Exactness of Semidefinite Relaxations for Nonlinear Optimization Problems with Underlying Graph Structure

Somayeh Sojoudi

Electrical Engineering and Computer Sciences
University of California, Berkeley

Javad Lavaei

Industrial Engineering and Operations Research
University of California, Berkeley

Polynomial Optimization

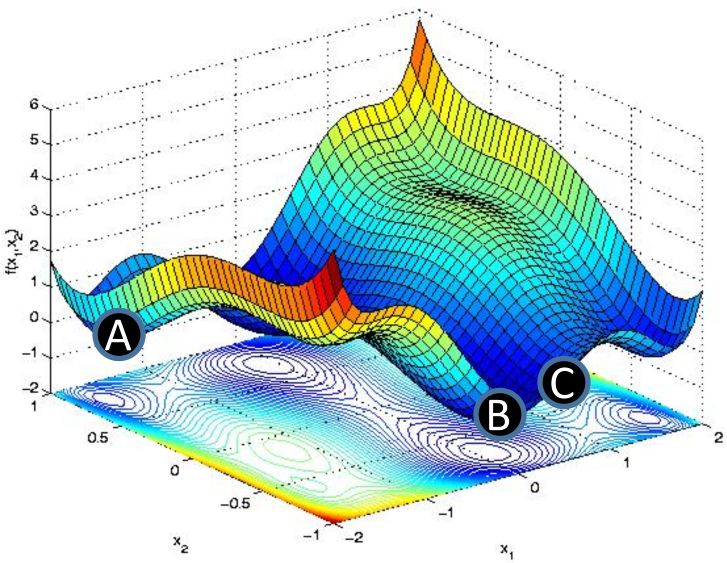
❑ **Polynomial Optimization:**

$$\begin{aligned} \min \quad & x^T M x \\ \text{s.t.} \quad & x_i^2 = 1, \quad i = 1, 2, ..., n \end{aligned}$$

Special case: Combinatorial optimization and integer programming problems

Very hard to solve

❑ **Different types of solutions:**



Point A: Local solution

Point B: Global solution

Point C: Near-global solution

Focus of our research

❖ **Approach:** Low-rank optimization, matrix completion, graph theory, convexification

Convexification

$\text{trace}\{M_0 x x^H\}$

$$\min_{x \in \mathbf{C}^n} \quad x^H M_0 x$$
$$\text{s.t.} \quad x^H M_i x \leq a_i, \quad i = 1, 2, \ldots, m$$

- **Transformation:** Replace xx^H with W.
- W is positive semidefinite and rank 1

SDP relaxation

$$\min_{W \in \mathbb{H}^n} \quad \text{trace}\{M_0 W\}$$
$$\text{s.t.} \quad \text{trace}\{M_i W\} \leq a_i, \quad i = 1, 2, \ldots, m$$
$$W \succeq 0$$

- **Rank-1 SDP:** Recovery of a global solution x

Penalized SDP

$$\min_{W} \quad \text{trace}\{M_0 W\} + \lambda\, g(W)$$
$$\text{s.t.} \quad \text{trace}\{M_i W\} \leq a_i, \quad i = 1, 2, \ldots, m$$
$$W \succeq 0$$

- **Rank-1 penalized SDP:** Recovery of a near-global solution x

Research Problems

Arbitrary Real/Complex Polynomial Optimization

↓ Conversion

$$\min_{x \in \mathbf{D}^n} \quad x^H M_0 x$$
$$\text{s.t.} \quad x^H M_i x \leq a_i, \quad i = 1, 2, ..., m$$

↓ SDP/ Penalized SDP

$$\min_{W} \quad \text{trace}\{M_0 W\} + \lambda\, g(W)$$
$$\text{s.t.} \quad \text{trace}\{M_i W\} \leq a_i, \quad i = 1, 2, ..., m$$
$$W \succeq 0$$

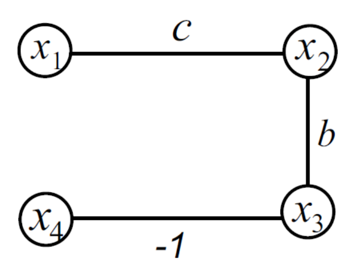
- How does structure make SDP relaxation exact? → Complexity analysis based on generalized weighted graph
- Connection between sparsity and rank? → Proof of existence of low-rank solution using OS and treewidth
- How to design penalized SDP? → Propose two methods to design penalty
- Design scalable numerical algorithm? → Cheap iterations for large-scale problems
- Power optimization problems → Finding near-global solutions using physics of power grids

