Reference [106] proposes two decompositions for the SDP relaxation derived from the primal and dual problem formulations, exploiting network sparsity through chordal extension. Computing agents solve SDP subproblems, one for each maximal clique, corresponding to small regions of the network and share primal or dual variables with the other connected subregions. The updates can be performed asynchronously. Reference [46] applies related techniques to the voltage regulation problem for distribution systems.

b) Applications of ADMM to the SDP Relaxation: ADMM techniques are applied to solve OPF problems for three-phase unbalanced models of radial distribution networks in [32], which shows improved convergence relative to dual decomposition approaches. A similar ADMM approach is applied in [108] to optimize distribution systems with large quantities of solar PV generation. Reference [109] applies ADMM to OPF problems for balanced mesh network models suitable for transmission systems. The heart of the approach in [109] consists of eigenvalue computations that can be performed in parallel. Reference [110] proposes an ADMM algorithm for unbalanced three-phase models of distribution systems. In the key step for the algorithm in [110], each of the agents' problems reduces to evaluating either a closed form expression or the eigendecomposition of a 6×6 matrix.

2) Distributed Optimization with the SOCP Relaxation: Reference [111] applies an ADMM technique to the SOCP relaxation (i.e., (3a)–(3c), (9)) for single-phase models of radial networks in a manner that creates subproblems associated with each bus. An analytical solution for each subproblem yields favorable computational characteristics. This is extended to the case of unbalanced radial networks in [112], where each subproblem either has a closed-form solution or is a small eigenvalue problem whose size is independent of the network size. Related work [113] considers methods for tuning the ADMM parameter ρ in (16), which can have a large impact on the convergence rate.

C. Distributed Algorithms for the Non-Convex OPF Problem

Other than the C+I technique, all other decomposition techniques described in Section II-C have been applied to non-convex formulations of the OPF problem. Note that the theoretical guarantees associated with convex formulations (i.e., the linear approximations reviewed in Section III-A and the relaxations reviewed in Section III-B) are generally not available for non-convex formulations. However, the papers reviewed below demonstrate that various distributed optimization techniques are capable of solving certain practical non-convex OPF problems.

1) Applications of Dual Decomposition to the Non-Convex OPF Problem: Early work [114] applies a dual decomposition method that dualizes the coupling constraints associated with the tie lines between regions. Each subproblem is a non-convex OPF problem with a penalization term in the objective associated with the coupling constraints. The approach in [114] uses an interior point algorithm in combination with cutting plane methods to solve these subproblems. In more recent work, [115] proposes a dual decomposition based algorithm for balanced radial networks using an augmented Lagrangian approach. The algorithm in [115] can be implemented asynchronously and has associated theory claiming a convergence guarantee.

2) Applications of ADMM to the Non-Convex OPF Problem: Reference [116] applies ADMM to a decoupled power flow model, which independently considers the active power/voltage angle and reactive power/voltage magnitude couplings. The algorithm in [116] decomposes the active and reactive power flows between regions.

Recent ADMM-based research efforts [75], [117]–[120] model the fully coupled AC power flow equations in terms of the voltage phasors. Reference [117] decomposes coupling constraints on the rectangular voltage components (i.e., e_i and f_i where the voltage phasor $V_i = e_i + \mathbf{j}f_i$, $i \in \mathcal{N}$). The dual variable updates (16c) can be computed locally by each agent in this approach. Reference [118] also describes an ADMM approach that regionally decomposes subproblems based on shared rectangular voltage coordinates. Subsequent work [119] proposes a decomposition using auxiliary variables that represent the sums and differences of voltage phasors between the terminals of lines that are shared by multiple regions. The sums and differences of the voltage phasors more closely represent the expressions found in the power flow equations, which results in improved convergence characteristics. Under the assumption that the solver applied to each subproblem is reliable in finding a local solution, an approach for updating the penalty parameter (ρ in (16)) gurantees convergence of the ADMM algorithm. In order to apply ADMM techniques to large problems, [75] proposes a spectral partitioning technique for determining the regional decomposition. In combination with the coupling approach proposed in [119] and a strategy for updating the penalty parameter, the spectral partitioning technique results in tractability for large problems (e.g., the 2383-bus Polish system in MATPOWER [121]) [75]. In recent work, [120] performs extensive numerical studies via application of an ADMM technique to a variety of test cases. The results show that the numerical performance of the ADMM algorithm is sensitive to the penalty parameter, with certain parameter values reducing the number of iterations required by an order of magnitude relative to other parameter values. The results also empirically demonstrate the existence of parameter values that yield near globally optimal solutions for all test cases considered. However, appropriate parameter values ranged over several orders of magnitude, and the paper does not provide a method for choosing appropriate parameter values.

The algorithm in [122] uses a power flow formulation that includes variables for both current and voltage phasors. Each iteration of the algorithm in [122] solves a quadratic program derived via applying linearization techniques, with the overall algorithm yielding a solution that satisfies the non-linear power flow equations.

3) Applications of ATC to the Non-Convex OPF Problem: A two-level ATC algorithm is applied in [123] to coordinate the operation of a distribution grid that contains micrgrids. The voltage magnitudes and angles at the boundaries of the distribution system and microgrid subproblems are coupled using an exponential penalty formulation [124].

4) Applications of APP to the Non-Convex OPF Problem: Early work in distributed optimization techniques for OPF problems includes the APP-based approach in [125]. The OPF problem is decomposed regionally using "dummy generators" whose active and reactive power outputs and voltage phasors model the neighboring regions. Subsequent work [126] demonstrates the capabilities of this decomposition using a 2587-line model of ERCOT. Case studies with multiple regions are presented in [127], which also provides guidance regarding the choice of penalty parameters in the APP formulation.

5) Applications of OCD to the Non-Convex OPF Problem: OCD techniques were first proposed in the context of the non-convex OPF problem [74], with a more detailed description and analysis of the convergence characteristics presened in [128]. Several advances are presented in [129], including parameter tuning and better consideration of the reference angle. The approach in [130] considers the coordinated operation of FACTS devices using an overlapping regional decomposition. In order to speed convergence rates, [76] proposes the use of "correction terms" that require some additional sharing of information between buses which are not directly connected in the power system network. Reference [131] describes a partitioning method based on a spectral analysis that results in computational improvements for the OCD approach.

6) Applications of Other Distributed Optimization Techniques to the Non-Convex OPF Problem: Two other distributed optimization techniques have also been applied to non-convex OPF problems: Gradient Dynamics and Dynamic Programming with Message Passing.

First proposed in [132] with more recent treatments in [133], [134], the Gradient Dynamics (GD) technique embeds the KKT conditions for an optimization problem in a dynamical system. The equilibria of the dynamical system correspond to KKT points for the original OPF problem. Assuming the satisfaction of certain technical conditions, the approach in [135] and [136] constructs a formulation which ensures that only the optima of the OPF problem are locally stable, with other KKT points being unstable. Thus, the OPF problem can be solved by integrating the dynamical system. This technique inherits the decomposibility associated with the network structure: when integrating the dynamical system, each bus can serve as a computing agent that only communicates with its neighbors. The Gradient Dynamics approach is applied to solve the non-convex OPF problem in [135]. Theoretical analyses of the proposed approach and comparison to convex relaxation techniques are presented in [136]–[138].

A Dynamic Programming technique proposed in [139] (see also the more general presentation in [140]) performs an interval-based discretization of the power flow variables in the DistFlow model (3). For tree networks, this discretization enables the application of tools from dynamic programming to compute both a lower bound on the optimal objective value of the OPF problem and an approximately feasible solution (to within a tolerance that depends on the discretization). Discrete variables can also be incorporated into this formulation. The tree topology enables a natural distributed implementation using a message passing approach. Extension to more general network topologies is also possible using more sophisticated ideas from constraint programming.

D. Comparison of Distributed Algorithms for Power System Optimization

Most of the existing numerical algorithms for solving power system optimization problems are based on either first-order methods relying on gradients of the objective and constraint functions or second-order methods relying on both gradients and hessians of the objective and constraint functions. Second-order methods benefit from a small number of iterations and a high convergence rate, but the complexity of each iteration is prohibitive for large-scale problems in general. In particular, these methods are not parallelizable unless the problem is highly sparse and structured. Conversely, first-order methods have cheap iterations that can often be parallelized, but the convergence rate is low and is highly affected by the condition number of the problem data.

Although first-order methods all have the same convergence rate in the worst case, they exhibit different performances on specific applications, with the empirical convergence and complexity of each method depending on the specifics of the underlying problem. This explains the large number of first-order methods surveyed in this paper. Each of these methods has some tunable parameters to improve the performance, and there is a trade-off between how many iterations are required to obtain a high-quality approximate solution and how many computing nodes (and how much communication) are used for parallelizing the computations. For instance, the performance of an ADMM-based algorithm depends on the step size parameter ρ in (16), which balances the convergence rates of the primal and dual residuals. While there are various strategies to select an appropriate value of ρ (see, e.g., the review of such strategies for general optimization problems in [68, Section 3.4]), performance is generally problem dependent [113].

With a strong dependence on the application and problem of interest as well as appropriate parameter tuning, quantitative analyses via numerical simulations are key for understanding performance in practice. See [51], [54], [55], [113] for further discussion and quantitative comparisons among some of the methods surveyed in this paper. Further empirical work is needed to better characterize the practical performance of distributed optimization algorithms and the selection of appropriate tuning parameters for various power system optimization problems.

IV. Online Optimization and Control

Section III focuses on offline algorithms for solving OPF problems. Even though these algorithms are distributed, they iterate on all variables in the cyberspace until they converge before their solutions are applied to the physical grid. In particular the intermediate iterates typically do not satisfy the power flow equations (Kirchhoff's laws) nor operational constraints. While offline algorithms have been widely used in traditional power system applications, they may become inadequate in some future applications that involve a large network of distributed energy resources, especially in the presence of fluctuating loads and volatile renewables.

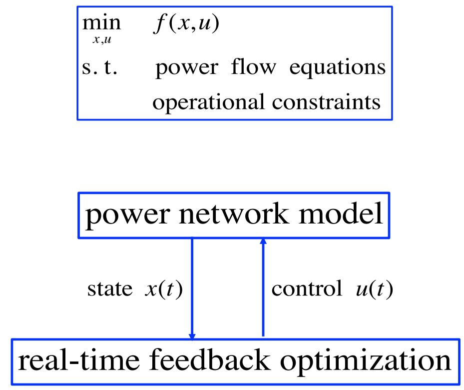

Fig. 2. General structure of real-time, or online, algorithms for optimization problems.

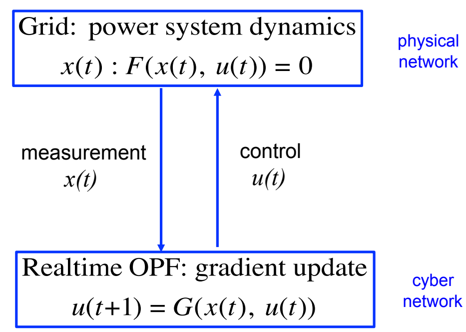

Fig. 3. Online or real-time algorithms for OPF problems where the controller updates the control variable $u(t)$ in each period and applies it to the grid. The grid implicitly solves the power flow equation $F(x, u(t)) = 0$ to determine the state variable $x(t)$, which is measured and used to compute the control $u(t + 1)$ in the next period.

In this section, we summarize recent research on real-time, or online, algorithms for solving power system optimization and control problems. These algorithms iterate only on variables corresponding to controllable devices (e.g., intelligent loads) in feedback interaction with the grid. The grid may be modeled by a set of algebraic power flow equations for slow timescale behavior or by a set of differential equations for fast timescale behavior. The general structure is illustrated in Fig. 2, where a model of the physical network is given and an optimization problem is specified as the control objective. Our task is to design a real-time feedback controller so that the closed-loop system converges to an equilibrium that solves the optimization problem. Different papers use different power flow models and different algorithms to compute the control in each iteration within this general framework.

There are two important advantages of real-time closed-loop implementation of optimization algorithms. First, this approach naturally tracks changing network conditions as these changes manifest themselves in the network state $x(t)$ that is used to calculate the control $u(t)$ (see Fig. 2). These algorithms therefore tend to be robust to uncertainties and disturbances, e.g, due to fluctuating loads and volatile renewables. Furthermore, as we will see below, many of the proposed algorithms are to some extent decentralized and model-free (e.g., independent of system parameters and relying only on local measurements) which makes them attractive in a plug-and-play scenario. Second, for some applications that involve a large network of distributed energy resources in the future, solving the optimization centrally will be infeasible because of the high cost of collecting and communicating the required state and parameter data and because of the desire to protect private information spread across multiple organizations. Real-time distributed solution may be the only viable strategy in these situations.

