Absence of spurious local trajectories in time-varying optimization

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Abstract

In this paper, we study the landscape of optimization problems where the input data vary over time. To this end, we introduce the notion of spurious local trajectory as a generalization to the notion of spurious local solution in nonconvex (time-invariant) optimization. As a motivating case study, we consider the problem of optimal power flow in electrical networks with real-world and time-varying input data. We show that, despite the existence of spurious local solutions at every time, the time-varying landscape of the problem is free of spurious local trajectories. Inspired by this example, we propose an ordinary differential equation (ODE) which, at limit, characterizes the spurious local solutions of the time-varying optimization problem. By building upon this connection, we show that the absence of spurious local trajectory is closely related to the stability of the proposed ODE. In particular, we show that: (1) if the problem is time-invariant, the spurious local trajectories are ubiquitous since any strict local minimum is a locally stable equilibrium point of the ODE, and (2) if the ODE is time-varying, the local minima of the optimization problem may neither be equilibrium nor stable for the proposed ODE. To illustrate the applicability of the developed results, we consider a class of univariate problems with spurious local minima and provide sufficient conditions under which they are free of spurious local trajectories.

1 Introduction

Nonconvex optimization is at the crux of most of the real-world problems: the nation-wide optimal power scheduling problem in electrical systems (also known as optimal power flow problem) can be cast as a nonconvex quadratically-constrained quadratic programming, where the nonconvexity stems from the underlying laws of physics [1]. The nonconvexity is also inherent to most of the problems in machine learning; from the classical compressive sensing and matrix completion/sensing [2–4], to the more recent problems on the training of deep neural networks [5], they often possess nonconvex landscapes. Reminiscent from the classical complexity theory, this nonconvexity is perceived to be the main contributor to the intractability of these problems. In many (albeit not all) cases, this intractability implies that in the worst-case instances of the problem, spurious local minima exist and there is no efficient algorithm capable of escaping them. However, a lingering question remains unanswered: are these worst-case instances common in practice or do they correspond to the pathological or rare cases?

Answering this question has been the subject of many recent studies. In particular, it has been shown that nearly-isotropic classes of problems in matrix completion/sensing [6–8], robust principle component analysis [9,10], and dictionary recovery [11] have benign landscape, implying that they are free of spurious local minima. It has also been shown recently in [12] that the stochastic gradient
descent can escape the small fluctuations in the landscape, provided that on average, the gradient of the objective function is one-point convex with respect to the globally optimal solution. On the other hand, several works have shown that the spurious local minima are ubiquitous in many problems, including neural networks [13,14] and the instances of the matrix completion that are not nearly-isotropic [15]. Therefore, in general, one cannot expect the absence of undesired local solutions in a given nonconvex problem. At the core of the aforementioned results is the assumption on the static and time-invariant nature of the landscape. In contrast, most of the real-world problems should be solved sequentially over time with time-varying input data. For instance, in the optimal power flow problem, the electricity consumption of the consumers changes hourly [16,17]. Therefore, it is natural to study the landscape of such time-varying nonconvex optimization problems, taking into account their dynamic nature.

In this paper, we consider a class of nonconvex optimization problems where the input data varies over time. As a motivating case study, we consider the optimal power flow problem with California load profile and empirically show that, despite having multiple point-wise local solutions at every time step, the local trajectories converge to the global solution over time. Inspired by this observation, we introduce the notion of spurious local trajectory as a generalization to the point-wise spurious local solutions. We show that a time-varying optimization can have point-wise spurious local minima at every time step and yet, it can be free of spurious local trajectory. By building upon this notion, we consider a class of nonconvex optimization problems with equality constraints and model their local trajectories as an ordinary differential equation (ODE). We show that the absence of the spurious local trajectories in this time-varying optimization is equivalent to the convergence of all solutions in its corresponding ODE. Based on this equivalence, we consider a class of time-varying univariate optimization problems and present sufficient conditions under which, despite having point-wise spurious local minima at all times, the time-varying problem is free of spurious local trajectory. Finally, by studying the stability of the proposed ODE on feasible manifolds, we show that every strict local minima of the optimization problem is locally stable on its feasible region. This implies that the time-varying nature of the problem is essential for the absence of spurious local trajectories.

It is worthwhile to mention that a fruitful connection has been made recently in [18] between ODEs and optimization of residual neural networks. Various other works have also highlighted the nice interplay between the fields (see for example [19–21]).

Comparison to online optimization. A common framework in machine learning for analyzing the time-varying optimization is online (convex or nonconvex) optimization (see [22] and [23] for a comprehensive survey). In general, the main goal in such problems is to propose a sequential algorithm and measure its performance through the notion of global regret, which is defined as the incurred sub-optimality error of the proposed algorithm compared to the optimal fixed algorithm in the hindsight [22,23]. It is well-known that in the nonconvex settings, such notion of global regret cannot be minimized. Therefore, different researchers have resorted to minimizing a surrogate notion of local regret [17,25,26], which measures the sub-optimality compared to a local point-wise solution to the problem. Contrary to this line of research, we focus on the global landscape of the time-varying and nonconvex optimization problems.

This paper is organized as follows. Section 2 presents a motivating case study on power systems optimization with real-world data. Section 3 presents a notion of spurious local trajectory for time-varying optimization. Section 4 analyzes the stability of local trajectories. Finally, we conclude in Section 5. To streamline the presentation, the proofs are deferred to the appendix.
2 Case study in electric power systems

In this section, we present an empirical study on the trajectory of the solutions for the optimal power flow problem with time-varying load profile, where the goal is to match the supply of electricity with demand while satisfying the network, physical, and technological constraints. In practice, the problem is solved sequentially over time with the constraint that at every time-step, the solution cannot be significantly different from the one obtained in the previous time-step due to the so-called ramping constraints of the generators. We consider the IEEE 9-bus system [27] and initialize the system from the global solution, as well as three different spurious local solutions. We then change the load over time based on the California average load profile for the month of January 2019 (Figure 1a). The optimal power flow problem is then solved sequentially every 15 minutes for a range of 24 hours, while taking into account the temporal couplings between solutions via the ramping constraints. The trajectories of the solutions for the optimal power flow problem with different initial points appear in Figure 1b. In this figure, the solid blue lines represent the cost obtained by the semidefinite programming (SDP) relaxation of the optimal power flow [28]. This curve is a lower bound to the globally optimal cost and serves as a certificate of the global optimality whenever it touches other trajectories.

The gray circles in these plots are some of the local solutions that were obtained via a Monte Carlo simulation. Based on Figure 1b, indeed there exist multiple local solutions at every time-step (some of them emerge over time). However, surprisingly, the trajectories of the local solutions that are initialized at different points all converge towards the global solution.

3 Notion of spurious local trajectory

Inspired by the above case study, we consider the effect of the variation in the input data on the landscape of the optimization problem. We focus on the following time-varying nonconvex optimization:

$$\inf_{x \in \mathbb{R}^n} f(x, t) \quad \text{s.t.} \quad h_i(x) = d_i(t), \quad i = 1, \ldots, m$$

(1)

where the objective function $f(x, t)$ and the right-hand side of the equality constraints vary over time $t \in [0, T]$. We assume that $f : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}$, $h_i : \mathbb{R}^n \rightarrow \mathbb{R}$, $d_i : \mathbb{R}^+ \rightarrow \mathbb{R}$ for $i = 1, \ldots, m$ are continuously twice differentiable functions, and that $T > 0$ is a finite time horizon. Moreover, we assume that $f$ is uniformly bounded from below (i.e., $f(x, t) \geq M$ for some constant $M$) and the problem is feasible for all $t \in [0, T]$. In practice, one can only hope to sequentially solve this problem at a finite number of times $0 = t_0 < t_1 < t_2 < \ldots < t_N = T$. However, notice that — as elucidated in our case study on the optimal power flow problem — in many real-world problems, it is neither practical nor realistic to have solutions that change abruptly over time. One way to circumvent this issue is to regularize the problem at time $t_{k+1}$ by penalizing the deviation of its solution from the one obtained at time $t_k$. Precisely, we employ the proximal regularization as is done in online learning (e.g., in [29]). We fix a regularization parameter $\alpha > 0$ in Definitions 1, 2, and 3 below.

**Definition 1.** Given evenly spaced-out time steps $0 = t_0 < t_1 < t_2 < \ldots < t_N = T$ for some integer $N$, a sequence $x_0, x_1, x_2, \ldots, x_N$ is said to be a **discrete local trajectory** of the time-varying optimization (1) if the following holds:

1. $x_0$ is a local solution to the time-varying optimization (1) at time $t_0 = 0$;
2. for $k = 0, 1, 2, \ldots, N - 1$, $x_{k+1}$ is local solution to

$$\inf_{x \in \mathbb{R}^n} f(x, t_{k+1}) + \alpha \frac{\|x - x_k\|^2}{2(t_{k+1} - t_k)}$$

s.t. \quad h_i(x) = d_i(t_{k+1}), \quad i = 1, \ldots, m. \quad (2)$$

Above, $\| \cdot \|$ denotes the Euclidian norm.

Note that in the above definition, the term local solution refers to any local minimum, local maximum, or a saddle point of (2). Naturally, due to the non-convexity of the objective and constraint functions, we do not have unicity of the sequence in the above definition\footnote{For example, there exists two discrete trajectories starting at $x_0 = 0$ and at time $t_0 = 0$ for the time-varying objective function $f(x, t) := x^2(T/2 - t)$. Indeed, the discrete trajectory stays at $x_k = 0$ for $t_k \leq T/2$ and then, due to the regularization, it bifurcates into two separate discrete trajectories.}. This makes its analysis quite difficult. In order to cope with this, we consider the continuous time limit as the number of time steps $N$ increases. Subsequently, we fully characterize the trajectory of these local solutions through an ordinary differential equation (ODE). Under appropriate assumptions, we show that the solution to this ODE exists, it is unique, and all discrete trajectories converge towards it. These properties should not be taken for granted, and as the reader will see, they are highly non-trivial. Taking the continuous limit yields the following ordinary differential equation

$$\dot{x} = -\frac{1}{\alpha} \eta(x, t) + \theta(x)d \quad \text{(ODE)}$$

where

$$\eta(x, t) := \left[I - \nabla h(x)^T (\nabla h(x) \nabla h(x)^T)^{-1} \nabla h(x)\right] \nabla_x f(x, t), \quad (3)$$

$$\theta(x) := \nabla h(x)^T (\nabla h(x) \nabla h(x)^T)^{-1}. \quad (4)$$
Above, \( J_h(x) \) denotes the Jacobian of the left hand side of the constraints \( h(x) = [h_1(x), \ldots, h_m(x)]^\top \) and \( d(t) \) denotes the right hand side of the constraints, that is to say \( d(t) = [d_1(t), \ldots, d_m(t)]^\top \). The function \( \eta(x, t) \) can be interpreted as the orthogonal projection of the gradient \( \nabla_x f(x, t) \) of the time-varying objective on the Kernel of \( J_h(x)^\top \).