Structured Optimization

- **Approach:** Map the structure into a *generalized weighted graph.*

$$\min_{x_1,x_2} \quad x_1^4 + a x_2^2 + b x_1^2 x_2 + c x_1 x_2$$

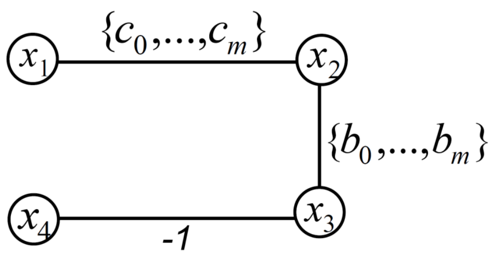
$$\begin{aligned} \min_{x \in \mathbb{R}^4} \quad & x_3^2 + a x_2^2 + b x_2 x_3 + c x_1 x_2 \\ \text{s.t.} \quad & x_1^2 - x_3 x_4 \le 1 \\ & x_4^2 - 1 = 0 \end{aligned}$$



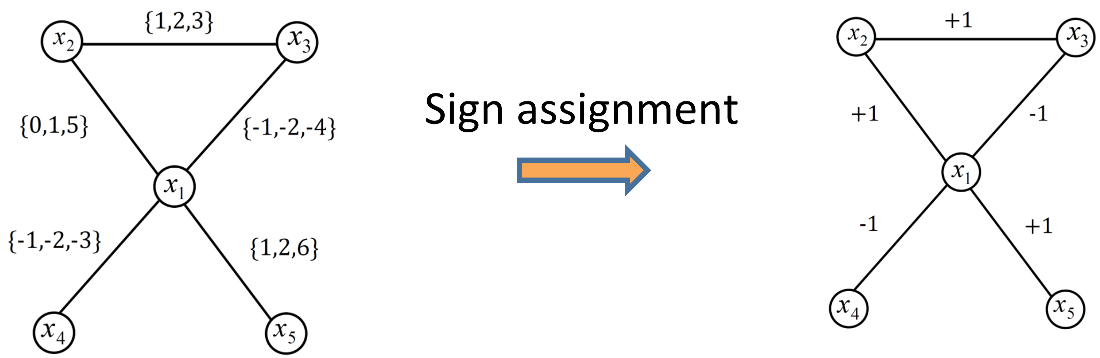
Due to structure, SDP is always exact.

Generalized weighted graph:

$$\begin{aligned} \min_{x_1,x_2} \quad & x_1^4 + a_0 x_2^2 + b_0 x_1^2 x_2 + c_0 x_1 x_2 \\ \text{s.t.} \quad & x_1^4 + a_i x_2^2 + b_i x_1^2 x_2 + c_i x_1 x_2 \le \alpha_i, \quad i = 1, 2, \ldots, m \end{aligned}$$



Real-Valued Optimization



Theorem

The SDP relaxation is exact if

$$\sigma_{ij} \neq 0, \qquad \forall (i,j) \in \mathcal{G}$$

$$\prod_{(i,j)\in\mathcal{O}_r} \sigma_{ij} = (-1)^{|\mathcal{O}_r|}, \qquad \forall r \in \{1, \dots, p\}$$

Edge

Cycle

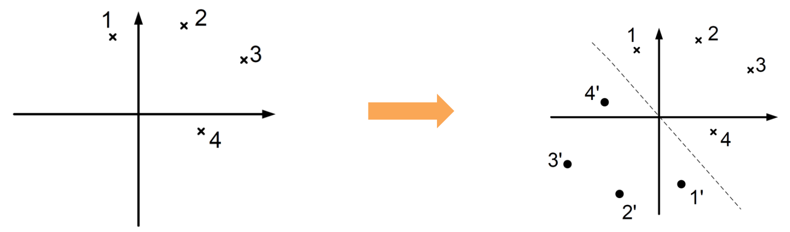
- Special cases:
  - **Positive optimization:** Bipartite graph
  - **Negative optimization:** Arbitrary graph

Interesting phenomena happen for complex optimization.

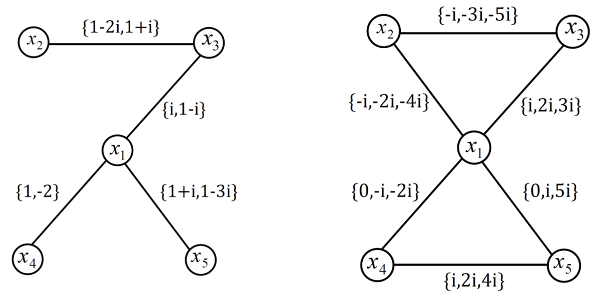
1. S. Sojoudi and J. Lavaei, "Exactness of Semidefinite Relaxations for Nonlinear Optimization Problems with Underlying Graph Structure," SIOPT, 2014.