In the following, we first summarize distributed control theory for general systems. We then review four prominent applications of online feedback optimization algorithms specific to power systems: real-time optimal power flow, optimal frequency control, optimal voltage regulation, and optimal wide-area control.

A. Overview of Distributed Control Theory

Classical control theory provides a rich mathematical foundation for the design of centralized controllers for an interconnected or multi-channel system composed of several (interconnected) subsystems. A centralized control framework is concerned with a single control unit responsible for collecting the outputs of all subsystems, processing the acquired information, and generating the inputs of those subsystems. This centralized control approach is an unattractive, if not infeasible, strategy for many real-world systems due in part to its computation and communication complexity.

The area of decentralized control has been created to address the challenges arising in the control of complex networks and large-scale systems [141]–[144]. The objective is to design a structurally constrained controller with the aim of reducing the computation and communication complexity of the overall controller. The control layer consists of a number of local controllers (sub-controllers), where each sub-controller is in charge of controlling only one of the subsystems of the interconnected system. The local controllers are often allowed to exchange limited information with one another. Recalling the definitions in the introduction, this type of controller is usually referred to as a *distributed controller* (especially when the local controllers are geographically distributed). In contrast, a *decentralized controller* has no information exchange among the local controllers.

Consider the problem of designing an optimal decentralized controller for a multi-channel deterministic or stochastic system, where the optimality is measured with respect to a linear-quadratic, H_2, or H_∞ performance index. It has long been known that this problem is computationally hard to solve and, in particular, NP-hard in the worst case [145]–[147]. Great effort has been devoted to investigating this highly complex problem for special types of systems, including spatially distributed systems [148]–[152], dynamically decoupled systems [153], [154], weakly coupled systems [155], and strongly connected systems [156]. Another special case that has received considerable attention is the design of an optimal static distributed controller [157], [158]. Early approaches for the optimal decentralized control problem were based on parameterization techniques [159], [160], which were then evolved into matrix optimization methods [161], [162].

Due to the recent advances in the area of convex optimization, the focus of the existing research efforts has shifted from deriving a closed-form solution for the above control synthesis problem to finding a convex formulation of the problem that can be efficiently solved numerically [163]–[170]. This has been carried out in the seminal work [171] by deriving a sufficient condition named quadratic invariance, which has been generalized in [172] by deploying the concept of partially ordered sets. These conditions have been further investigated in several other papers [173]–[175]. A different approach is taken in the recent papers [176], [177], where it has been shown that the decentralized control problem can be cast as a convex optimization for positive systems. More recently, conic optimization has been applied to the optimal distributed control problem, and it has been shown that a semidefinite programming (SDP) relaxation of this problem always has a low-rank solution [178], [179]. Finally, another stream of research attempts to overcome the complexity of decentralized optimal control problems by appropriately regularizing centralized problems so that they are either convexified (even in presence of structural constraints) [180] or admit a sparse solution [158], [170], [181], [182].

The optimal distributed control problem in a general setting deals with the minimization of a cost functional that is composed of both terminal and transient (stage) costs. Moreover, the state and input constraints are requited to belong to pre-specified sets at all times. Although this problem has a high computational complexity, a special case of the problem is much more tractable where the stage cost is zero and there is no hard constraint on the state and input trajectories. The latter problem has been studied in the context of electric power systems for various applications such as real-time optimal power flow control, frequency control, and voltage control. We will survey these papers in the rest of this section.

B. Real-Time Optimal Power Flow

We first consider the problem of solving OPF in closed loop. While this problem has emerged only recently, it has already led to several parallel research developments by different groups. Here we focus on slow timescale behavior where the network is modeled by a set of power flow equations. In each iteration, the real-time optimization algorithm computes a control $u(t)$ and applies it to the grid, which then computes the state $x(t)$ by implicitly solving the power flow equations in real time at scale, as illustrated in Fig. 3. *This approach hence explicitly exploits the law of physics as a power flow solver.* Unlike offline algorithms, the intermediate iterates not only satisfy the power flow equations by design, but may also satisfy operational constraints, depending on the specific algorithm.

In [183], a radial network is modeled by the DistFlow equations (3) and a first-order gradient algorithm is used to compute the control in each iteration. Barrier functions are used to ensure that operational constraints are satisfied at all times. Sufficient conditions for convergence to a local or a global optimum of the non-convex OPF problem are both established. The same problem is studied in [184] using a completely different approach that does not require the controller to know the network model. This approach uses a gradient-based extremum seeking algorithm where a sinusoidal probing signal is injected into the network in order to estimate the gradient of the cost function with respect to control variable $u(t)$. Reference [184] proves a sufficient condition for the cost function to be convex in the control u and, under this condition, it shows that the algorithm converges to a neighborhood of the optimum. Incidentally, this sufficient condition also guarantees that the gradient algorithm of [183] attains global optimality. A general network is modeled in [185] using the AC power flow equations (1) in the complex form. The paper proposes a first-order distributed subgradient algorithm for solving the SDP relaxation of the OPF problem and proves its global convergence. The methods in both [183] and [185] extend to the case of multiphase unbalanced networks. In [186], an online projected-gradient approach is proposed that steers the closed-loop system on the power flow manifold (the space of solutions of the power flow equations [22]) towards a locally optimal solution. The performance and robustness of this approach have been demonstrated in [187].

A key assumption in all of the above papers is that the OPF problem is static. The time-varying case where the OPF problem changes in each iteration is studied in [188]. A linearized power flow model closely related to (5) is used and a first-order primal-dual algorithm is proposed based on a regularized Lagrangian. The paper characterizes the tracking performance of the proposed algorithm in terms of the rate at which OPF drifts and errors due to regularization and in gradient updates. In [189], the AC power flow model (1) is used and a quasi-Newton method is proposed for better tracking of the time-varying OPF problem. Tracking performance is characterized in terms of the rate at which OPF drifts, the error in the Hessian estimation, and the condition number of the approximate Hessian. A different approach is proposed in [190] to deal with time-varying and random OPF problems using the DistFlow model (3) and its linearization (6) for radial networks. Stochastic dual subgradient algorithms are proposed and certain operational constraints are enforced in an average sense.

Since the problem is motivated by the control of a large network of distributed energy resources in the future, decentralized or distributed algorithms are desirable. Most schemes proposed, however, are centralized. Notable exceptions are online algorithms for volt/var control that are decentralized (e.g. [191], [192]) or distributed (e.g. [193], [194]). Many of these papers are discussed in Section IV-D on optimal voltage regulation.

In summary, real-time online optimal power flow is a relatively young problem, but it has already triggered a widespread interest. Most fundamental questions concerning convergence, robustness properties, and the cyber-physical implementation, especially distributed versions of online algorithms, are wide open to date.

C. Optimal Frequency Control

Maintaining the system frequency close to its nominal value despite fluctuating loads and generation is one of the central tasks in controlling power systems. At its core, frequency control is an optimal resource allocation problem, where generation and load have to be balanced in the economically most efficient way. Traditionally, this is achieved using a hierarchical control scheme consisting of primary (Droop Control), secondary (AGC), and tertiary (Economic Dispatch) layers operating at different time scales [195]. Droop controllers installed in synchronous generators and in voltage-source inverters are fully decentralized and operate on a fast time scale, but cannot by themselves restore the system frequency to its nominal value following load changes. To ensure a correct steady-state frequency and a fair power sharing among generators and inverters, centralized AGC and Economic Dispatch schemes are traditionally employed. Developing more flexible distributed schemes to replace, or complement, these traditional control layers has been a very active research area in the past few years. The interest is explained by the wide-spread integration of distributed power generation, the deployment of smart frequency-responsive loads, and the increasing interest in microgrids with a need for independent operation.

In the following, we focus primarily on schemes supporting the secondary and tertiary layers, that is, asymptotic frequency regulation in an economically efficient way and possibly subject to operational constraints. We begin by reviewing integral-control strategies focusing on the optimal balancing problem and then discuss primal-dual control strategies that directly attack the resource allocation optimization problem.

1) Distributed Averaging for Optimal Frequency Control: To obtain correct steady-state frequency without a centralized controller, it has been proposed to complement the droop controllers with fully decentralized integral control [196]–[198]. Although these schemes ensure system stability and correct steady-state frequency in theory, in practice they suffer from poor robustness to measurement bias and clock drifts [197], [199]–[201]. Furthermore, the injections of such decentralized integral controllers generally do not lead to an efficient allocation of generation resources.

To alleviate this shortcoming, distributed averaging-based integral control can be implemented [198], [199], [202]–[207]. These schemes are no longer decentralized and require communication between the local controllers, but they can, on the other hand, also ensure a fair sharing of power generation by equalizing the marginal prices, so-called active power sharing. Hence, they can also perform the duties normally assigned to the tertiary power dispatch layer. Coordination of generation via discrete-time consensus [208] or via ratio consensus [209], [210] have also been considered in the literature. It should be pointed out that while centralized schemes suffer from a single point of failure, the distributed schemes require retrofitting of the communication architecture, which may prevent implementation in practice. It has also been demonstrated that the distributed schemes can be sensitive to faults and misbehavior of agents [200].

To combine the advantages of centralized and distributed frequency-regulation schemes, semi-decentralized schemes based on a single average of the local measurements have been proposed [197], [200], [211], [212]. These methods can be derived from a dual-gradient approach, and stability and optimal economic dispatch can be ensured [200]. The papers [213]–[218] have characterized and compared the transient control performances of certain semi-decentralized, distributed, and decentralized control schemes under varying network topologies and parameters. In particular, for decentralized and semi-decentralized schemes, losses due to non-equilibrium power flows have been show to be equal, and, for uniform network parameters, independent of the network's connectivity. However, in the case of distributed averaging-based integral control, there is a dependence on network connectivity that can be exploited to decrease transient resistive losses [219].

Extensions of some of the above distributed frequency-control schemes to more general dynamical models have been pursued in [220]–[225] by means of a passivity-based analysis.

2) Primal-Dual Methods for Optimal Frequency Control: In parallel to integral-control strategies, a rich literature has emerged that directly attacks the optimal generator dispatch problem by means of optimization strategies which can be implemented online as frequency controllers. Typically, these strategies are based on primal-dual gradient methods dating back to [226]–[228] that seek the saddle points of the Lagrangian function of the underlying optimization problem.

To the best of our knowledge, the earliest work that adopted this approach was [229], which also exploited the pricing interpretation of the Lagrange multipliers as a byproduct of the dualization-based method. This pricing aspect of frequency control has also been picked up in the recent literature on so-called transactive control, bridging the gap between real-time control, offline optimization, and market aspects [230], [231].

Aside from pricing, another interpretation of dualization-based methods is that the primal-dual dynamics of a carefully crafted generator dispatch optimization problem are formally equivalent to the power system physics plus additional controller dynamics [232]–[237], e.g., certain Lagrange multipliers formally correspond to generator frequencies. Thus, part of the frequency control problem is already solved by the power system physics, and additional controllers enforce operational constraints. Interestingly, part of these additionally needed control loops already exist (e.g., droop controllers) whereas others are novel. Once having understood that optimization algorithms can be used to reverse-engineer the power system physics and controllers already in place, the existing controllers can also be tuned accordingly so that the closed-loop system optimizes a desired cost function [238]–[240].

Finally, independently of pricing interpretations and reverse engineering of the power system physics and controls, primal-dual-based optimal frequency controllers have also been deployed in [220], [241], [242] for fairly general, detailed, and non-linear power system dynamics.

3) Conclusions on Optimal Frequency Control: Frequency control is a well studied and mature problem area that has seen various contributions from different directions. In the following, we review a few open problems. It is generally not known whether economically efficient frequency control is possible in an entirely decentralized fashion. Furthermore, the analysis of the widely adopted distributed averaging-based integral controllers is thus far restricted to symmetric communication topologies and the case of quadratic optimization problems subject to power balance constraints. Hence, the existing setup precludes uni-directional communication, non-quadratic objectives as well as inequality constraints arising, e.g., from generation limits. Finally, primal-dual algorithms are very appealing since their implementation can partially be outsourced to the system physics. If we take this striking idea from frequency control to general power system operation, it is yet unclear which control actions can be outsourced to the system physics and which have to be implemented in a cyber-layer.

D. Optimal Voltage Control

Optimal voltage regulation is generally considered to be a more complex task than frequency regulation. Whereas in frequency regulation the system can be balanced in the economically most efficient way while respecting operational constraints, the task of regulating the voltages is inherently not aligned (and sometimes even in contradiction) with economic objectives such as minimizing power losses, achieving a desirable fair power sharing (or curtailment), or other operational objectives such as maximizing the distance to voltage collapse [207], [243]–[246]. Due to this multi-objective nature of voltage regulation, typically weighted sums of cost functions are considered, voltages are regulated only outside certain safe deadbands, or voltage bands are imposed as constraints rather than as objectives.