The standard results in ordinary differential equations, namely the Picard-Lindelöf Theorem\cite{30} Theorem 3.1, the Cauchy-Peano Theorem\cite{30} Theorem 1.2, and the Carathéodory Theorem\cite{30} Theorem 1.1, cannot be applied to prove the existence of a solution to (ODE) on \([0, T]\). Indeed, the classical Lipschitz property fails to hold due to the matrix inversion. More importantly, those standard results only provide the existence of a solution defined locally, that is to say on a neighborhood \([0, \alpha]\) where \( \alpha < T \) is potentially very small. To overcome these limitations, we identify assumptions on the set of discrete local trajectories that enable us to prove the existence and unicity on \([0, T]\). Similar to the Cauchy-Peano Theorem, our proof relies on the Arzelà–Ascoli Theorem (see Appendix for the various lemmas involved).

**Assumption 1** (Uniform Boundedness). There exist constants \( R_1 > 0 \) and \( R_2 > 0 \) such that, for any discrete local trajectory \( x_0, x_1, x_2, \ldots, \|x_k\| \) and the objective function of (2) at \( x_k \) are upper bounded by \( R_1 \) and \( R_2 \), respectively, for every \( k = 0, 1, 2, \ldots \).

**Assumption 2** (Non-singularity). There exists a constant \( c > 0 \) such that, for any discrete local trajectory \( x_0, x_1, x_2, \ldots \), it holds that \( \sigma_{\min}(\nabla f(x_k)) \geq c, \ k = 0, 1, 2, \ldots \) where \( \sigma_{\min} \) denotes the minimal singular value.

Note that Assumption 2 implies that linear independence constraint qualification holds at every point of a discrete local trajectory.

**Theorem 1** (Existence and Uniqueness). Let Assumption 1 and Assumption 2 hold. If the initial condition of (ODE) is a local solution to the time-varying optimization at \( t = 0 \), then there exists a unique continuously differentiable solution \( x : [0, T] \to \mathbb{R}^n \).

**Theorem 2** (Convergence). Let Assumption 1 and Assumption 2 hold. Given a fixed initial point \( x_0 \), any sequence of discrete local trajectories \((x^\Delta t_k)\), where the difference \( \Delta t \) between the time steps approaches zero, converges towards the solution \( x : [0, T] \to \mathbb{R}^n \) with \( x(0) = x_0 \), in the sense that

\[
\lim_{\Delta t \to 0^+} \sup_{0 \leq k \leq [T/\Delta t]} \|x_k^\Delta t - x(t_k)\| = 0. \tag{5}
\]

Above, \([\cdot]\) denotes the ceiling of the real number.

Now that we have established the connection between the discrete local trajectories to their continuous limit, we naturally propose the following definition.

**Definition 2.** A continuously differentiable function \( x : [0, T] \to \mathbb{R} \) is said to be a **continuous local trajectory** of the time-varying optimization (1) if the following holds:

1. \( x(0) \) is a local solution to the time-varying optimization (1) at time \( t = 0 \);
2. \( x \) is a solution to the ordinary differential equation (ODE).

We next introduce the central notion in this paper. It would be tempting to say that a continuous local trajectory is non-spurious if the final state is a global solution, or near a global solution upon a small perturbation of the initial condition. That would be in line with the notion of stability in dynamical systems. However, this notion is ill-suited in our setting because it is unrelated to the landscape of the time-varying optimization (1) at \( t = T \). Recall that we use a regularization
parameter $\alpha$ in order to prevent the terms in the discrete local trajectory from changing abruptly from one time-step to the next. Thus, $\alpha$ acts like inertia in the continuous local trajectory. Consider the case where a unique point-wise global minimum forms a continuously differentiable function that varies over time faster than the continuous local trajectory. Then, as long as the time horizon is big enough, all trajectories would be considered spurious. This would be true even for the trajectory initialized at the global minimum. See Figure 2 for an illustration of this phenomena. The following definition remediates the issue.

**Definition 3.** A continuous local trajectory $x : [0, T] \rightarrow \mathbb{R}$ is said to be a **spurious local trajectory** if its final state $x(T)$ does not belong to the region of attraction of a global solution to the time-varying optimization $(1)$ at time $t = T$. In other words, the trajectory is non-spurious if the initial value problem

\[
\begin{align*}
\dot{x} &= -\frac{1}{\alpha} \eta(x, T), \\
x(0) &= x(T).
\end{align*}
\]

admits a continuously differentiable solution $\bar{x} : \mathbb{R}_+ \rightarrow \mathbb{R}^n$ such that $\bar{x}(t)$ converges towards a global solution as $t \rightarrow +\infty$.

The above definition makes a link between time-varying optimization and the theory of switched systems [31–33]. This link is new to the best of our knowledge. Indeed, the question of whether a continuous local trajectory is spurious can be formulated using the following switched system

\[
\dot{x} = -\frac{1}{\alpha} \eta(x, \sigma(t)) + \theta(x) \dot{\sigma}'(t)
\]

where $\sigma$ is referred to as switching signal in the literature. The fact that its derivative is not defined at $t = T$ poses no problem. Indeed, we are interested in finding continuous solutions in the extended sense

\[
x(t) = x(0) + \int_0^t \left[ -\frac{1}{\alpha} \eta(x(\tau), \sigma(\tau)) + \theta(x(\tau)) \dot{\sigma}'(\tau) \right] d\tau.
\]

Non-spurious then means that every local solution at time $t = 0$ of the time-varying optimization $(1)$ belongs to the region of attraction of a global minimum of $(1)$ at time $t = T$. Note that there exists software for computing reachable sets of switched systems [34]. Empirical identification of this region for $(7)$ is considered as an enticing challenge for future work.

## 4 Stability analysis of local trajectories

In this section, we show that the time-varying nature of the optimization is crucial for the absence spurious local trajectories. In particular, we illustrate an intriguing connection between the landscape of the time-varying optimization and the stability of the (ODE). We show that by starting from a spurious local solution, the solution to the (ODE) can escape the basin of attraction of this local minimum over time and converge to the global one. From a control theoretical perspective, this suggests that the trajectory defined by the point-wise local solutions is not stable for the (ODE). In what follows, we formalize this observation by showing that the time-varying nature of the optimization is essential for the instability of the (ODE) around the spurious local minima of the time-varying optimization $(1)$. 


We begin by assuming that the time-varying optimization actually does not change over the time interval $[0,T]$. Then, we may simplify the notations and omit $t$ from the optimization, as follows:
\[
\inf_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad h(x) = d
\]
where $h(x) = [h_1(x), \ldots, h_m(x)]^T$ and $d = [d_1, \ldots, d_m]^T$. Likewise, we may drop $t$ from the dynamics:
\[
\dot{x} = -\frac{1}{\alpha} \left[ I - \mathcal{J}_h(x)^T (\mathcal{J}_h(x)\mathcal{J}_h(x)^T)^{-1} \mathcal{J}_h(x) \right] \nabla f(x).
\]

In this case, we show that all continuous local trajectories initialized at a spurious solution to $(10)$ are spurious trajectories. This is a direct implication of the following proposition. Its proof is based on arguments akin to the stability theory of Lyapunov [35]. We rely on the key observation that in the time-invariant case, the objective function $f(x)$ decreases along the continuous local trajectories.

Proposition 1 (Local stability). Any strict local minimum $x^*$ of the time-invariant optimization $(10)$ is locally stable for $(11)$, in the sense that
\[
\forall \epsilon > 0, \exists \delta > 0: \quad (\|x(0) - x^*\| \leq \delta \text{ and } h(x(0)) = d) \quad \Rightarrow \quad \forall t \in [0,T], \quad \|x(t) - x^*\| \leq \epsilon \quad (12)
\]

where $x : [0,T] \rightarrow \mathbb{R}^n$ satisfies the ordinary differential equation $(11)$.

The stability of time-varying systems is quite convoluted, even in the linear case. For example, in [36] it is shown that a linear time-varying system can be asymptotically stable despite having strictly positive eigenvalues at all times. We note that several necessary and sufficient conditions for the stability of linear time-varying system were proposed recently in [37]. A generalized time-varying Lyapunov function was proposed in [38] and has been applied in [39] to study the stability of an averaged system. Slowly time varying systems is investigated in [40]. The main reasoning behind the failure of Proposition 1 in time-varying problems is that $f(x,t)$ does not necessarily decrease along
the continuous trajectories. As a preliminary step for further study, we focus on uni-dimensional time-varying optimization. We prove that the time-varying nature is the key to the absence of spurious trajectories. Consider the following optimization

$$\inf_{x \in \mathbb{R}} f(x, t) := g(x - \beta \sin(t))$$

(13)

where $g : \mathbb{R} \rightarrow \mathbb{R}$ is continuously twice differentiable and $\beta > 0$ models the variation of the data over time. Note that this problem can also be reformulated as minimizing $g(x_2)$ over $x_1, x_2 \in \mathbb{R}$ with the constraint $x_1 - x_2 = \beta \sin(t)$. Only the right hand side varies over time, and therefore, this problem fits well in our introduced framework. We assume that $g$ admits only three stationary points $g'(y_1) = g'(y_2) = g'(y_3)$ with $y_1 < y_2 < y_3$. We assume also that $y_1, y_3$ are local minima such that $g(y_1) > g(y_3)$, while $y_2$ is a local maximum. Finally, we assume that $g$ is coercive (its limit at $\pm \infty$ is $+\infty$). Thus, its global infimum is reached in $y_3$. The following proposition identifies sufficient conditions for the absence of spurious local trajectories.