Complex-Valued Optimization

- ❑ **Real-valued case: "T "** is sign definite if **T** and **–T** are separable in **R**:
- ❑ **Complex-valued case: "T "** is sign definite if **T** and **–T** are separable in **R**^{2}:



Theorem: SDP is exact for acyclic graphs with sign definite sets and certain cyclic graphs.

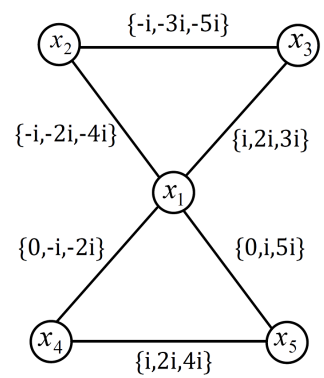


- ❑ The proposed conditions include several existing ones ([Kim and Kojima, 2003], [Padberg, 1989], [Bose, Gayme, Chandy, and Low, 2012], etc.).

1 S. Sojoudi and J. Lavaei, "Exactness of Semidefinite Relaxations for Nonlinear Optimization Problems with Underlying Graph Structure," SIOPT 2014.

Complex-Valued Optimization

- **Purely imaginary weights** (lossless power grid):



Theorem

Exact relaxation for weakly cyclic graphs with homogeneous weight sets.

- Consider a real matrix **M**:

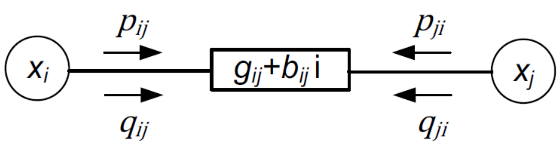
$$\min_{\mathbf{x}\in\mathcal{C}^n} \quad \mathbf{x}^* M \mathbf{x}$$
$$\text{s.t.} \quad |x_j| = 1, \quad j = 1, 2, ..., m$$

- Polynomial-time solvable for weakly-cyclic bipartite graphs.

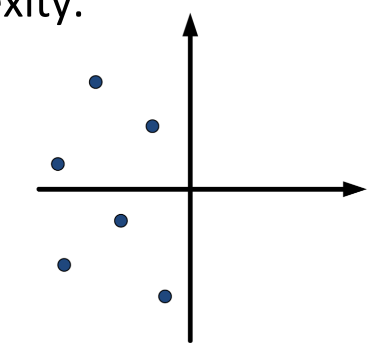
1. S. Sojoudi and J. Lavaei, "Exactness of Semidefinite Relaxations for Nonlinear Optimization Problems with Underlying Graph Structure," SIOPT, 2014.

Example

Example: Physics of power grids reduces computational complexity.