The settings in the literature vary between transmission and distribution scenarios, dominantly inductive or resistive grids, and accordingly compensators provide either active or reactive power to support the voltages. In the following, we will (with slight abuse of notation) employ the colloquial terminology of reactive power support. The relevant literature is rich in terms of centralized approaches that aim at transferring existing centralized transmission-level solutions to distributed generation scenarios and distribution systems. On the other hand, it has been recognized that the task of voltage regulation is to a large part a truly localized problem and fully decentralized controllers [243], [247] can in certain instances perform equally well as centralized strategies. However, it is also known that mere decentralized strategies [248]–[250] cannot successfully regulate the voltages in the presence of certain constraints. In this case, the local compensators need to be coordinated through a communication infrastructure.

1) Decentralized Optimal Voltage Control Strategies: We begin our literature survey with *fully decentralized control strategies*. It has been broadly recognized that the unconstrained optimization problem of minimizing a combination of power losses and sum-of-squared voltage deviations admits an entirely decentralized optimal solution described by a linear trade-off between the local reactive power injection and the local voltage deviation – colloquially also known as *droop control*. This finding that droop-like behaviors are *inverse optimal* is fairly robust to modeling assumptions and has been made in meshed networks (modeled by the non-linear reactive power flow (2b) with fixed angles) [251] and in radial networks using the Linearized DistFlow model (6) [252]. Similar droop behaviors are being incorporated into national grid codes as technical specifications for grid connected generators [253], [254]. The gains of these droop control laws can be optimized, according to both the grid topology and the operating point of the grid, especially with respect to the active power injection of the same generators [255]. Likewise, the *IEEE 1547.8 control standard* proposing piece-wise linear droop behavior has been found to be inverse optimal to a cost composed of sum-of-squared voltage deviations and reactive power provisioning in a Linearized DistFlow setting [191]. Even for the full AC power flow (2), a variety of local control strategies give rise to gradient-type closed-loop dynamics that are implicitly optimal to cost functions composed of power losses, voltage deviations, and injection costs [245]. For example, a popular theme throughout the literature is that the gradient of a power loss cost gives rise to power flows according to the principle of least action.

Given these insights, different *optimality-seeking controllers* can be engineered. References [192], [256] provide projected (sub)gradient algorithms that can be implemented as fully decentralized control strategies. The resulting closed-loop dynamics converge to the same optimizers as the droop-like IEEE 1547.8 standard, but under less restrictive conditions and with better transient performance. These results have been extended in [257] towards asynchronous updates and dynamically changing network conditions and in [258] towards pseudo-gradient algorithms easing the implementation. Different cost functions have been considered and optimized through local gradient-based control strategies: [259] considers reactive power loss minimization via projected integral controllers and dual gradient ascent methods (13), [260] considers the objective of power transfer maximization through a projected gradient scheme, [261] develops local proximal gradient schemes to minimize sum-of-squared voltage deviations and the cost of reactive power provisioning, and [262] provides a gradient-based algorithm that changes the reactive power provisioning only when voltages are outside the admissible range. All of these schemes consider linearized power flow models. Finally, inverse optimal droop-like controllers with quadratic nonlinearities are advocated in [263] for the quadratic power flow formulation (1), in [251] for a non-linear reactive power flow model (2b) neglecting angles, and in [245] for the full AC power flow model (2).

2) Distributed Optimal Voltage Control: Despite the widespread success of fully decentralized control strategies, a number of recent references [248]–[250] observed that a large class of local controllers cannot successfully regulate the voltages within prescribed bounds when the compensators are also limited in terms of their reactive power injection. The reason for this shortfall is the same that allowed the previous references to prove their convergence statements: namely, in a linearized system setting, monotone droop-like strategies give rise to a unique closed-loop voltage and injection profile. The latter may be feasible or not depending on the constraints and system loading. Hence, in such scenarios the local compensators need to be coordinated through a communication infrastructure. We refer to such strategies as *distributed*.

In [193], an optimal reactive power flow problem is formulated for a linearized power flow model that gives rise to a linearly-constrained quadratic program whose optimizer can be computed in a distributed fashion. A distributed online control algorithm is tasked with tracking and stabilizing this optimizer in closed loop. In order to incorporate the generator constraints on reactive power injection, a distributed gradient projection approach has been proposed in [264]. For the same task, a distributed dual ascent method is proposed [194] that guarantees convergence to the operating region where both reactive power limits and voltage constraints are satisfied. A projected dual ascent and an accelerated version are presented in [249], [250]. Another set of distributed strategies target the objective of fair reactive power sharing based on average consensus of the injection ratios [207], [245], [265]. Yet another objective is that of maximizing the distance to voltage collapse, which is approached by means of regularization and a distributed dual ascent method [246]. Finally, there are many approaches decomposing centralized voltage optimization problems into local subproblems that need to be coordinated through communication. Examples are decompositions of SDPs [32], [46], ADMM schemes [108], [266], broadcast communication [267], and leader-follower schemes [268]. Whereas these approaches can be used for distributed closed-loop control, the communication and computation load is quite high making them more suitable for parallel and offline computation.

3) Conclusions on Optimal Voltage Control: In conclusion, the field of distributed optimal voltage control is rich in terms of objectives, architectures, and algorithms. It is yet to be understood which problems admit fully decentralized solutions and when communication is needed. Another open question is whether the design of these control strategies can be performed in a decentralized manner, i.e., based only on the system's local parameters, enabling scalable and adaptive plug-and-play deployments of these solutions.

E. Optimal Wide-Area Control for Oscillation Damping

Inter-area oscillations in bulk power systems are associated with the dynamics of synchronous machines oscillating relative to each other. These system-wide oscillations arise from modular network topologies, adversely interacting controllers, and large inter-area power transfers. Inter-area oscillations induce severe stress and performance limitations on the transmission network and may even cause instabilities and outages.

These oscillations are conventionally damped by generator excitation control via power system stabilizers (PSSs) or as proposed more recently via HVDC links or FACTS devices. However, mere decentralized control actions can interact in an adverse way and destabilize the overall system. Furthermore, even when decentralized controllers provide stability they may result in poor performance, and their optimal tuning presents non-trivial design challenges [269], [270].

The deployment of renewables in remote locations, the increasingly deregulated operation of power systems, the advent of low-inertia generation, and transmission network expansions put inter-area oscillations back in the spotlight. The monitoring and analysis of inter-area oscillations has recently been enhanced by advances in wide-area measurement and communication technologies as well as scientific advances in large-scale and multi-agent systems. These advances pave the way to wide-area control (WAC), where control loops are closed from remote phasor measurements to local synchronous machine excitation controllers enabling real-time distributed control on a continental scale. We refer to the surveys [271]–[274] and the articles in [275] for further information.

Several efforts have been directed towards the selection of few but critical WAC channels [274], [276] and decentralized or distributed WAC design based on robust and optimal control methods; see [277]–[282] and references therein. However, it is to be noted the majority of the existing approaches are either based on centralized (output feedback) control or pre-parametrized and sub-optimally tuned controllers. More recently, several approaches emerged that use the decentralized control techniques reviewed in Section IV-A. Particularly, sparsity-promoting approaches have been applied very successfully: the ℓ_1-regularized H_2-control approach developed in [170] has been applied to WAC by means of PSS [283]–[285] and HVDC links [286], and it has also been adopted for pricing in WAC [287]. An ℓ_1-regularized H_∞-control approach is presented in [288]. A static H_2-output feedback problem for PSS design has been proposed in [289] using the methods in [158]. Finally, it has recently been observed that DC-segmented power systems are poset-causal [290] making them amenable to decentralized H_2 control as in [172].

The above recent references indicate that optimal decentralized and distributed control techniques are very much suited for wide-area damping control. We firmly believe that further powerful optimal design methods will be successfully applied to WAC problems in the near future. However, we emphasize optimality is merely one side of the story, and robustness of WAC to communication issues and changing system configurations should not be sacrificed for performance.

V. Conclusions

After summarizing various power flow models and techniques for distributed optimization, this paper has surveyed the literature regarding offline distributed optimization and control algorithms for a variety of power system applications. Algorithms based on Dual Decomposition, the Alternating Direction Method of Multipliers, Analytical Target Cascading, the Auxiliary Problem Principle, Optimality Condition Decomposition, and Consensus+Innovation have shown promise in solving a variety of power system optimization and control problems. This paper then reviewed progress on online optimization and control algorithms for the purposes of real-time optimal power flow, optimal frequency control, and optimal voltage control. Recent developments suggest the great potential of these approaches for power system control, but further work is required to address a variety of open questions. Aside from specific algorithmic questions, more general concerns of distributed strategies relate to privacy issues, cyber-physical security, as well as robustness to communication uncertainties and failures.

References

[1] Z. Qiu, G. Deconinck, and R. Belmans, "A Literature Survey of Optimal Power Flow Problems in the Electricity Market Context," in *Power Systems Conference and Exposition, 2009. PSCE '09. IEEE/PES*, March 2009, pp. 1–6.

[2] J. A. Momoh, R. Adapa, and M. E. El-Hawary, "A Review of Selected Optimal Power Flow Literature to 1993. I. Nonlinear and Quadratic Programming Approaches," *IEEE Trans. Power Syst.*, vol. 14, no. 1, pp. 96–104, Feb. 1999.

[3] J. A. Momoh, M. E. El-Hawary, and R. Adapa, "A Review of Selected Optimal Power Flow Literature to 1993. II. Newton, Linear Programming and Interior Point Methods," *IEEE Trans. Power Syst.*, vol. 14, no. 1, pp. 105–111, Feb. 1999.

[4] S. Frank, I. Steponavice, and S. Rebennack, "Optimal Power Flow: A Bibliographic Survey, Parts I and II," *Energy Syst.*, vol. 3, no. 3, pp. 221–289, 2012.

[5] A. Castillo and R. P. O'Neill, "Survey of Approaches to Solving the ACOPF (OPF Paper 4)," US Federal Energy Regulatory Commission, Tech. Rep., Mar. 2013.

[6] B. Stott and O. Alsaç, "Optimal Power Flow–Basic Requirements for Real-Life Problems and their Solutions," in *SEPOPE XII Symposium, Rio de Janeiro, Brazil*, 2012.

[7] P. Panciatici, M. C. Campi, S. Garatti, S. H. Low, D. K. Molzahn, A. X. Sun, and L. Wehenkel, "Advanced Optimization Methods for Power Systems," *18th Power Syst. Comput. Conf. (PSCC)*, Aug. 2014.

[8] F. Capitanescu, J. L. M. Ramos, P. Panciatici, D. Kirschen, A. M. Marcolini, L. Platbrood, and L. Wehenkel, "State-of-the-Art, Challenges, and Future Trends in Security Constrained Optimal Power Flow," *Electric Power Syst. Res.*, vol. 81, no. 8, pp. 1731 – 1741, 2011.

[9] J. C. Mukherjee and A. Gupta, "A Review of Charge Scheduling of Electric Vehicles in Smart Grid," *IEEE Syst. J.*, vol. 9, no. 4, pp. 1541–1553, Dec. 2015.

[10] J. S. Vardakas, N. Zorba, and C. V. Verikoukis, "A Survey on Demand Response Programs in Smart Grids: Pricing Methods and Optimization Algorithms," *IEEE Communications Surveys Tutorials*, vol. 17, no. 1, pp. 152–178, First Quarter 2015.

[11] D. Zhu and G. Hug, "Decomposed stochastic model predictive control for optimal dispatch of storage and generation," *IEEE Transactions on Smart Grid*, vol. 5, no. 4, pp. 2044–2053, 2014.

[12] S. Riverso and G. Ferrari-Trecate, "Plug-and-play distributed model predictive control with coupling attenuation," *Optimal Control Applications and Methods*, vol. 36, no. 3, pp. 292–305, 2015.

[13] M. E. Baran and F. F. Wu, "Optimal Capacitor Placement on Radial Distribution Systems," *IEEE Trans. Power Del.*, vol. 4, no. 1, pp. 725–734, Jan. 1989.

[14] M. Farivar and S. H. Low, "Branch Flow Model: Relaxations and Convexification (Part I)," *IEEE Trans. Power Syst.*, vol. 28, no. 3, pp. 2554–2564, Aug. 2013.

[15] ——, "Branch Flow Model: Relaxations and Convexification (Part II)," *IEEE Trans. Power Syst.*, vol. 28, no. 3, pp. 2565 – 2572, Aug. 2013.