**Proposition 2.** If $\alpha, \beta > 0$ are such that

1. $\alpha \beta \geq C := \max_{y_1 \leq y \leq y_3} g'(y)$,
2. $\exists m_1, m_2 \in \mathbb{R} : m_1 < y_1 < m_2$ and $g'(m_1) = g'(m_2) = -\alpha \beta$,
3. $-C/\alpha(t_2 - t_1) - \beta(\sin(t_2) - \sin(t_1)) + m_1 \geq m_2$ where $0 < t_1 \leq t_2$ satisfy $\cos(t_1) = \cos(t_2) = -C/(\alpha \beta),$

then the time-varying optimization (13) has no spurious local trajectories.

We highlight the implications of the above propositions through a numerical example.

**Example 1.** Consider the objective function $f(x, t) := g(x - \beta \sin(t))$ where

$$g(y) := 1/4y^4 + 1/8y^3 - 2y^2 - 3/2y + 8.$$  

(14)

The time-varying objective $f(x, t)$ has the following stationary points: it admits a spurious local minimum at $-2 + \beta \sin(t)$, a local maximum at $-3/8 + \beta \sin(t)$, and a global minimum at $2 + \beta \sin(t)$. The three sufficient conditions of Proposition 2 can be brought to bear on this example. They yield three inequalities, as shown in Figure 3a whose feasible region is represented in Figure 3b. Taking a point in that feasible region, we confirm numerically in Figure 3a that a trajectory initialized at a local minimum of $f(\cdot, 0)$ winds up in the region of attraction of the global solution to $f(\cdot, T)$ at the final time $T = 2\pi$. In contrast, taking a point outside the feasible region, we observe in Figure 3d that a trajectory initialized at a local minimum of $f(\cdot, 0)$ does not end up in the region of attraction of the global solution to $f(\cdot, T)$.

We make a few remarks regarding Figure 3a. Note that $k_1$ and $k_2$ are integers in $\{0, 1, 2\}$ such that $k_1$ minimizes the line it appears in, and $k_2$ minimizes the line it appears in while not being equal to $k_1$. These numbers come from Viète’s solution to a cubic equation [41]. Furthermore, the second inequality corresponds to minus the discriminant of a fourth order polynomial.

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5 Conclusion

In this work, we study the landscape of time-varying nonconvex optimization problems. We introduce the notion of spurious local trajectory as a counterpart to the notion of spurious local minima in the time-invariant optimization. The key insight to this new notion is the fact that a regularized version of the time-varying optimization problem is naturally endowed with an ordinary differential equation (ODE) at its limit. This close interplay enables us to study the solutions of this ODE to certify the absence of the spurious local trajectories in the problem. Through different case studies and theoretical results, we show that a time-varying optimization can have multiple spurious local minima, and yet its landscape can be free of spurious local trajectories. We further show that the variation of the landscape over time is the main reason behind the absence of spurious local trajectories. In particular, we prove that any spurious strict local minimum in time-invariant optimization problem is a locally stable equilibrium of its corresponding ODE, thereby giving rise to a spurious local trajectory. Avenues for future work include the extension of the notion of spurious local trajectories to the time-varying optimization over infinite time horizon. Furthermore, it would worthwhile to derive necessary and sufficient conditions for the absence of spurious local trajectories in more general settings using the theory of switched systems.
1. \( \alpha \beta \geq C := -255/256 + 259 \sqrt{201}/768 - \sqrt{201^3}/1728 \)

2. 
   \[ -27\alpha^2 \beta^2 + 6885/128\alpha \beta + 61009/256 \geq 0 \]

3. 
   \[ -\frac{C}{\alpha} \left[ 2\pi - 2 \arccos \left( -\frac{C}{\sqrt{3}} \right) \right] + \ldots \]
   \[ -\beta \sin \left[ 2\pi - \arccos \left( -\frac{C}{\sqrt{3}} \right) \right] + \beta \sin \left[ \arccos \left( -\frac{C}{\sqrt{3}} \right) \right] + \ldots \]
   \[ 2 \sqrt{\frac{250}{192}} \cos \left[ \frac{1}{2} \arccos \left( \frac{48900 - 69152\alpha \beta}{132068} \sqrt{162/259} - 2k\pi}{3} \right] + \ldots \]
   \[ -2 \sqrt{\frac{250}{192}} \cos \left[ \frac{1}{2} \arccos \left( \frac{48900 - 69152\alpha \beta}{132068} \sqrt{162/259} - 2k\pi}{3} \right] \geq 0 \]

(a) Inequalities in function of \( \alpha, \beta \) guaranteeing absence of spurious trajectories.

(b) Sufficient condition in blue in function of \( \alpha, \beta \) for absence of spurious trajectories.

(c) Non-spurious trajectory for \( \alpha = 0.4 \) and \( \beta = 10 \).

(d) Spurious trajectory for \( \alpha = 0.2 \) and \( \beta = 5 \).

Figure 3: Illustration of the notion of spurious trajectory

References


A Proofs of Theorems 1 and 2

In this section, we provide the formal versions of Theorems 1 and 2 together with their proofs. First, we give a more precise statement of the problem, as well as the required assumptions. For clarity, we may use different notations from the ones used in the main body of the paper. Consider the following optimization

\[
\begin{align*}
\min_x f(x,t) \quad (15) \\
\text{s.t. } h_i(x) = d_i(t) \quad i = 1, 2, \ldots, r \quad (16)
\end{align*}
\]

where the objective function \( f(x,t) \) and the right-hand side of the constraints vary over time. We assume that \( f : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}, \ h_i : \mathbb{R}^n \rightarrow \mathbb{R}, \) and \( d_i : \mathbb{R}_+ \rightarrow \mathbb{R} \) are continuously twice differentiable functions in their domains. Furthermore, we assume that \( f \) is uniformly bounded from below and moreover, the problem is feasible for every \( t \in [0,T] \). Suppose that we are interested to solve this optimization problem from time \( t = 0 \) to \( t = T \). In practice, one can only hope to sequentially solve the discretized analog of this problem, defined as

\[
\begin{align*}
\min_x f(x,s_k) \quad (17) \\
\text{s.t. } h_i(x) = d_i(s_k) \quad i = 1, 2, \ldots, r \quad (18)
\end{align*}
\]

where \( s_k = \min\{k\Delta t, T\} \) for \( k = 0, \ldots, \lceil T/\Delta t \rceil \), and \( \Delta t > 0 \) is the discretization time. However, notice that—as elucidated in our case study on the optimal power flow problem—in many real-world problems, it is neither practical nor realistic to have a solution trajectory for \( 15 \) that changes abruptly over time. One way to circumvent this issue is to regularize the discretized problem \( 17 \) at time \( s_k \) by penalizing the deviation of its solution from the one obtained at time \( s_{k-1} \). Accordingly, consider an initial point \( x_0 \) that satisfies the Karush–Kuhn–Tucker (KKT) conditions for \( 17 \) at \( s_k = 0 \) (we simply refer to this point as a feasible initial point). We propose to solve the following regularized problem sequentially for every \( k = 1, \ldots, \lceil T/\Delta t \rceil \):

\[
\begin{align*}
\min_x f(x,s_k) + \frac{\alpha}{s_k - s_{k-1}}\|x - x_{k-1}\|_2^2 \quad (19) \\
\text{s.t. } h_i(x) = d_i(s_k) \quad (20)
\end{align*}
\]

We refer to the above optimization problem as \( \text{OPT}(k, \Delta t, x_{k-1}) \).
Let the Jacobian of the constraint set be defined as
\[ J_h(x) = \begin{bmatrix} \nabla_x h_1(x)^T \\ \nabla_x h_2(x)^T \\ \vdots \\ \nabla_x h_r(x)^T \end{bmatrix} \] (21)

**Definition 4.** Given a feasible initial point \( x_0 \), we say the pair \( (x_0, \Delta t, \{x^\Delta t_k\}_{k=1}^{\lfloor T/\Delta t \rfloor}) \) is an admissible KKT (AKKT) tuple if it has a continuously differentiable and unique solution

Assumption 3. There exists \( \ell > 0 \) such that any \( \Delta t \leq \ell \) is endowed with at least one AKKT tuple \( (x_0, \Delta t, \{x^\Delta t_k\}_{k=1}^{\lfloor T/\Delta t \rfloor}) \). Furthermore, for any AKKT tuple \( (x_0, \Delta t, \{x^\Delta t_k\}_{k=1}^{\lfloor T/\Delta t \rfloor}) \), the sequence \( \{x_0, \{x^\Delta t_k\}_{k=1}^{\lfloor T/\Delta t \rfloor}\} \) is uniformly bounded.

According to the Definition 4, the Jacobian matrix \( J_h(x^\Delta t_k) \) is non-singular for every \( k = 0, \ldots, \lfloor T/\Delta t \rfloor \) and every AKKT tuple \( (x_0, \Delta t, \{x^\Delta t_k\}_{k=1}^{\lfloor T/\Delta t \rfloor}) \). In this work, we impose a slightly stronger condition on the singular values of \( J_h(x^\Delta t_k) \).

Assumption 4. There exists a universal constant \( c > 0 \) such that \( \sigma_{\min}(J_h(x^\Delta t_k)) \geq c \) for every \( k = 0, \ldots, \lfloor T/\Delta t \rfloor \) and every AKKT tuple \( (x_0, \Delta t, \{x^\Delta t_k\}_{k=1}^{\lfloor T/\Delta t \rfloor}) \).

Now, we are ready to present our main theorem.

**Theorem 3.** Consider the following ODE
\[ \dot{x} = -\frac{1}{2\alpha} \left( I - J_h(x)^T (J_h(x)J_h(x)^T)^{-1}J_h(x) \right) \nabla_x f(x, t) + J_h(x)^T \nabla_x f(x, t) \] (22)
with a feasible initial condition \( x(0) = x_0 \). The following statements hold:

1. (Existence and uniqueness) \( (22) \) has a continuously differentiable and unique solution \( x : [0, T] \rightarrow \mathbb{R}^n \).