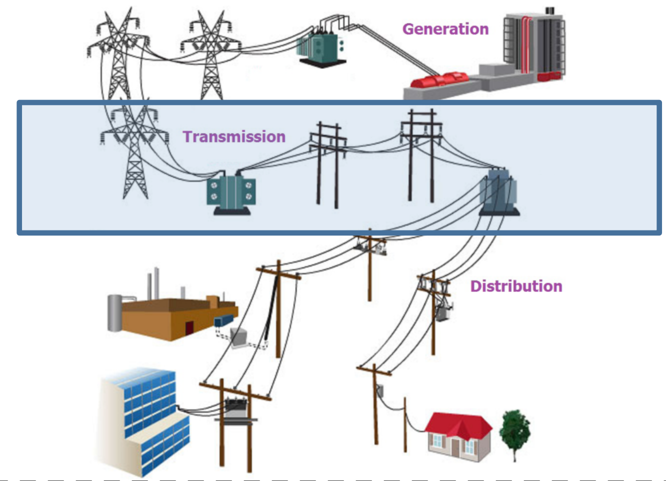
Coefficients of $x_i\, x_j$

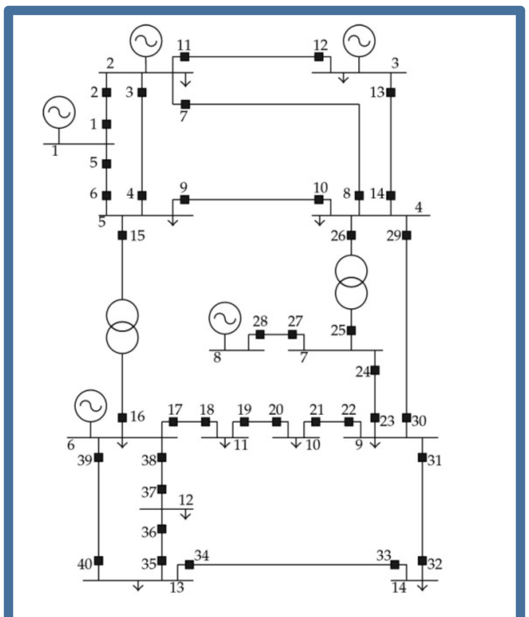


Sign definite due to passivity

Power Systems

- **Power system:**
  - A large-scale system consisting of generators, loads, lines, etc.
  - Used for generating, transporting and distributing electricity.





ISO, RTO, TSO

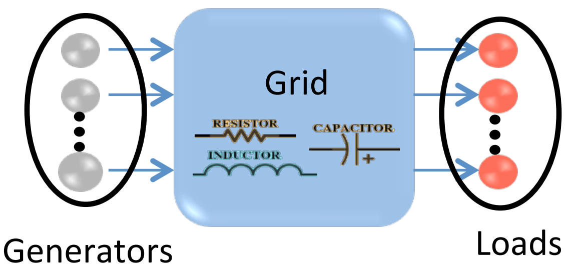
1. Optimal power flow (OPF)
2. Security-constrained OPF
3. State estimation
4. Network reconfiguration
5. Unit commitment
6. Dynamic energy management

NP-hard

(real-time operation and market)

Optimal Power Flow

Optimal Power Flow: Optimally match supply with demand

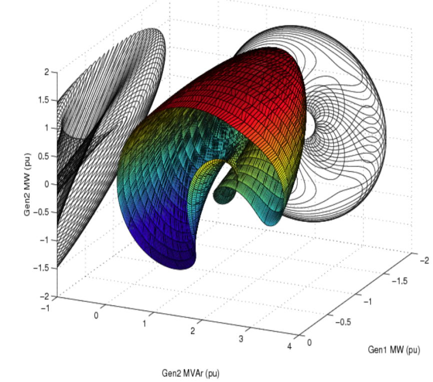


Vector of complex voltages

$$\min_{x \in \mathbf{C}^n} \quad x^H M_0 x$$
$$\text{s.t.} \quad x^H M_i x \leq a_i, \quad i = 1, 2, \ldots, m$$

- **Real-time operation:** OPF is solved every 5-15 minutes.
- **Market:** Security-constrained unit-commitment OPF
- **Complexity:** Strongly NP-complete with long history since 1962.
- **Common practice:** Linearization
- **FERC and NETSS Study:** Annual cost of approximation > $ 1 billion

A multi-billion critical system depends on optimization.

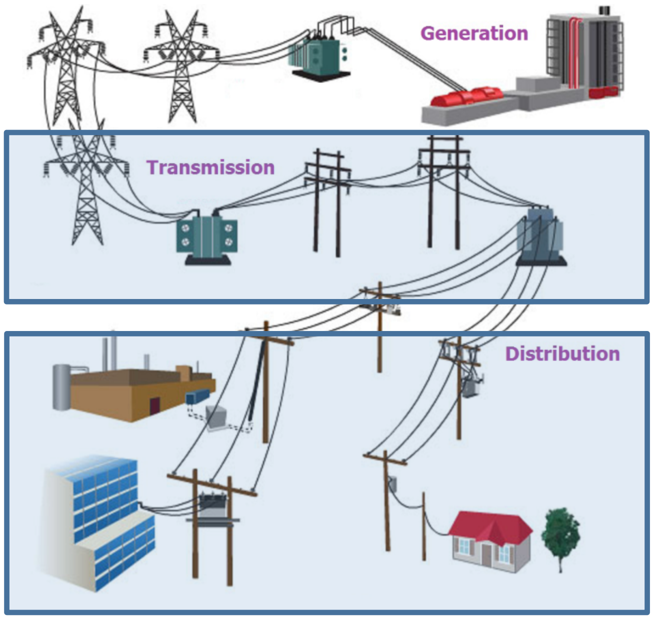


OPF feasible set

(Ian Hisken et al. 2003)

Exactness of Relaxation

- SDP is exact for IEEE benchmark examples and several real data sets.



cyclic → **Theorem:** Exact under positive LMPs with many transformers.

acyclic → **Theorem:** Exact under positive LMPs.