[16] S. Bose, S. H. Low, T. Teeraratkul, and B. Hassibi, "Equivalent Relaxations of Optimal Power Flow," *IEEE Trans. Automatic Control*, vol. 60, no. 3, pp. 729–742, Mar. 2015.

[17] S. H. Low, "Convex Relaxation of Optimal Power Flow–Parts I: Formulations and Equivalence," *IEEE Trans. Control Network Syst.*, vol. 1, no. 1, pp. 15–27, Mar. 2014.

[18] ——, "Convex Relaxation of Optimal Power Flow–Part II: Exactness," *IEEE Trans. Control Network Syst.*, vol. 1, no. 2, pp. 177–189, June 2014.

[19] B. Stott, J. Jardim, and O. Alsaç, "DC Power Flow Revisited," *IEEE Trans. Power Syst.*, vol. 24, no. 3, pp. 1290–1300, Aug. 2009.

[20] S. Bolognani and S. Zampieri, "On the Existence and Linear Approximation of the Power Flow Solution in Power Distribution Networks," *IEEE Trans. Power Syst.*, vol. 31, no. 1, pp. 163–172, Jan. 2016.

[21] S. V. Dhople, S. S. Guggilam, and Y. C. Chen, "Linear Approximations to AC Power Flow in Rectangular Coordinates," in *53rd Annu. Allerton Conf. Commun., Control, Comput.*, Sept. 2015, pp. 211–217.

[22] S. Bolognani and F. Dörfler, "Fast Power System Analysis via Implicit Linearization of the Power Flow Manifold," in *53rd Annu. Allerton Conf. Commun., Control, Comput.*, Sept. 2015.

[23] M. E. Baran and F. F. Wu, "Optimal Sizing of Capacitors Placed on A Radial Distribution System," *IEEE Trans. Power Delivery*, vol. 4, no. 1, pp. 735–743, 1989.

[24] X. Bai, H. Wei, K. Fujisawa, and Y. Wang, "Semidefinite Programming for Optimal Power Flow Problems," *Int. J. Electr. Power Energy Syst.*, vol. 30, no. 6-7, pp. 383–392, 2008.

[25] J. Lavaei and S. H. Low, "Zero Duality Gap in Optimal Power Flow Problem," *IEEE Trans. Power Syst.*, vol. 27, no. 1, pp. 92–107, Feb. 2012.

[26] M. Farivar, C. R. Clarke, S. H. Low, and K. M. Chandy, "Inverter VAR Control for Distribution Systems with Renewables," in *IEEE SmartGridComm Conf.*, Oct. 2011.

[27] R. A. Jabr, "Radial Distribution Load Flow using Conic Programming," *IEEE Trans. Power Syst.*, vol. 21, no. 3, pp. 1458–1459, Aug. 2006.

[28] S. Sojoudi and J. Lavaei, "Physics of Power Networks Makes Hard Optimization Problems Easy to Solve," in *IEEE PES General Meeting*, July 2012, pp. 1–8.

[29] C. Coffrin, H. L. Hijazi, and P. Van Hentenryck, "The QC Relaxation: A Theoretical and Computational Study on Optimal Power Flow," *IEEE Trans. Power Syst.*, vol. 31, no. 4, pp. 3008–3018, July 2016.

[30] B. Kocuk, S. S. Dey, and X. A. Sun, "Strong SOCP Relaxations of the Optimal Power Flow Problem," *Operations Research*, vol. 64, no. 6, pp. 1177–1196, 2016.

[31] D. K. Molzahn and I. A. Hiskens, "Sparsity-Exploiting Moment-Based Relaxations of the Optimal Power Flow Problem," *IEEE Trans. Power Syst.*, vol. 30, no. 6, pp. 3168–3180, Nov. 2015.

[32] E. Dall'Anese, H. Zhu, and G. B. Giannakis, "Distributed Optimal Power Flow for Smart Microgrids," *IEEE Trans. Smart Grid*, vol. 4, no. 3, pp. 1464–1475, Sept. 2013.

[33] L. Gan and S. H. Low, "Convex Relaxations and Linear Approximation for Optimal Power Flow in Multiphase Radial Networks," *18th Power Syst. Comput. Conf. (PSCC)*, Aug. 2014.

[34] B. C. Lesieutre, D. K. Molzahn, A. R. Borden, and C. L. DeMarco, "Examining the Limits of the Application of Semidefinite Programming to Power Flow Problems," in *49th Annu. Allerton Conf. Commun., Control, Comput.*, 28-30 Sept. 2011.

[35] W. A. Bukhsh, A. Grothey, K. McKinnon, and P. A. Trodden, "Local Solutions of the Optimal Power Flow Problem," *IEEE Trans. Power Syst.*, vol. 28, no. 4, pp. 4780–4788, 2013.

[36] R. Louca, P. Seiler, and E. Bitar, "A Rank Minimization Algorithm to Enhance Semidefinite Relaxations of Optimal Power Flow," in *51st Annu. Allerton Conf. Commun., Control, Comput.*, Oct. 2013.

[37] B. Kocuk, S. S. Dey, and X. A. Sun, "Inexactness of SDP Relaxation and Valid Inequalities for Optimal Power Flow," *IEEE Trans. Power Syst.*, vol. 31, no. 1, pp. 642–651, Jan. 2016.

[38] K. Christakou, D. C. Tomozei, J. Y. Le Boudec, and M. Paolone, "AC OPF in Radial Distribution Networks – Part I: On the Limits of the Branch Flow Convexification and the Alternating Direction Method of Multipliers," *arXiv:1503.06809*, July 2016.

[39] D. Bienstock and A. Verma, "Strong NP-hardness of AC Power Flows Feasibility," *arXiv:1512.07315*, Dec. 2015.

[40] K. Lehmann, A. Grastien, and P. Van Hentenryck, "AC-Feasibility on Tree Networks is NP-Hard," *IEEE Trans. Power Syst.*, vol. 31, no. 1, pp. 798–801, Jan. 2016.

[41] S. Bose, D. Gayme, S. H. Low, and K. M. Chandy, "Optimal Power Flow Over Tree Networks," in *49th Annu. Allerton Conf. Commun., Control, Comput.*, Oct. 2011.

[42] S. Bose, D. Gayme, K. M. Chandy, and S. H. Low, "Quadratically Constrained Quadratic Programs on Acyclic Graphs with Application to Power Flow," *IEEE Trans. Control Network Syst.*, vol. 2, no. 3, pp. 278–287, 2015.

[43] B. Zhang and D. Tse, "Geometry of feasible injection region of power networks," in *49th Annu. Allerton Conf. Commun., Control, Comput.*, Sept. 2011, pp. 1508–1515.

[44] ——, "Geometry of Injection Regions of Power Networks," *IEEE Trans. Power Syst.*, vol. 28, no. 2, pp. 788–797, May 2013.

[45] J. Lavaei, D. Tse, and B. Zhang, "Geometry of Power Flows and Optimization in Distribution Networks," *IEEE Trans. Power Syst.*, vol. 29, no. 2, pp. 572–583, Mar. 2014.

[46] B. Zhang, A. Y. Lam, A. D. Domnguez-Garca, and D. Tse, "An Optimal and Distributed Method for Voltage Regulation in Power Distribution Systems," *IEEE Trans. Power Syst.*, vol. 30, no. 4, pp. 1714–1726, July 2015.

[47] S. Sojoudi and J. Lavaei, "Convexification of Optimal Power Flow Problem by Means of Phase Shifters," in *IEEE Int. Conf. Smart Grid Commun. (SmartGridComm)*, Oct. 2013, pp. 756–761.

[48] N. Li, L. Chen, and S. H. Low, "Exact Convex Relaxation of OPF for Radial Networks using Branch Flow Models," in *IEEE International Conf. Smart Grid Commun. (SmartGridComm)*, Nov. 2012.

[49] L. Gan, N. Li, U. Topcu, and S. H. Low, "Optimal Power Flow in Distribution Networks," in *52nd IEEE Conf. Decis. Control (CDC)*, Dec. 2013.

[50] ——, "Exact Convex Relaxation of Optimal Power Flow in Radial Networks," *IEEE Trans. Automatic Control*, vol. 60, no. 1, pp. 72–87, Jan. 2015.

[51] A. Kargarian, J. Mohammadi, J. Guo, S. Chakrabarti, M. Barati, G. Hug, S. Kar, and R. Baldick, "Toward Distributed/Decentralized DC Optimal Power Flow Implementation in Future Electric Power Systems," to appear in *IEEE Trans. Smart Grid*, 2016.

[52] J. Nocedal and S. Wright, *Numerical Optimization*, 2nd ed. Springer, 2000.

[53] E. Castillo, R. Mínguez, A. Conejo, and R. Garcia-Bertrand, *Decomposition Techniques in Mathematical Programming*. Springer Berlin, 2006.

[54] B. H. Kim and R. Baldick, "A Comparison of Distributed Optimal Power Flow Algorithms," *IEEE Trans. Power Syst.*, vol. 15, no. 2, pp. 599–604, May 2000.

[55] M. Arnold, S. Knopfli, and G. Andersson, "Improvement of OPF Decomposition Methods Applied to Multi-Area Power Systems," in *IEEE Lausanne PowerTech*, July 2007, pp. 1308–1313.

[56] D. Gabay and B. Mercier, "A Dual Algorithm for the Solution of Nonlinear Variational Problems via Finite Element Approximation," *Computers & Mathematics with Applications*, vol. 2, no. 1, pp. 17 – 40, 1976.

[57] M.-A. Glowinski, R., "Sur l'approximation, par lments finis d'ordre un, et la rsolution, par pnalisation-dualit d'une classe de problmes de dirichlet non linaires," *ESAIM: Mathematical Modelling and Numerical Analysis - Modlisation Mathmatique et Analyse Numrique*, vol. 9, no. R2, pp. 41–76, 1975.

[58] D. Gabay, "Applications of the Method of Multipliers to Variational Inequalities," in *Augmented Lagrangian Methods: Applications to the Numerical Solution of Boundary-Value Problems*, ser. Studies in Mathematics and Its Applications, M. Fortin and R. Glowinski, Eds. Elsevier, 1983, vol. 15, pp. 299–331.

[59] J. Eckstein and D. P. Bertsekas, "On the Douglas—Rachford Splitting Method and the Proximal Point Algorithm for Maximal Monotone Operators," *Math. Prog.*, vol. 55, no. 1, pp. 293–318, 1992.

[60] J. Douglas and H. H. Rachford, "On the Numerical Solution of Heat Conduction Problems in Two and Three Space Variables," *Trans. American Math. Soc.*, vol. 82, no. 2, pp. 421–439, 1956.

[61] P. L. Lions and B. Mercier, "Splitting algorithms for the sum of two nonlinear operators," *SIAM J. Numerical Analysis*, vol. 16, no. 6, pp. 964–979, 1979.

[62] M. R. Hestenes, "Multiplier and Gradient Methods," *J. Optimiz. Theory Appl.*, vol. 4, no. 5, pp. 303–320, 1969.

[63] M. J. Powell, "A Method for Nonlinear Constraints in Minimization Problems," in *Optimization*, R. Fletcher, Ed. New York: Academic Press, 1969, pp. 283–298.

[64] H. Everett III, "Generalized Lagrange Multiplier Method for Solving Problems of Optimum Allocation of Resources," *Operations Research*, vol. 11, no. 3, pp. 399–417, 1963.

[65] G. B. Dantzig and P. Wolfe, "Decomposition principle for linear programs," *Oper. Res.*, vol. 8, no. 1, pp. 101–111, Feb. 1960.

[66] J. F. Benders, "Partitioning Procedures for Solving Mixed-Variables Programming Problems," *Numerische Mathematik*, vol. 4, no. 1, pp. 238–252, Dec. 1962.

[67] G. B. Dantzig, *Linear Programming and Extensions*, ser. Princeton Landmarks in Mathematics and Physics. Princeton University Press, 1963.

[68] S. Boyd, N. Parikh, E. Chu, B. Peleato, and J. Eckstein, "Distributed Optimization and Statistical Learning via the Alternating Direction Method of Multipliers," *Found. Trends Machine Learning*, vol. 3, no. 1, pp. 1–122, 2011.

[69] A. Teixeira, E. Ghadimi, I. Shames, H. Sandberg, and M. Johansson, "The ADMM Algorithm for Distributed Quadratic Problems: Parameter Selection and Constraint Preconditioning," *IEEE Trans. Signal Proc.*, vol. 64, no. 2, pp. 290–305, Jan. 2016.

[70] M. Kraning, E. Chu, J. Lavaei, and S. Boyd, "Dynamic Network Energy Management via Proximal Message Passing," *Found. Trends Optimiz.*, vol. 1, no. 2, pp. 70–122, 2013.