2. (Convergence) Any AKKT tuple \( (x_0, \Delta t, \{x^\Delta t_k\}_{k=1}^{\lfloor T/\Delta t \rfloor}) \) with uniformly bounded sequence \( \{x^\Delta t_k\}_{k=1}^{\lfloor T/\Delta t \rfloor} \) satisfies
\[ \lim_{\Delta t \to 0^+} \sup_{0 \leq k \leq \lfloor T/\Delta t \rfloor} \|x^\Delta t_k - x(s_k)\| = 0, \] (23)

**A.1 A Preliminary Lemma**

**Lemma 1** (Lipschitz property on a ball). Given a continuously differentiable function \( p(x) : \mathbb{R}^n \rightarrow \mathbb{R}^n \), we have
\[ \|p(x) - p(y)\| \leq L(\epsilon)\|x - y\| \text{ for every } x, y \in \mathcal{B}(\epsilon) \] (24)
where \( L(\epsilon) \) is a universal constant independent of \( x, y \) and \( \mathcal{B}(\epsilon) \) is the Euclidean ball centered at zero with a radius \( \epsilon \).
Proof. Note that $B(\epsilon)$ is compact for any finite $\epsilon > 0$. Therefore, for every $i = 1, \ldots, m$, the maximum of $\|\nabla p_i(x)\|$ on $B(\epsilon)$ is finite with (at least) one attainable maximizer $x^i \in B(\epsilon)$. Therefore, we have

$$\|p(x) - p(y)\| \leq \left( \max_{1 \leq i \leq n} \sqrt{n} \|\nabla p_i(x^i)\| \right) \|x - y\|$$

for every $x, y \in B(\epsilon)$ (25)

which completes the proof.

We will regularly refer to the above lemma in our subsequent analysis.

A.2 Existence and Uniqueness Proof.

Next, we show the existence and uniqueness of the solution to the proposed ODE. To streamline the presentation and without loss of generality, we assume that $\lceil T/\Delta t \rceil$ is a natural number and hence, $s_k - s_{k-1} = \Delta t$ for every $k = 1, \ldots, T/\Delta t$. Furthermore, to simplify the notation, we may use the same symbols to refer to different universal constants throughout the proofs. The following lemmas will be useful in proving the existence of a solution (22).

Lemma 2. There exist universal constants $\bar{t}$ and $c > 0$ such that for every AKKT tuple $(x_0, \Delta t, \{x^*_k\}_{k=0}^{\lceil T/\Delta t \rceil})$ with $\Delta t \leq \bar{t}$, we have $\|x^*_k - x^*_k - 1\| \leq c \Delta t$ for every $k = 1, \ldots, [T/\Delta t]$.

Proof. See Appendix B

When necessary for a AKKT tuple $(x_0, \Delta t, \{x^*_k\}_{k=1}^{T/\Delta t})$, the sequence $\{x^*_k\}_{k=0}^{T/\Delta t}$ is extended to $\{x^*_k\}_{k=0}^{T/\Delta t}$ with $x^*_0 = x_0$.

Lemma 3. Given $x_0$, there exist

1. $\{t_n\}_{n=1}^\infty$ with $\lim_{n \to \infty} t_n = 0$ such that each $t_n$ is endowed with an AKKT tuple $(x_0, t_n, \{x^*_k\}_{k=0}^{T/t_n})$,

and

2. a continuously differentiable and uniformly bounded function $\bar{x} : \[0, T\] \to \mathbb{R}^n$ that satisfies $\bar{x}(0) = x_0$,

with the following properties:

$$\lim_{n \to \infty} \sup_{1 \leq k \leq T/t_n} \left\| x^*_k - \bar{x}(kt_n) \right\| = 0,$$

$$\lim_{n \to \infty} \sup_{1 \leq k \leq T/t_n} \left\| \frac{x^*_k - x^*_k - 1}{t_n} - \dot{\bar{x}}(kt_n) \right\| = 0.$$  

(26) (27)

Moreover, there exists a universal constant $c > 0$ such that $\sigma_{\min}(J_{t_n}(\bar{x}(t))) \geq c$ for every $t \in [0, T]$.

Proof. See Appendix B

Lemma 4. Consider two continuous functions $g_1 : [0, T] \to \mathbb{R}^n$ and $g_2 : [0, T] \to \mathbb{R}^n$. We have $g_1 = g_2$ if and only if

$$\lim_{\Delta t \to 0^+} \sup_{0 \leq k \leq [T/\Delta t]} \|g_1(k\Delta t) - g_2(k\Delta t)\| = 0$$

(29)
Proof. Suppose that there exists \( \hat{x} \in [0,T] \) for which \( g_1(\hat{x}) \neq g_2(\hat{x}) \). Then, due to the continuity of \( g_1 \) and \( g_2 \), there must exist \( \delta > 0 \) such that \( g_1(x) \neq g_2(x) \) for every \( x \in S := [\hat{x} - \delta, \hat{x} + \delta] \). Note that the interval \( S \) has nonzero Lebesgue measure and hence \( \lim_{\Delta t \to 0^+} \sup_{0 \leq t \leq \frac{\Delta t}{\hat{t}}} \|g_1(k\Delta t) - g_2(k\Delta t)\| \neq 0 \) which is a contradiction. This completes the proof.

We now provide the proof for the existence of the solution for (22).

Proof of existence and uniqueness: Consider the sequence \( \{t_n\}_{n=1}^{\infty} \) and its corresponding AKKT tuple \( \{(x_{0, t_n}, \{(x_{k}^{T/t_n})_{k=0}^{\infty}\}_{k=0}^{\infty}\}_{n=1}^{\infty} \) that is introduced in Lemma 3. Due to Assumption 4, the linear independence constraint qualification (LICQ) holds at \( x_k^{*} \) for every \( k = 0, \ldots, T/t_n \) and \( n = 1, \ldots, \infty \). Therefore, for any \( n \), there exists a sequence of Lagrangian vectors \( \{\mu_k^{T/t_n}\}_{k=0}^{\infty} \) such that \( \{(x_k^{T/t_n})_{k=0}^{\infty}, \{\mu_k^{T/t_n}\}_{k=0}^{\infty}\} \) satisfies the KKT conditions:

\[
\nabla_x f_k(x_k^{*}) + \mathcal{J}_h(x_k^{*})^T \mu_k^{T} + \frac{2\alpha}{t_n}(x_k^{*} - x_{k-1}^{*}) = 0 \tag{Stationarity}
\]

\[
h_i(x_k^{*}) = d_{i,k} \tag{feasibility}
\]

for every \( k = 1, \ldots, T/t_n \), where \( f_k(x_k^{*}) = f(x_k^{*}, kt_n) \) and \( d_{i,k} = d_i(kt_n) \). The feasibility condition implies that

\[
\frac{1}{t_n} (h_i(x_k^{*}) - h_i(x_{k-1}^{*})) = \frac{d_{i,k} - d_{i,k-1}}{t_n}
\]

\[
\implies \nabla h_i(x_k^{*})^T \left( \frac{x_k^{*} - x_{k-1}^{*}}{t_n} \right) = \frac{d_{i,k} - d_{i,k-1}}{t_n} \tag{30}
\]

for some \( \tilde{x}_i^{T/t_n} = (1 - \alpha_i)x_k^{*} + \alpha_i x_{k-1}^{*} \) with \( \alpha_i \in [0,1] \), where the last implication is due to the differentiability of \( h_i(x) \) and the Mean Value Theorem. For simplicity and with a slight abuse of notation, define

\[
\mathcal{J}\{(\tilde{x}_i^{T/t_n})_{i=1}^{r}\} \equiv \begin{bmatrix} \nabla h_1(\tilde{x}_i^{T/t_n})^T \\ \vdots \\ \nabla h_r(\tilde{x}_i^{T/t_n})^T \end{bmatrix}, \quad d_k = \begin{bmatrix} d_{1,k} \\ \vdots \\ d_{r,k} \end{bmatrix} \tag{31}
\]

This implies that

\[
\mathcal{J}_h(\{(\tilde{x}_i^{T/t_n})_{i=1}^{r}\}) \left( \frac{x_k^{*} - x_{k-1}^{*}}{t_n} \right) = \frac{d_k - d_{k-1}}{t_n} \tag{32}
\]

Combining this equality with the stationarity condition leads to

\[
\mathcal{J}_h(\{(\tilde{x}_i^{T/t_n})_{i=1}^{r}\}) \nabla_x f_k(x_k^{*}) + \mathcal{J}_h(\{(\tilde{x}_i^{T/t_n})_{i=1}^{r}\}) \mathcal{J}_h(x_k^{*})^T \lambda_k^{T} + 2\alpha \left( \frac{d_k - d_{k-1}}{t_n} \right) = 0 \tag{33}
\]

Now, note that, due to Assumption 4, \( \sigma_{\min}(\mathcal{J}_h(x_k^{*})) \geq c \) for some universal constant \( c > 0 \). Therefore, for any \( y \) sufficiently close to \( x_k^{*} \), \( \mathcal{J}_h(y) \) remains full-row rank. Together with the definition of \( \{(\tilde{x}_i^{T/t_n})_{i=1}^{r}\} \) and the result of Lemma 2, this implies that \( \mathcal{J}_h(\{(\tilde{x}_i^{T/t_n})_{i=1}^{r}\}) \mathcal{J}_h(x_k^{*})^T \) is invertible for sufficiently small \( \Delta t \). Therefore,

\[
\lambda_k^{T} = -\left( \mathcal{J}_h(\{(\tilde{x}_i^{T/t_n})_{i=1}^{r}\}) \mathcal{J}_h(x_k^{*})^T \right)^{-1} \left( \mathcal{J}_h(\{(\tilde{x}_i^{T/t_n})_{i=1}^{r}\}) \nabla_x f_k(x_k^{*}) + 2\alpha \left( \frac{d_k - d_{k-1}}{t_n} \right) \right) \tag{34}
\]
Substituting this into the stationarity condition and performing the necessary simplifications lead to
\[
\frac{x_k^n - x_{k-1}^n}{t_n} = -\frac{1}{2\alpha} \left( I - J_h(x_k^n) \right)^\top \left( J_h(\{ x_{i,k}^n \}_{i=1}^r) J_h(x_k^n) \right) ^{-1} J_h(\{ x_{i,k}^n \}_{i=1}^r) \nabla f_k(x_k^n) \\
+ J_h(x_k^n) \left( J_h(\{ x_{i,k}^n \}_{i=1}^r) J_h(x_k^n) \right) ^{-1} \left( \frac{d_k - d_{k-1}}{t_n} \right) := g \left( \{ x_{i,k}^n \}_{i=1}^r, x_k^n, \frac{d_k - d_{k-1}}{t_n} \right) 
\] (35)

Consider the continuously differentiable function \( \bar{x}(t) \) that is introduced in Lemma 3. The above equality together with (27) implies that
\[
\lim_{n \to \infty} \sup_{0 \leq k \leq \lfloor \frac{t_n}{\Delta t} \rfloor} \left\| \dot{x}(kt_n) - g \left( \{ \bar{x}(kt_n) \}_{i=1}^r, \bar{x}(kt_n), \bar{d}(kt_n) \right) \right\| = 0 
\] (36)

Therefore, one can write
\[
\lim_{n \to \infty} \sup_{0 \leq k \leq \lfloor \frac{t_n}{\Delta t} \rfloor} \left\| \dot{x}(kt_n) - g \left( \{ \bar{x}(kt_n) \}_{i=1}^r, \bar{x}(kt_n), \bar{d}(kt_n) \right) \right\| 
\leq \lim_{n \to \infty} \sup_{0 \leq k \leq \lfloor \frac{t_n}{\Delta t} \rfloor} \left\| \dot{x}(kt_n) - g \left( \{ \bar{x}(kt_n) \}_{i=1}^r, \bar{x}(kt_n), \bar{d}(kt_n) \right) \right\| 
+ \lim_{n \to \infty} \sup_{0 \leq k \leq \lfloor \frac{t_n}{\Delta t} \rfloor} \left\| g \left( \{ \bar{x}(kt_n) \}_{i=1}^r, \bar{x}(kt_n), \bar{d}(kt_n) \right) - g \left( \{ \bar{x}(kt_n) \}_{i=1}^r, \bar{x}(kt_n), \bar{d}(kt_n) \right) \right\| (37)
\]

We present the following lemma.