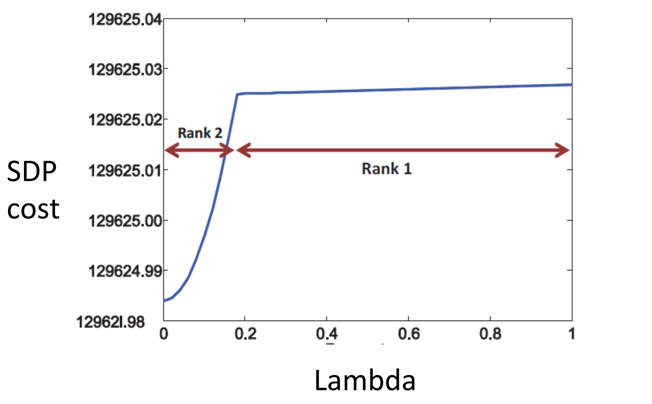
Physics of power networks (e.g., passivity) reduces computational complexity for power optimization problems.

1. S. Sojoudi and J. Lavaei, "Exactness of Semidefinite Relaxations for Nonlinear Optimization Problems with Underlying Graph Structure," SIOPT, 2014.
2. S. Sojoudi and J. Lavaei, "Physics of Power Networks Makes Hard Optimization Problems Easy to Solve," PES 2012.
3. J. Lavaei and S. Low, "Zero Duality Gap in Optimal Power Flow Problem," IEEE Transactions on Power Systems, 2012.
4. J. Lavaei, D. Tse and B. Zhang, "Geometry of Power Flows and Optimization in Distribution Networks," IEEE Transactions on Power System, 2014.
5. R. Madani, S. Sojoudi and J. Lavaei, "Convex Relaxation for Optimal Power Flow Problem: Mesh Networks," IEEE Transactions on Power Systems, 2015.

Example: Near-Global Solutions

Strategy: Penalize reactive loss over problematic lines

- **Modified IEEE 118-bus:**
  - 3 local solutions
  - Costs: 129625, 177984, 195695



Case	TW	Cost	Guarantee	Time (sec)
Chow's 9 bus	2	5296.68	100%	≤ 5
IEEE 14 bus	2	8081.53	100%	≤ 5
IEEE 24 bus	4	63352.20	100%	≤ 5
IEEE 30 bus	3	576.89	100%	≤ 5
NE 39 bus	3	41864.40	99.994%	≤ 5
IEEE 57 bus	5	41737.78	100%	≤ 5
IEEE 118 bus	4	129660.81	99.995%	≤ 5
IEEE 300 bus	6	719725.10	99.998%	13.9
Polish 2383wp	23	1874322.65	99.316%	529
Polish 2736sp	23	1308270.20	99.970%	701
Polish 2737sop	23	777664.02	99.995%	675
Polish 2746wop	23	1208453.93	99.985%	801
Polish 2746wp	24	1632384.87	99.962 %	699
Polish 3012wp	24	2608918.45	99.188%	814
Polish 3120sp	24	2160800.42	99.073 %	910

Case	Minima	Cost	Guarantee
WB2	2	877.78	100%
WB3	2	417.25	100%
WB5	2	946.58	99.995%
WB5 Mod	3	1482.22	100%
LMBM3	5	5694.54	100%
LMBM3_50	2	5823.86	99.807%
case22loop	2	4538.80	100 %
case30loop	2	2863.06	100%
case30loop Mod	3	2861.88	100%
case39 Mod4	3	557.15	99.999%
case118 Mod1	3	129625.19	99.999%
case118 Mod2	2	85987.59	100 %
case300 Mod2	2	474643.46	99.996%

7000 simulations

1. R. Madani, S. Sojoudi and J. Lavaei, "Convex Relaxation for Optimal Power Flow Problem: Mesh Networks," IEEE Transactions on Power Systems, 2015.
2. R. Madani, M. Ashraphijuo and J. Lavaei, "Promises of Conic Relaxation for Contingency-Constrained Optimal Power Flow Problem," Allerton 2014.

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- **Google:** Numerical algorithms for nonlinear optimization
- **Siebel:** Computational methods for maximizing efficiency, reliability and resiliency of power systems

Collaborators

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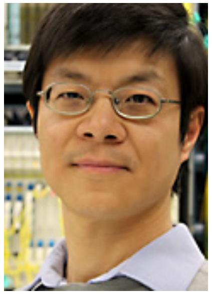
- Richard Murray
- Steven Low
- Ross Baldick



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- Stephen Boyd
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Research Group:

- Ramtin Madani
- Abdulrahman Kalbat
- Salar Fattahi
- Morteza Ashraphijuo