[71] S. Tosserams, L. F. Etman, P. Y. Papalambros, and J. E. Rooda, "An Augmented Lagrangian Relaxation for Analytical Target Cascading using the Alternating Directions Method of Multipliers," *Struct. Multidisc. Optim.*, vol. 31, no. 3, pp. 176–189, 2006.

[72] S. DorMohammadi and M. Rais-Rohani, "Comparison of Alternative Strategies for Multilevel Optimization of Hierarchial Systems," *Appl. Math.*, vol. 3, no. 10A, pp. 1448–1462, 2012.

[73] G. Cohen, "Auxiliary Problem Principle and Decomposition of Optimization Problems," *J. Optimiz. Theory Appl.*, vol. 32, no. 3, pp. 277–305, Nov. 1980.

[74] A. J. Conejo, F. J. Nogales, and F. J. Prieto, "A Decomposition Procedure based on Approximate Newton Directions," *Math. Program.*, vol. 92, pp. 495–515, 2002.

[75] J. Guo, G. Hug, and O. Tonguz, "Intelligent Partitioning in Distributed Optimization of Electric Power Systems," to appear in *IEEE Trans. Smart Grid*, 2016.

[76] K. Baker, J. Guo, G. Hug, and X. Li, "Distributed MPC for Efficient Coordination of Storage and Renewable Energy Sources Across Control Areas," *IEEE Trans. Smart Grid*, no. 2, pp. 992–1001, Mar. 2016.

[77] S. Kar, G. Hug, J. Mohammadi, and J. M. Moura, "Distributed State Estimation and Energy Management in Smart Grids: A Consensus + Innovations Approach," *IEEE J. Sel. Topics Signal Proc.*, vol. 8, no. 6, pp. 1022–1038, Dec. 2014.

[78] J. Mohammadi, G. Hug, and S. Kar, "Distributed Approach for DC Optimal Power Flow Calculations," *arXiv:1410.4236*, Oct. 2014.

[79] ——, "Role of Communication on the Converence Rate of Fully Distribued DC Optimal Power Flow," *IEEE Int. Conf. Smart Grid Comm.*, pp. 43–48, Nov. 2014.

[80] ——, "Asynchronous Distributed Approach for DC Optimal Power Flow," *IEEE Eindhoven PowerTech*, June 2015.

[81] A. J. Conejo and J. A. Aguado, "Multi-Area Coordinated Decentralized DC Optimal Power Flow," *IEEE Trans. Power Syst.*, vol. 13, no. 4, pp. 1272–1278, Nov. 1998.

[82] C. H. Lin and S. Y. Lin, "Distributed Optimal Power Flow With Discrete Control Variables of Large Distributed Power Systems," *IEEE Trans. Power Syst.*, vol. 23, no. 3, pp. 1383–1392, Aug. 2008.

[83] N. Gatsis and G. B. Giannakis, "Decomposition Algorithms for Market Clearing With Large-Scale Demand Response," *IEEE Trans. Smart Grid*, vol. 4, no. 4, pp. 1976–1987, Dec. 2013.

[84] J. Y. Joo and M. D. Ili'c, "Multi-Layered Optimization Of Demand Resources Using Lagrange Dual Decomposition," *IEEE Trans. Smart Grid*, vol. 4, no. 4, pp. 2081–2088, Dec. 2013.

[85] E. Miehling and D. Teneketzis, "A Decentralized Mechanism for Computing Competitive Equilibria in Deregulated Electricity Markets," in *American Control Conf. (ACC)*, July 2016, pp. 4107–4113.

[86] A. Abboud, R. Couillet, M. Debbah, and H. Siguerdidjane, "Asynchronous Alternating Direction Method of Multipliers Applied to the Direct-Current Optimal Power Flow Problem," in *IEEE Int. Conf. Acoustics, Speech Signal Proc. (ICASSP)*, May 2014, pp. 7764–7768.

[87] S. Chakrabarti, M. Kraning, E. Chu, R. Baldick, and S. Boyd, "Security Constrained Optimal Power Flow via Proximal Message Passing," in *Clemson Univ. Power Syst. Conf. (PSC)*, 2014.

[88] A. Kargarian, Y. Fu, and Z. Li, "Distributed Security-Constrained Unit Commitment for Large-Scale Power Systems," *IEEE Trans. Power Syst.*, vol. 30, no. 4, pp. 1925–1936, July 2015.

[89] A. Kargarian and Y. Fu, "System of Systems Based Security-Constrained Unit Commitment Incorporating Active Distribution Grids," *IEEE Trans. Power Syst.*, vol. 29, no. 5, pp. 2489–2498, Sept. 2014.

[90] K. Chung, B. Kim, and D. Hur, "Distributed Implementation of Generation Scheduling Algorithm on Interconnected Power Systems," *Energy Conversion and Management*, vol. 52, no. 12, pp. 3457 – 3464, 2011.

[91] A. Ahmadi-Khatir, A. J. Conejo, and R. Cherkaoui, "Multi-Area Unit Scheduling and Reserve Allocation Under Wind Power Uncertainty," *IEEE Trans. Power Syst.*, vol. 29, no. 4, pp. 1701–1710, July 2014.

[92] A. G. Bakirtzis and P. N. Biskas, "A Decentralized Solution to the DC-OPF of Interconnected Power Systems," *IEEE Trans. Power Syst.*, vol. 18, no. 3, pp. 1007–1013, Aug. 2003.

[93] P. N. Biskas, A. G. Bakirtzis, N. I. Macheras, and N. K. Pasialis, "A Decentralized Implementation of DC Optimal Power Flow on a Network of Computers," *IEEE Trans. Power Syst.*, vol. 20, no. 1, pp. 25–33, Feb. 2005.

[94] Z. Li, Q. Guo, H. Sun, and J. Wang, "Coordinated Economic Dispatch of Coupled Transmission and Distribution Systems Using Heterogeneous Decomposition," *IEEE Trans. Power Syst.*, vol. 31, no. 6, pp. 4817–4830, Nov. 2016.

[95] X. Lai, L. Xie, Q. Xia, H. Zhong, and C. Kang, "Decentralized Multi-Area Economic Dispatch via Dynamic Multiplier-Based Lagrangian Relaxation," *IEEE Trans. Power Syst.*, vol. 30, no. 6, pp. 3225–3233, Nov. 2015.

[96] Z. Li, Q. Guo, H. Sun, and J. Wang, "A New LMP-Sensitivity-Based Heterogeneous Decomposition for Transmission and Distribution Coordinated Economic Dispatch," to appear in *IEEE Trans. Smart Grid*, 2016.

[97] J. Mohammadi, G. Hug, and S. Kar, "Agent-Based Distributed Security Constrained Optimal Power Flow," to appear in *IEEE Trans. Smart Grid*, 2016.

[98] Z. Wang, B. Chen, J. Wang, and J. Kim, "Decentralized Energy Management System for Networked Microgrids in Grid-Connected and Islanded Modes," *IEEE Trans. Smart Grid*, vol. 7, no. 2, pp. 1097–1105, Mar. 2016.

[99] P. Šulc, S. Backhaus, and M. Chertkov, "Optimal Distributed Control of Reactive Power Via the Alternating Direction Method of Multipliers," *IEEE Trans. Energy Conv.*, vol. 29, no. 4, pp. 968–977, Dec. 2014.

[100] M. Bazrafshan and N. Gatsis, "Decentralized Stochastic Programming for Real and Reactive Power Management in Distribution Systems," in *IEEE Int. Conf. Smart Grid Commun. (SmartGridComm)*, Nov. 2014, pp. 218–223.

[101] W. J. Ma, J. Wang, V. Gupta, and C. Chen, "Distributed Energy Management for Networked Microgrids Using Online Alternating Direction Method of Multipliers with Regret," to appear in *IEEE Trans. Smart Grid*, 2016.

[102] S. S. Guggilam, E. DallAnese, Y. C. Chen, S. V. Dhople, and G. B. Giannakis, "Scalable Optimization Methods for Distribution Networks With High PV Integration," *IEEE Trans. Smart Grid*, vol. 7, no. 4, pp. 2061–2070, July 2016.

[103] B. A. Robbins, H. Zhu, and A. D. Domnguez-Garca, "Optimal Tap Setting of Voltage Regulation Transformers in Unbalanced Distribution Systems," *IEEE Trans. Power Syst.*, vol. 31, no. 1, pp. 256–267, Jan. 2016.

[104] X. Bai and H. Wei, "A Semidefinite Programming Method with Graph Partitioning Technique for Optimal Power Flow Problems," *Int. J. Electrical Power & Energy Syst.*, vol. 33, no. 7, pp. 1309–1314, 2011.

[105] R. A. Jabr, "Exploiting Sparsity in SDP Relaxations of the OPF Problem," *IEEE Trans. Power Syst.*, vol. 27, no. 2, pp. 1138–1139, May 2012.

[106] A. Y. Lam, B. Zhang, and D. N. Tse, "Distributed Algorithms for Optimal Power Flow Problem," in *IEEE 51st Conf. Decis. Control (CDC)*, Dec. 2012, pp. 430–437.

[107] L. Vandenberghe and M. S. Andersen, "Chordal Graphs and Semidefinite Optimization," *Found. Trends Optimiz.*, vol. 1, no. 4, pp. 241–433, 2015.

[108] E. DallAnese, S. V. Dhople, B. B. Johnson, and G. B. Giannakis, "Decentralized Optimal Dispatch of Photovoltaic Inverters in Residential Distribution Systems," *IEEE Trans. Energy Conv.*, vol. 29, no. 4, pp. 957–967, Dec. 2014.

[109] R. Madani, A. Kalbat, and J. Lavaei, "ADMM for Sparse Semidefinite Programming with Applications to Optimal Power Flow Problem," in *54th IEEE Conf. Decis. Control (CDC)*, Dec. 2015, pp. 5932–5939.

[110] Q. Peng and S. H. Low, "Distributed Algorithm for Optimal Power Flow on Unbalanced Multiphase Distribution Networks," *arXiv:1512.06482*, Dec. 2015.

[111] ——, "Distributed Optimal Power Flow Algorithm for Radial Networks, I: Balanced Single Phase Case," to appear in *IEEE Trans. Smart Grid*, pp. 1–11, 2016.

[112] ——, "Distributed Algorithm for Optimal Power Flow on an Unbalanced Radial Network," *IEEE 54th Annu. Conf. Decis. Control (CDC)*, Dec. 2015.

[113] W. Zheng, W. Wu, B. Zhang, H. Sun, and Y. Liu, "A Fully Distributed Reactive Power Optimization and Control Method for Active Distribution Networks," *IEEE Trans. Smart Grid*, vol. 7, no. 2, pp. 1021–1033, Mar. 2016.

[114] J. Aguado and V. Quintana, "Inter-Utilities Power-Exchange Coordination: A Market-Oriented Approach," *IEEE Trans. Power Syst.*, vol. 16, no. 3, pp. 513–519, Aug. 2001.

[115] K. Christakou, D. C. Tomozei, J. Y. Le Boudec, and M. Paolone, "AC OPF in Radial Distribution Networks – Part II: An Augmented Lagrangian-based OPF Algorithm, Distributable via Primal Decomposition," *arXiv:1503.06809*, July 2016.

[116] X. Wang and Y. H. Song, "Apply Lagrangian Relaxation to Multi-Zone Congestion Management," in *IEEE Power Engineering Society Winter Meeting*, vol. 2, 2001, pp. 399–404.

[117] A. X. Sun, D. T. Phan, and S. Ghosh, "Fully Decentralized AC Optimal Power Flow Algorithms," in *IEEE Power Energy Society General Meeting*, July 2013.

[118] T. Erseghe, "Distributed Optimal Power Flow Using ADMM," *IEEE Trans. Power Syst.*, vol. 29, no. 5, pp. 2370–2380, Sept. 2014.

[119] ——, "A Distributed Approach to the OPF Problem," *EURASIP J. Advances Signal Proc.*, vol. 45, 2015.

[120] S. Mhanna, G. Verbic, and A. Chapman, "A Component-Based Dual Decomposition Method for the OPF Problem," *arXiv:1704.03647*, Apr. 2017.

[121] R. D. Zimmerman, C. E. Murillo-Sánchez, and R. J. Thomas, "MATPOWER: Steady-State Operations, Planning, and Analysis Tools for Power Systems Research and Education," *IEEE Trans. Power Syst.*, vol. 26, no. 1, pp. 12–19, 2011.

[122] S. Magnússon, P. C. Weeraddana, and C. Fischione, "A Distributed Approach for the Optimal Power-Flow Problem Based on ADMM and Sequential Convex Approximations," *IEEE Trans. Control Network Syst.*, vol. 2, no. 3, pp. 238–253, Sept. 2015.