**Lemma 5.** Given \( \{ \bar{x}_i \}_{i=1}^r, \bar{y}, \bar{z} \) with \( (\sum_{i=1}^r \| \bar{x}_i \|) + \| \bar{y} \| + \| \bar{z} \| \leq c_1 \) for some \( c_1 > 0 \), suppose that \( \sigma_{\min} \left( J(\{ x_i \}_{i=1}^r, J(y) \right) \geq c_2 \) for some \( c_2 > 0 \). Then, there exist universal constants \( L, r > 0 \), independent of \( \{ \bar{x}_i \}_{i=1}^r, \bar{y}, \bar{z} \), such that \( g(\{ \bar{x}_i \}_{i=1}^r, \bar{y}, \bar{z}) \) is locally \( L \)-Lipschitz continuous in a ball \( B = \{ (\{ x_i \}_{i=1}^r, y, z) \ | \ (\sum_{i=1}^r \| x_i - \bar{x}_i \|) + \| y - \bar{y} \| + \| z - \bar{z} \| \leq r \} \).

**Proof.** Due to continuous differentiability of \( J(x) \) and Lemma 1, it is easy to see that, independent of \( \{ \bar{x}_i \}_{i=1}^r, \bar{y}, \bar{z}, \) \( r \) can be chosen such that \( \sigma_{\min} \left( J(\{ x_i \}_{i=1}^r, J(y) \right) \geq c_2/2 \) for every \( \{ (x_i)_{i=1}^r, y, z) \in B(r) \). This observation, together with the definition of \( g(\cdot, \cdot, \cdot) \) in (35) can be used to complete the proof. The details are omitted for brevity.

According to Lemma 5, the function \( g(\cdot, \cdot, \cdot) \) is locally \( L \)-Lipschitz continuous on a ball with nonzero radius and centered at \( \left( \{ x_{i,k}^n \}_{i=1}^r, x_k^n, \frac{d_k - d_{k-1}}{t_n} \right) \) for every \( 0 \leq k \leq \lfloor \frac{t_n}{\Delta t} \rfloor \) and \( n = 1, \ldots, \infty \). This together with the definition of \( \{ \bar{x}_{i,k}^n \}_{i=1}^r \), the differentiability of \( d(t) \), and Lemma 3 implies that for sufficiently large \( n \) (or, equivalently, for sufficiently small \( t_n \)), there exists a constant \( c \) such that
\[
\left\| g \left( \{ \bar{x}(kt_n) \}_{i=1}^r, \bar{x}(kt_n), \bar{d}(kt_n) \right) - g \left( \{ \bar{x}_{i,k}^n \}_{i=1}^r, x_k^n, \frac{d_k - d_{k-1}}{t_n} \right) \right\| \leq c \left( \sum_{i=1}^r \| \bar{x}(kt_n) - \bar{x}_{i,k}^n \| + \| \bar{x}(kt_n) - x_k^n \| + \| \bar{d}(kt_n) - \left( \frac{d_k - d_{k-1}}{t_n} \right) \| \right) 
\leq c \left( (r + 1) \| \bar{x}(kt_n) - x_k^n \| + r \| \bar{x}((k-1)t_n) - x_{k-1}^n \| + r \| \bar{x}(kt_n) - \bar{x}((k-1)t_n) \| \right) 
+ \| \bar{d}(kt_n) - \left( \frac{d_k - d_{k-1}}{t_n} \right) \| (38)
\]
where we used the definition of \( \{\bar{x}^r_t\}_{r=1}^\tau \) and triangle inequality. Combining \( \text{(38)} \) and \( \text{(36)} \) with \( \text{(37)} \), Lemma 3 and Lemma 2 implies that

\[
\lim_{n \to \infty} \sup_{0 \leq k \leq \lfloor \frac{T}{\Delta t} \rfloor} \| \hat{\dot{x}}(kt_n) - g(\{\bar{x}(kt_n)\}_{i=1}^{\tau}, \bar{x}(kt_n), \bar{d}(kt_n)) \| = 0 \tag{39}
\]

Furthermore, due to Lemma 3, \( J(\hat{x}(t)) \) is full-row rank at every \( t \in [0, T] \) and therefore, \( g(\{\bar{x}(t)\}_{i=1}^{\tau}, \bar{x}(t), \bar{d}(t)) \) is continuous as a function of \( t \) in \([0, T]\\). Invoking Lemma 4 yields

\[
\hat{x}(t) = g(\{\bar{x}(t)\}_{i=1}^{\tau}, \bar{x}(t), \bar{d}(t)) \tag{40}
\]

at every \( t \in [0, T] \). This shows that \( \hat{x} : [0, T] \to \mathbb{R}^n \) is a solution to \( \text{(22)} \). Finally, due to Lemma 3, we have \( \sigma_{\min} (\mathcal{J}_k(\hat{x}(t))) \geq c \) for a universal constant \( c > 0 \). Therefore, Lemma 5 can be used to verify the existence of an open and connected set \( \mathcal{D} \) such that \( g(\cdot, \cdot, \cdot) \) is locally \( L \)-Lipschitz continuous on \( \mathcal{D} \) and \( (\hat{x}(t), t) \in \mathcal{D} \) for every \( t \in [0, T] \). Therefore, Theorem 2.2 in \[30\] can be used to show that \( \hat{x} : [0, T] \to \mathbb{R}^n \) is the unique solution to \( \text{(22)} \). This completes the proof. \( \square \)

A.3 Convergence Proof.

Lemma 6. There exists a universal constant \( \bar{t} \) such that for every \( \Delta t \leq \bar{t} \), there exists a pair \( (\Delta t, \{y^\Delta t_k\}_{k=0}^{\lfloor T/\Delta t \rfloor}) \) that satisfies the following statements:

- We have \( y^\Delta t_0 = x_0 \)
- \[
y^\Delta t_k = y^\Delta t_{k-1} + \Delta t \cdot g(\{y^\Delta t_i\}_{i=1}^{\tau}, y^\Delta t_i, b(s_k)) \tag{41}
\]
  for every \( k = 1, \ldots, \lfloor T/\Delta t \rfloor \).
- There exists a universal constant \( c_2 > 0 \) such that \( \|y^\Delta t_k - y^\Delta t_{k-1}\| \leq c_2 \Delta t \) for every \( k = 1, \ldots, \lfloor T/\Delta t \rfloor \).
- We have
  \[
  \lim_{\Delta t \to 0^+} \sup_{0 \leq k \leq \lfloor T/\Delta t \rfloor} \|y^\Delta t_k - x(s_k)\| = 0 \tag{42}
  \]
  where \( x : [0, T] \to \mathbb{R}^n \) is the unique solution to \( \text{(22)} \).

- We have \( \sigma_{\min}(\mathcal{J}(y^\Delta t_k)) \geq c_1 \) for some universal \( c_1 \) and every \( k = 1, \ldots, \lfloor T/\Delta t \rfloor \).

Proof. Note that \( \text{(41)} \) is the backward Euler iterations for \( \text{(22)} \). Furthermore, we have already shown the existence of a continuously differentiable and uniformly bounded solution to \( \text{(22)} \). The proof of the first three statements are immediately followed by the classical results on convergence of the backward Euler method; see \[42\] for more details. To verify the correctness of the last statement, note that we have shown in the previous subsection that the function \( \hat{x} : [0, T] \to \mathbb{R}^n \) introduced in Lemma 3 is indeed the unique solution to the proposed ODE and we have \( \mathcal{J}(\hat{x}(t)) \geq c \) for some universal \( c > 0 \) and every \( t \in [0, T] \). This together with \( \text{(42)} \) and Lemma 1 concludes the proof. \( \square \)