[123] A. K. Marvasti, Y. Fu, S. DorMohammadi, and M. Rais-Rohani, "Optimal Operation of Active Distribution Grids: A System of Systems Framework," *IEEE Trans. Smart Grid*, vol. 5, no. 3, pp. 1228–1237, May 2014.

[124] S. DorMohammadi and M. Rais-Rohani, "Exponential Penalty Function Formulation for Multilevel Optimization using the Analytical Target Cascading Framework," *Struct. Multidisc Optim.*, vol. 47, no. 11, pp. 559–612, 2013.

[125] B. H. Kim and R. Baldick, "Coarse-Grained Distributed Optimal Power Flow," *IEEE Trans. Power Syst.*, vol. 12, no. 2, pp. 932–939, 1997.

[126] R. Baldick, B. H. Kim, C. Chase, and Y. Luo, "A Fast Distributed Implementation of Optimal Power Flow," *IEEE Trans. Power Syst.*, vol. 14, no. 3, pp. 858–864, 1999.

[127] D. Hur, J. K. Park, and B. H. Kim, "Evaluation of Convergence Rate in the Auxiliary Problem Principle for Distributed Optimal Power Flow," *IEE Proc.–Generation, Transmission and Distribution*, vol. 149, no. 5, pp. 525–532, Sept. 2002.

[128] F. J. Nogales, F. J. Prieto, and A. J. Conejo, "A Decomposition Methodology Applied to the Multi-Area Optimal Power Flow Problem," *Annals of Operations Research*, vol. 120, no. 1, pp. 99–116, 2003.

[129] P. N. Biskas and A. G. Bakirtzis, "Decentralised OPF of Large Multiarea Power Systems," *IEE Proc.–Generation, Transmission and Distribution*, vol. 153, no. 1, pp. 99–105, Jan. 2006.

[130] G. Hug-Glanzmann and G. Andersson, "Decentralized Optimal Power Flow Control for Overlapping Areas in Power Systems," *IEEE Trans. Power Syst.*, vol. 24, no. 1, pp. 327–336, Feb. 2009.

[131] J. Guo, G. Hug, and O. K. Tonguz, "A Case for Non-Convex Distributed Optimization in Large-Scale Power Systems," to appear in *IEEE Trans. Power Syst.*, 2016.

[132] K. Arrow, L. Hurwicz, and H. Uzawa, *Studies in Linear and Non-Linear Programming*. Stanford University Press, 1958.

[133] D. Feijer and F. Paganini, "Stability of Primal-Dual Gradient Dynamics and Applications to Network Optimization," *Automatica*, vol. 46, no. 12, pp. 1974–1981, Dec. 2010.

[134] J. Wang and N. Elia, "A Control Perspective for Centralized and Distributed Convex Optimization," in *IEEE 50th Annu. Conf. Decis. Control (CDC)*, Dec. 2011, pp. 3800–3805.

[135] X. Ma and N. Elia, "A Distributed Continuous-Time Gradient Dynamics Approach for the Active Power Loss Minimizations," in *51st Annu. Allerton Conf. Commun., Control, Comput.*, Oct. 2013, pp. 100–106.

[136] ——, "Convergence Analysis for the Primal-Dual Gradient Dynamics Associated with Optimal Power Flow Problems," in *European Control Conf. (ECC)*, July 2015, pp. 1261–1266.

[137] ——, "Existence of Strong Lagrange Duals to Certain Optimal Power Flows," in *22nd Mediterranean Conf. Control Automation (MED)*, June 2014, pp. 640–645.

[138] ——, "A Sufficient Saddle Point Characterization for the Lagrangian Associated with General OPF Problems," in *53rd IEEE Conf. Decis. Control (CDC)*, Dec. 2014, pp. 1119–1124.

[139] K. Dvijotham, M. Chertkov, P. Van Hentenryck, M. Vuffray, and S. Misra, "Graphical Models for Optimal Power Flow," *Constraints*, vol. 22, no. 1, pp. 24–49, 2017.

[140] M. Chertkov, S. Misra, M. Vuffray, D. Krishnamurty, and P. Van Hentenryck, "Graphical Models and Belief Propagation-hierarchy for Optimal Physics-Constrained Network Flows," *arXiv:1702.01890*, Feb. 2017.

[141] D. D. Siljak, *Decentralized Control of Complex Systems*. Academic Press, 1994.

[142] L. Bakule, "Decentralized Control: An Overview," *Annual Reviews in Control*, vol. 32, no. 1, pp. 87 – 98, 2008.

[143] S.-H. Wang and E. J. Davison, "On the Stabilization of Decentralized Control Systems," *IEEE Trans. Automatic Control*, vol. AC-18, no. 5, pp. 473–478, 1973.

[144] N. Sandell Jr, P. Varaiya, M. Athans, and M. Safonov, "Survey of Decentralized Control Methods for Large Scale Systems," *IEEE Trans. Automatic Control*, vol. 23, no. 2, pp. 108–128, 1978.

[145] H. Witsenhausen, "A Counterexample in Stochastic Optimum Control," *SIAM J. Control*, vol. 6, no. 1, pp. 131–147, 1968.

[146] C. Papadimitriou and J. Tsitsiklis, "Intractable Problems in Control Theory," *SIAM J. Control Optimiz.*, vol. 24, no. 4, pp. 639–654, 1986.

[147] V. D. Blondel and J. N. Tsitsiklis, "A Survey of Computational Complexity Results in Systems and Control," *Automatica*, vol. 36, no. 9, pp. 1249 – 1274, 2000.

[148] R. D'Andrea and G. Dullerud, "Distributed Control Design for Spatially Interconnected Systems," *IEEE Trans. Automatic Control*, vol. 48, no. 9, pp. 1478–1495, 2003.

[149] B. Bamieh, F. Paganini, and M. A. Dahleh, "Distributed Control of Spatially Invariant Systems," *IEEE Trans. Automatic Control*, vol. 47, no. 7, pp. 1091–1107, 2002.

[150] C. Langbort, R. Chandra, and R. D'Andrea, "Distributed Control Design for Systems Interconnected over an Arbitrary Graph," *IEEE Trans. Automatic Control*, vol. 49, no. 9, pp. 1502–1519, 2004.

[151] N. Motee and A. Jadbabaie, "Optimal Control of Spatially Distributed Systems," *IEEE Trans. Automatic Control*, vol. 53, no. 7, pp. 1616–1629, 2008.

[152] G. Dullerud and R. D'Andrea, "Distributed Control of Heterogeneous Systems," *IEEE Trans. Automatic Control*, vol. 49, no. 12, pp. 2113–2128, 2004.

[153] T. Keviczky, F. Borrelli, and G. J. Balas, "Decentralized receding horizon control for large scale dynamically decoupled systems," *Automatica*, vol. 42, no. 12, pp. 2105–2115, 2006.

[154] F. Borrelli and T. Keviczky, "Distributed LQR Design for Identical Dynamically Decoupled Systems," *IEEE Trans. Automatic Control*, vol. 53, no. 8, pp. 1901–1912, 2008.

[155] D. D. Siljak, "Decentralized Control and Computations: Status and Prospects," *Annual Reviews in Control*, vol. 20, pp. 131–141, 1996.

[156] J. Lavaei, "Decentralized Implementation of Centralized Controllers for Interconnected Systems," *IEEE Trans. Automatic Control*, vol. 57, no. 7, pp. 1860–1865, 2012.

[157] M. Fardad, F. Lin, and M. R. Jovanovic, "On the Optimal Design of Structured Feedback Gains for Interconnected Systems," in *48th IEEE Conf. Decis. Control*, 2009, pp. 978–983.

[158] F. Lin, M. Fardad, and M. R. Jovanovic, "Augmented lagrangian approach to design of structured optimal state feedback gains," *IEEE Transactions on Automatic Control*, vol. 56, no. 12, pp. 2923–2929, 2011.

[159] J. Geromel, J. Bernussou, and P. Peres, "Decentralized Control through Parameter Space Optimization," *Automatica*, vol. 30, no. 10, pp. 1565 – 1578, 1994.

[160] R. A. Date and J. H. Chow, "A Parametrization Approach to Optimal H_2 and H_∞ Decentralized Control Problems," *Automatica*, vol. 29, no. 2, pp. 457 – 463, 1993.

[161] G. Scorletti and G. Duc, "An LMI Approach to Decentralized H_∞ Control," *Int. J. Control*, vol. 74, no. 3, pp. 211–224, 2001.

[162] G. Zhai, M. Ikeda, and Y. Fujisaki, "Decentralized H_∞ Controller Design: A Matrix Inequality Approach using a Homotopy Method," *Automatica*, vol. 37, no. 4, pp. 565 – 572, 2001.

[163] G. A. de Castro and F. Paganini, "Convex Synthesis of Localized Controllers for Spatially Invariant Systems," *Automatica*, vol. 38, no. 3, pp. 445 – 456, 2002.

[164] B. Bamieh and P. G. Voulgaris, "A Convex Characterization of Distributed Control Problems in Spatially Invariant Systems with Communication Constraints," *Systems & Control Letters*, vol. 54, no. 6, pp. 575 – 583, 2005.

[165] X. Qi, M. Salapaka, P. Voulgaris, and M. Khammash, "Structured Optimal and Robust Control with Multiple Criteria: A Convex Solution," *IEEE Trans. Automatic Control*, vol. 49, no. 10, pp. 1623–1640, 2004.

[166] N. Matni and J. C. Doyle, "A Dual Problem in H_2 Decentralized Control Subject to Delays," in *American Control Conf.*, 2013.

[167] L. Lessard, "A Separation Principle for Decentralized State-Feedback Optimal Control," in *51st Annual Allerton Conf. Commun., Control, Comput.*, 2013.

[168] L. Lessard and A. Nayyar, "Structural Results and Explicit Solution for Two-Player LQG Systems on a Finite Time Horizon," in *52nd IEEE Conf. Decis. Control*, 2013.

[169] M. Fardad and M. R. Jovanovic, "On the Design of Optimal Structured and Sparse Feedback Gains via Sequential Convex Programming," in *American Control Conf.*, 2014.

[170] F. Lin, M. Fardad, and M. R. Jovanović, "Design of optimal sparse feedback gains via the alternating direction method of multipliers," *IEEE Transactions on Automatic Control*, vol. 58, no. 9, pp. 2426–2431, 2013.

[171] M. Rotkowitz and S. Lall, "A Characterization of Convex Problems in Decentralized Control," *IEEE Trans. Automatic Control*, vol. 51, no. 2, pp. 274–286, 2006.

[172] P. Shah and P. A. Parrilo, "H_2-Optimal Decentralized Control over Posets: A State-Space Solution for State-Feedback," *IEEE Trans. Automatic Control*, vol. 58, no. 12, pp. 3084–3096, Dec. 2013.

[173] L. Lessard and S. Lall, "Optimal Controller Synthesis for the Decentralized Two-Player Problem with Output Feedback," *American Control Conf.*, 2012.

[174] A. Lamperski and J. C. Doyle, "Output Feedback H_2 Model Matching for Decentralized Systems with Delays," *American Control Conf.*, 2013.

[175] M. Rotkowitz and N. Martins, "On the Nearest Quadratically Invariant Information Constraint," *IEEE Trans. Automatic Control*, vol. 57, no. 5, pp. 1314–1319, 2012.

[176] T. Tanaka and C. Langbort, "The Bounded Real Lemma for Internally Positive Systems and H-infinity Structured Static State Feedback," *IEEE Trans. Automatic Control*, vol. 56, no. 9, pp. 2218–2223, 2011.

[177] A. Rantzer, "Distributed Control of Positive Systems," in *50th IEEE Conf. Decis. Control, European Control Conf. (CDC-ECC)*. IEEE, 2011, pp. 6608–6611.

[178] J. Lavaei, "Optimal Decentralized Control Problem as a Rank-Constrained Optimization," in *51st Annual Allerton Conf. Commun., Control, Comput. (Allerton)*, Oct. 2013, pp. 39–45.

[179] G. Fazelnia, R. Madani, A. Kalbat, and J. Lavaei, "Convex Relaxation for Optimal Distributed Control Problems," *IEEE Trans. Automatic Control*, vol. 62, no. 1, pp. 206–221, 2017.

[180] K. Dvijotham, E. Todorov, and M. Fazel, "Convex structured controller design in finite horizon," *IEEE Transactions on Control of Network Systems*, vol. 2, no. 1, pp. 1–10, 2015.

[181] S. Schuler, P. Li, J. Lam, and F. Allgöwer, "Design of structured dynamic output-feedback controllers for interconnected systems," *International Journal of Control*, vol. 84, no. 12, pp. 2081–2091, 2011.

[182] M. R. Jovanović and N. K. Dhingra, "Controller architectures: Tradeoffs between performance and structure," *European Journal of Control*, vol. 30, pp. 76–91, 2016.

[183] L. Gan and S. H. Low, "An Online Gradient Algorithm for Optimal Power Flow on Radial Networks," *IEEE J. Sel. Areas Commun.*, vol. 34, no. 3, pp. 625–638, 2016.

[184] D. B. Arnold, M. Negrete-Pincetic, M. D. Sankur, D. M. Auslander, and D. S. Callaway, "Model-Free Optimal Control of VAR Resources in Distribution Systems: An Extremum Seeking Approach," *IEEE Trans. Power Syst.*, vol. 31, no. 5, pp. 3583–3593, Sept. 2016.

[185] E. Dall'Anese, S. V. Dhople, and G. B. Giannakis, "Photovoltaic Inverter Controllers Seeking AC Optimal Power Flow Solutions," *IEEE Trans. Power Systems*, vol. 31, no. 4, pp. 2809–2823, 2016.

[186] A. Hauswirth, S. Bolognani, G. Hug, and F. Dörfler, "Projected Gradient Descent on Riemannian Manifolds with Applications to Online Power System Optimization," in *54th Annu. Allerton Conf. Commun., Control, Comput.*, Sept. 2016.

[187] A. Hauswirth, A. Zanardi, S. Bolognani, F. Drfler, and G. Hug, "Online optimization in closed loop on the power flow manifold," in *IEEE PES PowerTech*, 2017, to appear.

[188] E. Dall'Anese and A. Simonetto, "Optimal Power Flow Pursuit," to appear in *IEEE Trans. Smart Grid*, 2016.

[189] Y. Tang, K. Dvijotham, and S. H. Low, "Online OPF with Quasi-Newton Algorithm," Nov. 2016, INFORMS Annual Meeting.

[190] G. Wang, V. Kekatos, A. J. Conejo, and G. B. Giannakis, "Ergodic Energy Management Leveraging Resource Variability in Distribution Grids," *IEEE Trans. Power Syst.*, vol. 31, no. 6, pp. 4765–4775, 2016.

[191] M. Farivar, L. Chen, and S. H. Low, "Equilibrium and Dynamics of Local Voltage Control in Distribution Systems," in *52nd IEEE Conf. Decis. Control*, Dec. 2013, pp. 4329–4334.

[192] M. Farivar, X. Zho, and L. Chen, "Local Voltage Control in Distribution Systems: An Incremental Control Algorithm," in *IEEE Int. Conf. Smart Grid Commun. (SmartGridComm)*, Nov. 2015, pp. 732–737.

[193] S. Bolognani and S. Zampieri, "A Distributed Control Strategy for Reactive Power Compensation in Smart Microgrids," *IEEE Trans. Automatic Control*, vol. 58, no. 11, pp. 2818–2833, 2013.

[194] S. Bolognani, R. Carli, G. Cavraro, and S. Zampieri, "Distributed Reactive Power Feedback Control for Voltage Regulation and Loss Minimization," *IEEE Trans. Automatic Control*, vol. 60, no. 4, pp. 966–981, 2015.

[195] J. Machowski, J. Bialek, and J. R. Bumby, *Power System Dynamics and Stability*. Wiley, 1997.

[196] N. Ainsworth and S. Grijalva, "Design and Quasi-Equilibrium Analysis of a Distributed Frequency-Restoration Controller for Inverter-Based Microgrids," in *North American Power Symp. (NAPS)*, 2013, pp. 1–6.

[197] M. Andreasson, D. V. Dimarogonas, H. Sandberg, and K. H. Johansson, "Distributed Control of Networked Dynamical Systems: Static Feedback, Integral Action and Consensus," *IEEE Trans. Automatic Control*, vol. 59, no. 7, pp. 1750–1764, 2014.

[198] C. Zhao, E. Mallada, and F. Dörfler, "Distributed Frequency Control for Stability and Economic Dispatch in Power Networks," in *American Control Conf. (ACC)*, July 2015, pp. 2359–2364.

[199] M. Andreasson, D. V. Dimarogonas, H. Sandberg, and K. H. Johansson, "Distributed PI-Control with Applications to Power Systems Frequency Control," in *American Control Conf. (ACC)*, 2014, pp. 3183–3188.

[200] F. Dörfler and S. Grammatico, "Gather-and-Broadcast Frequency Control in Power Systems," *Automatica*, 2016, to appear. Available at *arXiv:1605.09560*.

[201] J. Schiffer, C. A. Hans, T. Kral, R. Ortega, and J. Raisch, "Modelling, Analysis and Experimental Validation of Clock Drift Effects in Low-Inertia Power Systems," *IEEE Trans. Industrial Electronics*, 2016.

[202] J. W. Simpson-Porco, F. Dörfler, and F. Bullo, "Synchronization and Power Sharing for Droop-Controlled Inverters in Islanded Microgrids," *Automatica*, vol. 49, no. 9, pp. 2603–2611, 2013.

[203] L.-Y. Lu, "Consensus-Based $P - f$ and $Q - \dot{V}$ Droop Control for Multiple Parallel-Connected Inverters in Lossy Networks," in *IEEE Int. Symp. Industrial Electronics*, May 2013.

[204] D. Burbano and M. di Bernardo, "Consensus and Synchronization of Complex Networks via Proportional-Integral Coupling," in *IEEE Int. Symp. Circu. Syst. (ISCAS)*, 2014, pp. 1796–1799.

[205] M. Andreasson, D. V. Dimarogonas, K. H. Johansson, and H. Sandberg, "Distributed vs. Centralized Power Systems Frequency Control," in *12th European Control Conf. (ECC)*, July 17-19 2013, pp. 3524–3529.

[206] F. Dörfler, J. W. Simpson-Porco, and F. Bullo, "Breaking the Hierarchy: Distributed Control & Economic Optimality in Microgrids," *IEEE Trans. Control Network Syst.*, 2016, in press.

[207] J. W. Simpson-Porco, S. Q., F. Dörfler, J. M. Vasquez, J. M. Guerrero, and F. Bullo, "Secondary Frequency and Voltage Control of Islanded Microgrids via Distributed Averaging," *IEEE Trans. Industrial Electronics*, vol. 62, no. 11, pp. 7025–7038, Nov. 2015.

[208] R. Mudumbai and S. Dasgupta, "Distributed Control for the Smart Grid: The Case of Economic Dispatch," in *Infor. Theory Appl. Workshop (ITA)*. IEEE, 2014, pp. 1–6.

[209] S. T. Cady, A. D. Domínguez-García, and C. N. Hadjicostis, "A Distributed Generation Control Architecture for Islanded AC Microgrids," *IEEE Trans. Control Syst. Technology*, vol. 23, no. 5, pp. 1717–1735, 2015.

[210] A. D. Dominguez-Garcia and C. N. Hadjicostis, "Coordination and Control of Distributed Energy Resources for Provision of Ancillary Services," in *First IEEE Int. Conf. Smart Grid Commun. (SmartGridComm)*, Oct. 2010, pp. 537–542.

[211] H. Bouattour, J. W. Simpson-Porco, F. Dörfler, and F. Bullo, "Further Results on Distributed Secondary Control in Microgrids," in *IEEE Conf. Decis. Control*, Dec. 2013, pp. 1514–1519.

[212] Q. Shafiee, J. M. Guerrero, and J. Vasquez, "Distributed Secondary Control for Islanded MicroGrids – A Novel Approach," *IEEE Trans. Power Electronics*, vol. 29, no. 2, pp. 1018–1031, 2014.

[213] B. Bamieh and D. F. Gayme, "The Price of Synchrony: Resistive Losses due to Phase Synchronization in Power Networks," in *American Control Conf. (ACC)*, June 2013, pp. 5815–5820.

[214] M. Siami and N. Motee, "Fundamental Limits on Robustness Measures in Networks of Interconnected Systems," in *52nd IEEE Conf. Decis. Control (CDC)*, Dec. 2013, pp. 67–72.

[215] E. Sjödin and D. F. Gayme, "Transient Losses in Synchronizing Renewable Energy Integrated Power Networks," in *American Control Conf. (ACC)*, June 2014, pp. 5217–5223.

[216] E. Tegling, B. Bamieh, and D. F. Gayme, "The Price of Synchrony: Evaluating the Resistive Losses in Synchronizing Power Networks," *IEEE Trans. Control Network Syst.*, vol. 2, no. 3, pp. 254–266, Sept. 2015.

[217] X. Wu, F. Dörfler, and M. R. Jovanović, "Topology Identification and Design of Distributed Integral Action in Power Networks," in *American Control Conference (ACC)*. IEEE, 2016, pp. 5921–5926.

[218] E. Tegling, M. Andreasson, J. W. Simpson-Porco, and H. Sandberg, "Improving Performance of Droop-Controlled Microgrids through Distributed PI-Control," in *American Control Conf. (ACC)*, July 2016, pp. 2321–2327.

[219] E. Tegling, "On Performance Limitations of Large-Scale Networks with Distributed Feedback Control," Licentiate Thesis, KTH Royal Institute of Technology, 2016.

[220] T. Stegink, C. De Persis, and A. van der Schaft, "A Unifying Energy-Based Approach to Optimal Frequency and Market Regulation in Power Grids," *arXiv:1510.05420*, 2015.

[221] S. Trip, M. Bürger, and C. De Persis, "An Internal Model Approach to (Optimal) Frequency Regulation in Power Grids with Time-Varying Voltages," *Automatica*, vol. 64, pp. 240–253, 2016.

[222] J. Schiffer, F. Dörfler, and E. Fridmann, "Robustness of distributed averaging control in power systems: Time delays & dynamic communication topology," *Automatica*, 2016, In press. Available at *arXiv:1607.07743*.

[223] N. Monshizadeh and C. De Persis, "Output Agreement in Networks with Unmatched Disturbances and Algebraic Constraints," *arXiv:1504.03609*, 2015, abridged version in *54th IEEE Conf. Decis. Control (CDC)*.

[224] C. De Persis, N. Monshizadeh, J. Schiffer, and F. Dörfler, "A Lyapunov Approach to Control of Microgrids with a Network-Preserved Differential-Algebraic Model," in *IEEE 55th Conf. Decis. Control (CDC)*, Dec. 2016, pp. 2595–2600.

[225] T. Stegink, C. De Persis, and A. van der Schaft, "Optimal Power Dispatch in Networks of High-Dimensional Models of Synchronous Machines," *arXiv:1603.06688*, 2016.

[226] T. Kose, "Solutions of Saddle Value Problems by Differential Equations," *Econometrica, J. Econometric Soc.*, pp. 59–70, 1956.

[227] K. J. Arrow, L. Hurwicz, H. Uzawa, and H. B. Chenery, "Studies in Linear and Non-Linear Programming," 1958.

[228] G. W. Brown and J. Von Neumann, "Solutions of Games by Differential Equations," DTIC Document, Tech. Rep., 1950.

[229] A. Jokić, M. Lazar, and P. P. Van den Bosch, "Real-Time Control of Power Systems using Nodal Prices," *Int. J. Electrical Power & Energy Syst.*, vol. 31, no. 9, pp. 522–530, 2009.

[230] A. K. Bejestani, A. Annaswamy, and T. Samad, "A Hierarchical Transactive Control Architecture for Renewables Integration in Smart Grids: Analytical Modeling and Stability," *IEEE Trans. Smart Grid*, vol. 5, no. 4, pp. 2054–2065, 2014.

[231] D. Cai, E. Mallada, and A. Wierman, "Distributed Optimization Decomposition for Joint Economic Dispatch and Frequency Regulation," in *54th IEEE Conf. Decis. Control (CDC)*, Dec. 2015, pp. 15–22.

[232] C. Zhao, U. Topcu, N. Li, and S. H. Low, "Power System Dynamics as Primal-Dual Algorithm for Optimal Load Control," *arXiv:1305.0585*, 2013.

[233] ——, "Design and Stability of Load-Side Primary Frequency Control in Power Systems," *IEEE Trans. Automatic Control*, vol. 59, no. 5, pp. 1177–1189, 2014.

[234] E. Mallada, C. Zhao, and S. H. Low, "Optimal load-side control for frequency regulation in smart grids," *IEEE Trans. on Automatic Control, to appear*, 2017.

[235] N. Li, L. Chen, C. Zhao, and S. H. Low, "Connecting Automatic Generation Control and Economic Dispatch from an Optimization View," in *American Control Conf.*, 2014, pp. 735–740.

[236] Z. Wang, F. Liu, S. H. Low, C. Zhao, and S. Mei, "Distributed frequency control with operational constraints, part I: Per-node power balance," 2017, submitted for publication.