Proof of convergence: The main idea behind the proof is to show that, given any AKKT tuple \( (x_0, \Delta t, \{x^\Delta t_k\}_{k=1}^{\lfloor T/\Delta t \rfloor}) \), we have

\[
\lim_{\Delta t \to 0^+} \sup_{0 \leq k \leq \lfloor T/\Delta t \rfloor} \|y^\Delta t_k - x^\Delta t_k\| = 0 \tag{43}
\]
Establishing this equality together with Lemma 6 is enough to complete the proof.
It is evident from [35] that the AKKT tuple \( (x_0, \Delta t, \{x^\Delta t_k\}_{k=1}^T) \) should satisfy
\[
x^\Delta t_k = x^\Delta t_{k-1} + \Delta t g \left( \{ \tilde{x}^\Delta t_{i,k} \}_{i=1}^r, x^\Delta t_k, \left( \frac{d_k - d_{k-1}}{\Delta t} \right) \right)
\]
where \( \tilde{x}^\Delta t_{i,k} = (1 - \alpha_i) x^\Delta t_k + \alpha_i x^\Delta t_{k-1} \) with \( \alpha_i \in [0, 1] \) for every \( i = 1, \ldots, n \). This implies that
\[
x^\Delta t_k - y^\Delta t_k = x^\Delta t_{k-1} - y^\Delta t_{k-1} + \Delta t \left( g \left( \{ \tilde{x}^\Delta t_{i,k} \}_{i=1}^r, x^\Delta t_k, \left( \frac{d_k - d_{k-1}}{\Delta t} \right) \right) - g \left( \{ y^\Delta t_{i,k} \}_{i=1}^r, y^\Delta t_k, \tilde{d}(s_k) \right) \right)
\]
\[
= x^\Delta t_{k-1} - y^\Delta t_{k-1} + \Delta t \left( g \left( \{ \tilde{x}^\Delta t_{i,k} \}_{i=1}^r, x^\Delta t_k, \left( \frac{d_k - d_{k-1}}{\Delta t} \right) \right) - g \left( \{ y^\Delta t_{i,k} \}_{i=1}^r, y^\Delta t_k, \tilde{d}(s_k) \right) \right)
\]
\[
+ \Delta t \left( g \left( \{ y^\Delta t_{i,k} \}_{i=1}^r, y^\Delta t_{k-1}, \tilde{d}(s_k) \right) - g \left( \{ y^\Delta t_{i,k} \}_{i=1}^r, y^\Delta t_k, \tilde{d}(s_k) \right) \right)
\]
(45)

Define \( E_k = \|x^\Delta t_k - y^\Delta t_k\| \) as the error at time-step \( k \). Note that, due to the Lemmas 3, 6 and 5, as well as the construction of \( \{ \tilde{x}^\Delta t_{i,k} \}_{i=1}^r \), there exist universal constants \( k, c, t > 0 \) such that \( g(\cdot, \cdot, \cdot) \) is locally \( L \)-Lipschitz continuous in the balls
\[
B_1 = \left\{ \{ x_i \}_{i=1}^r, y, z \mid \left( \sum_{i=1}^r \|x^\Delta t_i - x_i\| \right) + \|\tilde{x}^\Delta t - y\| + \left\| \left( \frac{d_k - d_{k-1}}{\Delta t} \right) - z \right\| \leq \tilde{c} \right\}
\]
(46)
and
\[
B_2 = \left\{ \{ x_i \}_{i=1}^r, y, z \mid \left( \sum_{i=1}^r \|y^\Delta t_i - x_i\| \right) + \|\tilde{y}^\Delta t - y\| + \left\| \tilde{d}(s_k) - z \right\| \leq \tilde{c} \right\}
\]
(47)

On the other hand, note that
\[
\left( \sum_{i=1}^r \|x^\Delta t_i - y^\Delta t_i\| \right) + \|x^\Delta t_k - y^\Delta t_k\| + \left\| \left( \frac{d_k - d_{k-1}}{\Delta t} \right) - \tilde{d}(s_k) \right\|
\leq r \|x^\Delta t_{k-1} - y^\Delta t_{k-1}\| + (r + 1) \|x^\Delta t_k - y^\Delta t_k\| + \left\| \left( \frac{d_k - d_{k-1}}{\Delta t} \right) - \tilde{d}(s_k) \right\|
\leq r \|x^\Delta t_{k-1} - y^\Delta t_{k-1}\| + (r + 1) \|x^\Delta t_k - x^\Delta t_{k-1}\| + (r + 1) \|x^\Delta t_{k-1} - y^\Delta t_{k-1}\| + \left\| \left( \frac{d_k - d_{k-1}}{\Delta t} \right) - \tilde{d}(s_k) \right\|
= (2r + 1)E_{k-1} + (r + 1) \|x^\Delta t_k - x^\Delta t_{k-1}\| + \left\| \left( \frac{d_k - d_{k-1}}{\Delta t} \right) - \tilde{d}(s_k) \right\|
\]
(48)

Due to Lemma 2 and the twice differentiability of \( d(t) \), there exist universal constants \( \tilde{t}_1, c_1, c_2 > 0 \) such that [48] is upper bounded by
\[
(2r + 1)E_{k-1} + (r + 1)c_1 \Delta t + c_2 \Delta t^2
\leq (2r + 1)E_{k-1} + ((r + 1)c_1 + c_2) \Delta t
\]
(49)
provided that \( \Delta t \leq \tilde{t}_1 \). Similarly, Lemma 6 can be used to show the existence of universal constants \( c_3, \tilde{t}_2 > 0 \) such that
\[
(r + 1) \|y^\Delta t_{k-1} - y^\Delta t_k\| \leq c_3 \Delta t
\]
(50)
provided that \( \Delta t \leq \tilde{t}_2 \). Now, we prove the validity of [23] by proving the following statements:
1. There exists a universal constant $\ell_3$ such that for every $\Delta t \leq \ell_3$ and $k = 0, \ldots, T/\Delta t$, [49] and [50] will be upper bounded by $\bar{c}$ which is defined as the radius of the balls (46) and (47). This together with the locally $L$-Lipschitz continuity of $g(\cdot, \cdot, \cdot)$ in the balls $B_1$ and $B_2$ lead to

$$\|A\| \leq (2r + 1)L\Delta t E_{k-1} + ((r + 1)c_1 + c_2)L\Delta t^2$$  \hspace{1cm} (51a)

$$\|B\| \leq c_3 L\Delta t^2$$  \hspace{1cm} (51b)

Combining these inequalities with (45) results in the following recursive inequality:

$$E_k \leq (1 + (2r + 1)L\Delta t) E_{k-1} + ((r + 1)c_1 + c_2 + c_3) L\Delta t^2$$ \hspace{1cm} (52)

2. We have

$$\lim_{\Delta t \to 0^+} \sup_{0 \leq k \leq T/\Delta t} E_k = 0$$

We prove the first statement using an inductive argument on $k$. In particular, we show that if the following inequality holds

$$\Delta t \leq \ell_3 = \min \left\{ \ell_{t_1}, \ell_{t_2}, \sqrt{\frac{c}{((r + 1)c_1 + c_2 + c_3)L}}, \frac{(2r + 1)c}{((r + 1)c_1 + c_2 + c_3)(e^{(2r + 1)TL} - 1)} \right\}$$ \hspace{1cm} (53)

Then, [49] and [50] remain in the balls $B_1$ and $B_2$, respectively and hence, [52] holds for every $k = 0, \ldots, T/\Delta t$.

**Base case:** $k = 1$. Note that in this case, $E_0 = 0$ and therefore, based on [53], we have $\Delta t \leq \ell_1$ and $\Delta t \leq \ell_2$. This implies that both [49] and [50] are upper bounded by $\bar{c}$ and, based on [52], we have

$$E_1 \leq (1 + (2r + 1)L\Delta t) E_0 + ((r + 1)c_1 + c_2 + c_3) L\Delta t^2$$

$$= ((r + 1)c_1 + c_2 + c_3) L\Delta t^2$$

$$\leq \bar{c}$$ \hspace{1cm} (54)

where the last inequality is due to [53].

**Inductive step.** Suppose that we have

$$(2r + 1)E_{k-1} + ((r + 1)c_1 + c_2)\Delta t \leq \bar{c}$$

$$c_3\Delta t \leq \bar{c}$$ \hspace{1cm} (55)

for every $k = 0, \ldots, m - 1$. This implies that [52] holds for every $k = 1, \ldots, m$. With some algebra, one can verify that

$$E_m \leq ((r + 1)c_1 + c_2 + c_3) L\Delta t^2 \sum_{i=0}^{m-1} (1 + (2r + 1)L\Delta t)^i$$

$$\leq ((r + 1)c_1 + c_2 + c_3) L\Delta t^2 \cdot \frac{(1 + (2r + 1)L\Delta t)^m - 1}{(2r + 1)L\Delta t}$$

$$\leq \frac{(r + 1)c_1 + c_2 + c_3}{2r + 1} \left( 1 + (2r + 1)L\Delta t \right)^{T/\Delta t} - 1 \Delta t$$

$$\leq \frac{(r + 1)c_1 + c_2 + c_3}{2r + 1} \left( e^{(2r + 1)LT} - 1 \right) \Delta t$$

$$\leq \bar{c}$$ \hspace{1cm} (56)
which completes the proof of the first statement. On the other hand, the above analysis implies that
\[
\sup_{0 \leq k \leq T/\Delta t} E_k \leq \frac{(r+1)c_1 + c_2 + c_3}{2^r + 1} \left( e^{(2r+1)LT} - 1 \right) \Delta t
\]
assuming that $\Delta t \leq \bar{t}_3$. Due to the fact that $\bar{t}_3 > 0$ and is independent of $\Delta t$, we have
\[
\lim_{\Delta t \to 0^+} \sup_{0 \leq k \leq T/\Delta t} E_k = 0
\]
thereby completing the proof of the convergence.

\section*{B Proof of Auxiliary Lemmas}

\subsection*{B.1 Proof of Lemma 2}

Note that $f(x, t)$ is uniformly bounded from below. Furthermore, for every AKKT tuple $(x_0, \Delta t, \{x^\Delta t\}_{k=0}^{T/\Delta t})$, the sequence $\{x^\Delta t\}_{k=0}^{T/\Delta t}$ is assumed to be uniformly bounded. This implies that $\|x^\Delta t - x^\Delta t_{k-1}\| = O(\sqrt{\Delta t})$. Therefore, due to Assumption 4 and the fact that $\mathcal{J}(x)$ is continuously differentiable, one can invoke Lemma [4] to show that there exist universal constants $t, c_1, c_2 > 0$ such that the following statements hold, provided that $\Delta t \leq \bar{t}$:

1. $\sigma_{\min}(\mathcal{J}(\bar{x}^\Delta t_{i,k}), \mathcal{J}(x^\Delta t_{k})) \geq c_1$ and hence, the function $g \left( \{\bar{x}^\Delta t_{i,k}\}_{i=1}^r, x^\Delta t_k, \left( \frac{d_k - d_{k-1}}{\Delta t} \right) \right)$ is well-defined (note that the function $g(\cdot, \cdot, \cdot)$ is defined in [43]). Furthermore, the sequence $\{\bar{x}^\Delta t_{i,k}\}_{i=1}^r$ is constructed similar to the one defined in (32).