[237] ——, "Distributed frequency control with operational constraints, part II: Network power balance," 2017, submitted for publication.

[238] S. You and L. Chen, "Reverse and Forward Engineering of Frequency Control in Power Networks," in *53rd IEEE Conf. Decis. Control (CDC)*, Dec. 2014, pp. 191–198.

[239] E. Mallada, C. Zhao, and S. H. Low, "Fair Load-Side Control for Frequency Regulation in Smart Grids," in *52nd Annu. Allerton Conf. Commun., Control, Comput.*, Sept. 2014.

[240] C. Zhao, E. Mallada, S. H. Low, and J. Bialek, "A Unified Framework for Frequency Control and Congestion Management," in *Power Syst. Comput. Conf. (PSCC)*. IEEE, 2016, pp. 1–7.

[241] X. Zhang and A. Papachristodoulou, "A Real-Time Control Framework for Smart Power Networks with Star Topology," in *American Control Conf. (ACC)*. IEEE, 2013, pp. 5062–5067.

[242] ——, "Distributed Dynamic Feedback Control for Smart Power Networks with Tree Topology," in *American Control Conf. (ACC)*. IEEE, 2014, pp. 1156–1161.

[243] K. Turitsyn, P. Sulc, S. Backhaus, and M. Chertkov, "Options for Control of Reactive Power by Distributed Photovoltaic Generators," *Proceedings of the IEEE*, vol. 99, no. 6, pp. 1063–1073, June 2011.

[244] H. G. Yeh, D. F. Gayme, and S. H. Low, "Adaptive VAR Control for Distribution Circuits with Photovoltaic Generators," *IEEE Trans. Power Syst.*, vol. 27, no. 3, pp. 1656–1663, 2012.

[245] C. De Persis and N. Monshizadeh, "Bregman Storage Functions for Microgrid Control with Power Sharing," *arXiv:1510.05811*, 2015, abridged version in *European Control Conference*, 2016.

[246] M. Todescato, J. W. Simpson-Porco, F. Dörfler, R. Carli, and F. Bullo, "Voltage Stress Minimization by Optimal Reactive Power Control," *IEEE Trans. Control Network Syst.*, Feb. 2016, Submitted. Available at *arXiv:1602.01969*.

[247] P. N. Vovos, A. E. Kiprakis, A. R. Wallace, and G. P. Harrison, "Centralized and Distributed Voltage Control: Impact on Distributed Generation Penetration," *IEEE Trans. Power Syst.*, vol. 22, no. 1, pp. 476–483, 2007.

[248] F. Olivier, P. Aristidou, D. Ernst, and T. Van Cutsem, "Active Management of Low-Voltage Networks for Mitigating Overvoltages due to Photovoltaic Units," *IEEE Trans. Smart Grid*, vol. 7, no. 2, pp. 926–936, 2016.

[249] G. Cavraro and R. Carli, "Local and Distributed Voltage Control Algorithms in Distribution Network." [Online]. Available: http://automatica.dei.unipd.it/tl_files/utenti2/cavraro/Pubblications/Local%20and%20distributed%20voltage%20control%20algorithms%20in%20distribution.pdf

[250] G. Cavraro, S. Bolognani, R. Carli, and S. Zampieri, "The Value of Communication in the Voltage Regulation Problem," in *55th IEEE Conf. Decis. Control*, Dec. 2016.

[251] J. W. Simpson-Porco, F. Dörfler, and F. Bullo, "Voltage Stabilization in Microgrids via Quadratic Droop Control," *IEEE Trans. Automatic Control*, 2015, In press. Available at *arXiv:1507.00431*.

[252] S. Kundu and I. A. Hiskens, "Distributed Control of Reactive Power from Photovoltaic Inverters," in *IEEE Int. Symp. Circu. Syst. (ISCAS)*, May 2013, pp. 249–252.

[253] DKE, "VDE-AR-N 4105: Generators connected to the LV distribution network - technical requirements for the connection to and parallel operation with low-voltage distribution networks," Tech. Rep., 2011.

[254] ENTSO-E Standard Draft, "Draft network code for requirements for grid connection applicable to all generators," Tech. Rep., 2012.

[255] A. Samadi, R. Eriksson, L. Söder, B. G. Rawn, and J. C. Boemer, "Coordinated Active Power-Dependent Voltage Regulation in Distribution Grids With PV Systems," *IEEE Trans. Power Del.*, vol. 29, no. 3, June 2014.

[256] H. Zhu and H. J. Liu, "Fast Local Voltage Control Under Limited Reactive Power: Optimality and Stability Analysis," *IEEE Trans. Power Syst.*, vol. 31, no. 5, pp. 3794–3803, Sept. 2016.

[257] H. J. Liu, W. Shi, and H. Zhu, "Decentralized Dynamic Optimization for Power Network Voltage Control," *IEEE Trans. Signal Information Processing Over Networks*, 2016.

[258] X. Zhou, M. Farivar, and L. Chen, "Pseudo-Gradient based Local Voltage Control in Distribution Networks," in *53rd Annu. Allerton Conf. Commun., Control, Comput.* IEEE, Sept. 2015, pp. 173–180.

[259] N. Li, G. Qu, and M. Dahleh, "Real-Time Decentralized Voltage Control in Distribution Networks," in *52nd Annual Allerton Conf. Commun., Control, Comput.* IEEE, 2014, pp. 582–588.

[260] P. Nahata, S. Mastellone, and F. Dörfler, "Decentralized Optimal Projected Control of PV Inverters in Residential Microgrids," in *IFAC World Congress*, Nov. 2017, submitted.

[261] V. Kekatos, L. Zhang, G. B. Giannakis, and R. Baldick, "Voltage Regulation Algorithms for Multiphase Power Distribution Grids," *IEEE Trans. Power Syst.*, vol. 31, no. 5, pp. 3913–3923, Sept. 2016.

[262] X. Zhou and L. Chen, "An Incremental Local Algorithm for Better Voltage Control in Distribution Networks," in *IEEE 55th Conf. Decis. Control (CDC)*, Dec. 2016, pp. 2396–2402.

[263] B. Zhang, A. D. Domínguez-García, and D. Tse, "A Local Control Approach to Voltage Regulation in Distribution Networks," in *North American Power Symp. (NAPS)*, 2013, pp. 1–6.

[264] S. Bolognani, A. Carron, A. Di Vittorio, D. Romeres, and L. Schenato, "Distributed Multi-Hop Reactive Power Compensation in Smart Micro-Grids subject to Saturation Constraints," in *IEEE 51st IEEE Conf. Decis. Control (CDC)*, Dec. 2012, pp. 1118–1123.

[265] J. Schiffer, T. Seel, J. Raisch, and T. Sezi, "Voltage Stability and Reactive Power Sharing in Inverter-Based Microgrids with Consensus-Based Distributed Voltage Control," *IEEE Trans. Control Syst. Technology*, vol. 24, no. 1, pp. 96–109, 2016.

[266] B. A. Robbins and A. D. Domínguez-García, "Optimal Reactive Power Dispatch for Voltage Regulation in Unbalanced Distribution Systems," *IEEE Trans. Power Syst.*, vol. 31, no. 4, pp. 2903–2913, 2016.

[267] K. Christakou, D. C. Tomozei, M. Bahramipanah, J. Y. Le Boudec, and M. Paolone, "Primary Voltage Control in Active Distribution Networks via Broadcast Signals: The Case of Distributed Storage," *IEEE Trans. Smart Grid*, vol. 5, no. 5, pp. 2314–2325, 2014.

[268] A. Bernstein, L. Reyes-Chamorro, J. Y. Le Boudec, and M. Paolone, "A Composable Method for Real-Time Control of Active Distribution Networks with Explicit Power Setpoints. Part I: Framework," *Electric Power Syst. Res.*, vol. 125, pp. 254–264, 2015.

[269] J. H. Chow, J. J. Sanchez-Gasca, H. Ren, and S. Wang, "Power system damping controller design-using multiple input signals," vol. 20, no. 4, pp. 82–90, 2000.

[270] R. A. Jabr, B. C. Pal, N. Martins, and J. C. R. Ferraz, "Robust and coordinated tuning of power system stabiliser gains using sequential linear programming," *IET Generation, Transmission & Distribution*, vol. 4, no. 8, pp. 893–904, 2010.

[271] A. Chakrabortty and P. P. Khargonekar, "Introduction to wide-area control of power systems," in *American Control Conference (ACC), 2013*. IEEE, 2013, pp. 6758–6770.

[272] K. Seethalekshmi, S. N. Singh, and S. C. Srivastava, "Wide-area protection and control: Present status and key challenges," in *Fifteenth National Power Systems Conference*, Bombay, India, Dec. 2008, pp. 169–175.

[273] J. Xiao, F. Wen, C. Y. Chung, and K. P. Wong, "Wide-area protection and its applications-a bibliographical survey," Atlanta, GA, USA, Oct. 2006, pp. 1388–1397.

[274] Y. Chompoobutrgool, L. Vanfretti, and M. Ghandhari, "Survey on power system stabilizers control and their prospective applications for power system damping using synchrophasor-based wide-area systems," *European Transactions on Electrical Power*, vol. 21, no. 8, pp. 2098–2111, 2011.

[275] M. Amin, "Special issue on energy infrastructure defense systems," vol. 93, no. 5, pp. 855–860, 2005.

[276] H. Nguyen-Duc, L. Dessaint, A. F. Okou, and I. Kamwa, "Selection of input/output signals for wide area control loops," Minneapolis, MN, USA, Jul. 2010, pp. 1–7.

[277] W. Qiu, V. Vittal, and M. Khammash, "Decentralized power system stabilizer design using linear parameter varying approach," *IEEE Transactions on Power Systems*, vol. 19, no. 4, pp. 1951–1960, 2004.

[278] Q. Liu, V. Vittal, and N. Elia, "LPV supplementary damping controller design for a thyristor controlled series capacitor (TCSC) device," *IEEE Transactions on Power Systems*, vol. 21, no. 3, pp. 1242–1249, 2006.

[279] K. Tomsovic, D. E. Bakken, V. Venkatasubramanian, and A. Bose, "Designing the next generation of real-time control, communication, and computations for large power systems," vol. 93, no. 5, pp. 965–979, 2005.

[280] M. Zima, M. Larsson, P. Korba, C. Rehtanz, and G. Andersson, "Design aspects for wide-area monitoring and control systems," vol. 93, no. 5, pp. 980–996, 2005.

[281] G. E. Boukarim, S. Wang, J. H. Chow, G. N. Taranto, and N. Martins, "A comparison of classical, robust, and decentralized control designs for multiple power system stabilizers," *IEEE Transactions on Power Systems*, vol. 15, no. 4, pp. 1287–1292, 2000.

[282] Y. Zhang and A. Bose, "Design of wide-area damping controllers for interarea oscillations," *IEEE Transactions on Power Systems*, vol. 23, no. 3, pp. 1136–1143, 2008.

[283] F. Dörfler, M. R. Jovanović, M. Chertkov, and F. Bullo, "Sparsity-promoting optimal wide-area control of power networks," *IEEE Transactions on Power Systems*, vol. 29, no. 5, pp. 2281–2291, 2014.

[284] X. Wu, F. Dörfler, and M. R. Jovanović, "Input-output analysis and decentralized optimal control of inter-area oscillations in power systems," *IEEE Transactions on Power Systems*, vol. 31, no. 3, pp. 2434–2444, 2016.

[285] M. Wytock and J. Z. Kolter, "A fast algorithm for sparse controller design," *arXiv preprint arXiv:1312.4892*, 2013.

[286] S. P. Azad, J. A. Taylor, and R. Iravani, "Decentralized supplementary control of multiple lcc-hvdc links," *IEEE Transactions on Power Systems*, vol. 31, no. 1, pp. 572–580, 2016.

[287] F. Lian, A. Duel-Hallen, and A. Chakrabortty, "Game-theoretic multi-agent control and network cost allocation under communication constraints," *arXiv preprint arXiv:1605.00620*, 2016.

[288] S. Schuler, U. Münz, and F. Allgöwer, "Decentralized state feedback control for interconnected systems with application to power systems," *Journal of Process Control*, vol. 24, no. 2, pp. 379–388, 2014.

[289] N. R. Naguru and V. Sarkar, "Optimal wide area control of a power system with limited measurements," in *Signal Processing, Informatics, Communication and Energy Systems (SPICES), 2015 IEEE International Conference on*. IEEE, 2015, pp. 1–5.

[290] J. A. Taylor and L. Scardovi, "Decentralized control of dc-segmented power systems," in *Communication, Control, and Computing (Allerton), 2014 52nd Annual Allerton Conference on*. IEEE, 2014, pp. 1046–1050.