2. We have
\[
\left\| g \left( \{\bar{x}^\Delta t_{i,k}\}_{i=1}^r, x^\Delta t_k, \left( \frac{d_k - d_{k-1}}{\Delta t} \right) \right) \right\| \leq c_2
\]

Similar to [43], one can verify that the following equality holds:
\[
\frac{x^\Delta t_k - x^\Delta t_{k-1}}{\Delta t} = g \left( \{\bar{x}^\Delta t_{i,k}\}_{i=1}^r, x^\Delta t_k, \left( \frac{d_k - d_{k-1}}{\Delta t} \right) \right)
\]

Combined with [59], this implies that $\|x^\Delta t_k - x^\Delta t_{k-1}\| \leq c_2 \Delta t$ and the proof is complete.

\subsection*{B.2 Proof of Lemma 3}

Consider a sequence $\{t_n\}_{n=1}^\infty$ such that $t_n > 0$ and $\lim_{n \to \infty} t_n = 0$. Furthermore, without loss of generality, we assume that $T/t_n$ is a natural number for every $n = 1, \ldots, \infty$. Given any $n$, consider an AKKT tuple $(x_0, t_n, \{x^\Delta t_k\}_{k=0}^n)$ and define a vector-valued function $\bar{x}_{t_n} : [0, T] \to \mathbb{R}^n$ whose $i^{th}$ element is the spline interpolation of the $i^{th}$ elements of the vectors $\{x^0_0, x^0_1, \ldots, x^0_{T/t_n}\}$. Notice that this interpolation can be made in such a way that $\bar{x}_{t_n}$ is continuously differentiable.

We prove this lemma by showing that there exists a continuously differentiable function $\bar{x}$ and a subsequence $\{\bar{x}_{t_{n_r}}\}_{r=1}^\infty$ of $\{\bar{x}_{t_n}\}_{n=1}^\infty$ such that $\{\bar{x}^r_{t_n}\}_{r=1}^\infty$ and $\{\bar{x}^r_{t_{n_r}}\}_{r=1}^\infty$ converge uniformly to $\bar{x}$ and $\bar{x}_{t_{n_r}}$, respectively.
We begin the proof by observing that for any solution \( \mathbf{x} = \{ \tilde{x}_{tn_k} \mid n = 1, \ldots, \infty \} \). \( \mathcal{X} \) is uniformly bounded (due to Assumption 4) and equicontinuous. Therefore, the Arzelà - Ascoli theorem can be invoked to show the existence of a uniformly convergent subsequence \( \{ \tilde{x}_{tn_k} \}_{k=1}^{\infty} \). Let \( \tilde{x} : [0, T] \to \mathbb{R}^n \) be the limit of \( \{ \tilde{x}_{tn_k} \}_{k=1}^{\infty} \). Now, consider the sequence \( \{ \tilde{x}_{tn_k} \}_{k=1}^{\infty} \). Notice that, due to the construction, \( \{ \tilde{x}_{tn_k} \}_{k=1}^{\infty} \) is continuous. Consider the class of functions \( \mathcal{X} = \{ \tilde{x}_{tn_k} \mid k = 1, \ldots, \infty \} \). Similar to the previous case, \( \mathcal{X} \) is uniformly bounded and equicontinuous. Therefore, another application of Arzelà - Ascoli theorem implies that \( \{ \tilde{x}_{tn_k} \}_{k=1}^{\infty} \) has a subsequence \( \{ \tilde{x}_{tn_{k_r}} \}_{r=1}^{\infty} \) that converges uniformly to a function \( y : [0, T] \to \mathbb{R}^n \). Since \( \{ n_r \}_{r=1}^{\infty} \subseteq \{ n_k \}_{k=1}^{\infty} \), we have that \( \{ \tilde{x}_{tn_r} \}_{r=1}^{\infty} \) converges uniformly to \( \tilde{x} \). Therefore, due to Theorem 7.17 of [42], we have \( \tilde{x} = y \).

Finally, recall that \( \{ x_{kn}^* \}_{n=1}^{\infty} \) is uniformly bounded and there exists a universal constant \( c \) such that \( J_h(x_{kn}^*) \geq c \) for every \( k = 0, \ldots, T/t_n \) and \( n = 1, \ldots, \infty \). This implies that the function sequence \( \{ \tilde{x}_{tn_r} \}_{r=1}^{\infty} \) is also uniformly bound and since they converge uniformly to \( \tilde{x} \), one can invoke Lemma 3 to verify the existence of a universal \( c' > 0 \) such that \( c \geq c' \) and \( J(\tilde{x}(t)) \geq c' \) for every \( t \in [0, T] \).

The details are omitted for brevity.

\[ \square \]

### C Proof of Proposition 1

We begin the proof by observing that for any solution \( x : [0, T] \to \mathbb{R}^n \), we have

\[
\frac{dh(x(t))}{dt} = J_h(x(t))x'(t)
\]

\[
= -\frac{1}{n} J_h(x(t)) [I - J_h(x(t))\mathcal{L}(J_h(x(t))J_h(x(t))^\top)^{-1}J_h(x(t))] \nabla f(x(t))
\]

\[
= 0
\]

and

\[
\frac{df(x(t))}{dt} = \nabla f(x(t))^\top x'(t)
\]

\[
= -\frac{1}{n} \nabla f(x(t))^\top [I - J_h(x(t))\mathcal{L}(J_h(x(t))J_h(x(t))^\top)^{-1}J_h(x(t))] \nabla f(x(t))
\]

\[
= -\frac{1}{n} \left( \| \nabla f(x(t)) \|^2 - \| \text{proj}_{\text{Ker} J_h(x(t)) \nabla f(x(t))} \|^2 \right) \leq 0
\]

where “\( \text{proj} \)” stands for orthogonal projection.

By definition of a strict local minimum, there exists \( r > 0 \) such that

\[
\forall x \in \mathbb{R}^n, \quad (0 < \| x - x^* \| \leq r \quad \text{and} \quad h(x) = d) \quad \implies \quad f(x^*) < f(x).
\]

(63)

It suffices to prove (62) for \( 0 < \epsilon \leq r \) so we choose such an \( \epsilon \). Consider

\[
m := \inf_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad h(x) = d \quad \text{and} \quad \| x - x^* \| = \epsilon
\]

(64)

We distinguish two cases.

1) The feasible set of (64) is empty. Let \( \delta := \epsilon/2 \) and consider an initial condition such that \( \| x(0) - x^* \| \leq \delta \) and \( h(x(0)) = d \). Due to equation (61), for all \( 0 \leq t \leq T \), it holds that \( h(x(t)) = d \). We reason by contradiction and assume that \( \| x(t) - x^* \| > \epsilon \) for some time \( 0 \leq t' \leq T \). Since \( \| x(t) - x^* \| \) is continuous and \( \| x(0) - x^* \| < \epsilon < \| x(t') - x^* \| \), there exists \( 0 \leq t'' < t' \) such that \( \| x(t'') - x^* \| = \epsilon \).
Together with \( h(x(t')) = d \), this implies that \( x(t') \) is a feasible point of (64). This is a contradiction, and thus \( \|x(t) - x^*\| \leq \epsilon \) for all \( 0 \leq t \leq T \).

2) The feasible set of (64) is not empty. The feasible is the non-empty intersection of a compact set and a closed set (\( h \) is continuous). It is thus compact. Since \( f \) is continuous, the optimization (64) reaches its infimum, say at \( \hat{x} \). According to (63), it holds that \( m = f(\hat{x}) > f(x^*) \). Using the definition of the continuity of \( f \), there exists \( \delta > 0 \) such that

\[
M := \sup_{x \in \mathbb{R}^n} f(x) \text{ s.t. } h(x) = d \text{ and } \|x - x^*\| \leq \delta
\]

satisfies \( M \leq (f(x^*) + m)/2 \). It holds that \( \delta < \epsilon \) or else the feasible set of (64) is a subset of the feasible set of (65), and then \( m \leq M \leq (f(x^*) + m)/2 < (m + m)/2 \leq m \). Now, consider an initial condition such that \( \|x(0) - x^*\| \leq \delta \) and \( h(x(0)) = d \). Due to equations (61) and (62), for all \( 0 \leq t \leq T \), we have that \( h(x(t)) = d \) and \( f(x(t)) \leq f(x(0)) \leq (f(x^*) + m)/2 \), where the second inequality holds because \( x(0) \) is feasible for (65). We reason by contradiction and assume that \( \|x(t) - x^*\| > \epsilon \) for some time \( 0 \leq t' \leq T \). Since \( \|x(t) - x^*\| \) is continuous and \( \|x(0) - x^*\| < \epsilon < \|x(t') - x^*\| \), there exists \( 0 \leq t'' \leq t' \) such that \( \|x(t'') - x^*\| \geq \epsilon \). Together with \( h(x(t'')) = d \), this implies that \( x(t'') \) is a feasible point of (64). This leads to the contradiction that \( m \leq f(x(t'')) \leq (f(x^*) + m)/2 < m \). Thus \( \|x(t) - x^*\| \leq \epsilon \) for all \( 0 \leq t \leq T \).

### D Proof of Proposition 2

A continuous local trajectory \( x : [0, 2\pi] \rightarrow \mathbb{R} \) satisfies

\[
x(0) \in \{y_1, y_2, y_3\}, \quad \dot{x} = -\frac{1}{\alpha} \nabla_x f(x, t),
\]

which, after the change of variable \( y := x - \beta \sin(t) \), reads

\[
y(0) \in \{y_1, y_2, y_3\}, \quad \dot{y} = -\frac{1}{\alpha} g'(y) - \beta \cos(t).
\]

We first show by contradiction that there exists \( t \in [0, 2\pi] \) such that \( y(t) \geq m_2 \). Assume that \( y(t) < m_2 \) for all \( t \in [0, 2\pi] \). Then, for all \( t \in [0, 2\pi] \), it holds that

\[
\dot{y} = -\frac{1}{\alpha} g'(y) - \beta \cos(t) \geq -\frac{C}{\alpha} - \beta \cos(t).
\]

Thus, we have

\[
y(t_2) \geq -\frac{C}{\alpha} (t_2 - t_1) - \beta (\sin(t_2) - \sin(t_1)) + y(t_1).
\]

We next show by contradiction that \( y(t_1) \geq m_1 \). Assume that \( y(t_1) < m_1 < y_1 \leq y(0) \). Let \( t_3 \) denote the maximal element of the compact set \( [0, t_1] \cap y^{-1}(m_1) \) where \( y^{-1}(b) := \{a \in \mathbb{R} \mid y(a) = b\} \). Thus \( y(t) \leq y(t_3) \) for all \( t \in [t_3, t_1] \). As a result, \( y'(t_3) \leq 0 \). Together with \( y'(t_3) = -1/\alpha g'(m_1) - \beta \cos(t_3) = \beta (1 - \cos(t_3)) \), this implies that \( t_3 = 0 \) or \( t_3 = 2\pi \). This is in contradiction with \( 0 < t_3 < t_1 < \pi \).

Now that we have proven that \( y(t_1) \geq m_1 \), equation (69) implies that \( y(t_2) \geq m_2 \). This is a contradiction. Therefore there exists \( t \in [0, 2\pi] \) such that \( y(t) \geq m_2 \). Using the same argument as in the previous paragraph, we obtain \( y(2\pi) \geq m_2 \). As a result, \( x(2\pi) = y(2\pi) - \beta \sin(2\pi) \geq m_2 \) as well.
Notice that $f(x, T) = g(x)$. We thus consider the initial value problem
\[
\dot{x} = -\frac{1}{\alpha} g'(\bar{x}) \quad \text{(70)}
\]
\[
\bar{x}(0) = x(2\pi) \quad \text{(71)}
\]
Since $g'$ is continuously differentiable, it is Lipschitz on any interval $[a, b]$ of $\mathbb{R}$. The existence of a local continuously differential solution is then guaranteed by the Picard-Lindelöf Theorem [30, Theorem 3.1]. Consider a maximal solution, that is to say $\bar{x} : [0, T] \to \mathbb{R}$ where $T \in \mathbb{R}$ or $T = +\infty$. We next show by contradiction that the latter holds. Without loss of generality, assume that $x(2\pi) < y_3$. We know that $g'(x) < 0$ for all $x(2\pi) < x < y_3$. As a result, $\bar{x}$ is an increasing function on $[0, t]$. It is also upper bounded by $y_3$. Indeed, it is upper bounded by any $y_3 + \epsilon$ for $\epsilon > 0$ small enough, so that $g'(y_3 + \epsilon) > 0$ (and then using the same argument from one of the above paragraphs for the third time). As a result, $\bar{x}$ has limit $\bar{x}(t)$ as $t$ converges towards $T$ from below. Since $g'$ is continuous, $\bar{x} : [0, T] \to \mathbb{R}$ is a solution to initial value problem, which is a contradiction. As a result, $T = +\infty$.

We next show that $\bar{x}''(t) = y_3$. Since $\bar{x}''(t) = -1/\alpha g'\left(\bar{x}(t)\right)$ for all $t \geq 0$, the derivative $\bar{x}$ has limit equal to $\bar{x}'(t) = -1/\alpha g'(\bar{x}(t))$. Since $\bar{x}''(t) \geq 0$ for all $t \geq 0$, it holds that $\bar{x}'(t) \geq 0$. Assume that $\bar{x}'(t) > 0$. Then there exists $T_0 \geq 0$ such that, for all $t \geq T_0$, we have $\bar{x}'(t) \geq \bar{x}'(t)/2$. Then $\bar{x}(t) \geq \bar{x}(T_0) + \bar{x}'(t)(t - T_0)/2$ diverges, which is a contradiction. Thus $\bar{x}'(t) = -1/\alpha g'(\bar{x}(t)) = 0$, which implies that $\bar{x}'(t)$ is equal to $y_1$, $y_2$, or $y_3$. Since $\bar{x}'(t) \geq \bar{x}'(0) = x(2\pi) \geq m_2 > y_2 > y_1$, we conclude that $\bar{x}'(t) = y_3$.

E Additional derivations

To streamline the presentation, we assume in this section that the optimization problem has only one equality constraint. The arguments made in the sequel can be naturally be extended to the case with multiple constraints. Suppose that $f(x, t)$ and $b(t)$ are time-invariant (we denote them by $f(x)$ and $b$ for simplicity). With this assumption, [15] and [22] are reduced to
\[
\text{min}_{x} \quad f(x) \quad \text{subject to} \quad h(x) = b \quad \text{(72)}
\]
and
\[
\dot{x} = -\frac{1}{\alpha} \left( \nabla_x f(x, t) - \frac{(\nabla h(x), \nabla_x f(x, t))}{\|
abla h(x)\|^2} \nabla h(x) \right) \quad \text{(73)}
\]
respectively.

Definition 5 (Locally stability on manifold). Consider a time-invariant dynamical system $\dot{x} = p(x)$ with equilibrium $\bar{x}$. We say that $\dot{x}$ is locally stable on manifold $\mathcal{U}$ if there exists a ball $\mathcal{B}(\bar{x}, \epsilon)$ centered at $\bar{x}$ with radius $\epsilon > 0$ such that for any $x_0 \in \mathcal{B}(\bar{x}, \epsilon) \cap \mathcal{U}$, the solution $z : [0, \infty) \to \mathbb{R}^n$ of $\dot{x} = p(x)$ with the initial condition $z(0) = x_0$ converges to $\bar{x}$.

Proposition 3. Suppose that $z$ satisfies the following conditions:
\[
\nabla h(z) \neq 0 \quad \text{(75)}
\]
\[
\nabla f(z) + \mu \nabla h(z) = 0 \quad \text{(76)}
\]
\[
\nabla^2 f(z) > 0 \quad \text{(77)}
\]
For some scalar $\mu$. Then,
1. $z$ is an equilibrium for (74).

2. $z$ is locally stable on the manifold $\mathcal{T} = \{y \mid h(y) = b\}$.

Intuitively, the above proposition says that the strict local minimum $z$ is stable along any feasible direction. This implies that there is no hope in escaping the strict local minima of (72) if the problem is time-invariant. In the following, we give a sketch of the proof. For a manifold $\mathcal{X}$, let $\dim(\mathcal{X})$ denote its dimension. First, we present the definition of stable manifold.

**Definition 6.** Consider a time-invariant dynamical system $\dot{x} = p(x)$ with equilibrium $x(t) = \bar{x}$ for every $t \in [0, \infty)$. The **stable manifold** $S_p(z)$ is defined as the set of all $x_0$ such that any solution $z : [0, \infty) \to \mathbb{R}^n$ with $z(0) = x_0$ to $\dot{x} = p(x)$ converges to $\bar{x}$.

Next, we present a modified version of the Stable Manifold Theorem.

**Theorem 4** (Stable Manifold Theorem). Given a time-invariant dynamical system $\dot{x} = p(x)$, consider $J_p(z)$ and let $V_s$ denote the subspace spanned by the negative eigenvalues of $J_p(z)$.

- $V_s$ is tangent to the stable manifold at $z$.
- The dimension of the stable manifold $S_p(z)$ in a neighborhood of $z$ is lower bounded by that of $V_s$.

Let the right hand side of (74) be denoted by $p(x)$. The next lemma is crucial in our proof.

\[
\dot{x} = -\frac{1}{\alpha} \left( \nabla f(x) - \frac{\langle \nabla h(x), \nabla_x f(x) \rangle}{\|\nabla h(x)\|^2} \nabla h(x) \right) := p(x) \quad (78)
\]

By Stable Manifold Theorem [43], it is enough to show that $T \subseteq V_s$, where $V_s$ is the subspace spanned by the eigenvectors of $J_p(z)$ corresponding to its negative eigenvalues.

**Lemma 7.** We have $S_p(z) \subseteq T := \{y : h(y) = b\}$.

**Proof.** Suppose there exists $\bar{y} \in S_p(z)$ such that $\bar{y} \not\in T$. This implies that $h(\bar{y}) \neq b$. Since the set $\bar{T} := \{y : h(y) = h(\bar{y})\}$ is invariant for (74), we must have $h(z) = h(\bar{y}) \neq b$, which is a contradiction. \qed

**Lemma 8.** For $p(x)$ defined as (78), we have

\[
J_p(z) = -\frac{1}{\alpha} \left( \nabla^2 f(z) + \mu \nabla^2 h(z) \right) \left( I - \frac{\nabla h(z) \nabla h(z)^\top}{\|\nabla h(z)\|^2} \right) \quad (79)
\]

where $\mu$ is defined as in Proposition 3. Moreover, $J_p(z)$ has $n - 1$ negative eigenvalues and 1 zero eigenvalue.

**Proof.** Due to (76), one can write

\[
\mu = -\frac{\langle \nabla h(z), \nabla f(z) \rangle}{\|\nabla h(z)\|^2} := r(z) \quad (80)
\]

On the other hand, one can write

\[
p(x) = -\frac{1}{\alpha} (\nabla f(x) + r(x) \nabla h(x)) \quad (81)
\]
Therefore, we have
\[
J_p(z) = -\frac{1}{\alpha} \left( \nabla^2 f(z) + \mu \nabla^2 h(z) + \nabla r(z) \nabla h(z)^\top \right)
\] (82)

Note that \(\nabla^2 f(z) + \mu \nabla^2 h(z)\) is symmetric and we have \(\nabla^2 f(z) + \mu \nabla^2 h(z) \succ 0\) due to (77). On the other hand, one can write
\[
\nabla r(z) = -\frac{\nabla^2 f(z) \nabla h(z)}{\|\nabla h(z)\|^2} - \frac{\nabla^2 h(z) \nabla f(z)}{\|\nabla h(z)\|^2} + \frac{2(\nabla h(z)^\top \nabla f(z)) \nabla^2 h(z) \nabla h(z)}{\|\nabla h(z)\|^4}
\] (83)

which implies that
\[
\nabla r(z) \nabla h(z)^\top = -\frac{\nabla^2 f(z) \nabla h(z) \nabla h(z)^\top}{\|\nabla h(z)\|^2} - \frac{\nabla^2 h(z) \nabla f(z) \nabla h(z)^\top}{\|\nabla h(z)\|^2} + \frac{2(\nabla h(z)^\top \nabla f(z)) \nabla^2 h(z) \nabla h(z)^\top}{\|\nabla h(z)\|^4}
\]
\[
= - \frac{\nabla^2 f(z) \nabla h(z) \nabla h(z)^\top}{\|\nabla h(z)\|^2} + \mu \cdot \frac{\nabla^2 h(z) \nabla h(z) \nabla h(z)^\top}{\|\nabla h(z)\|^2} - 2\mu \cdot \frac{\nabla^2 h(z) \nabla h(z) \nabla h(z)^\top}{\|\nabla h(z)\|^2}
\]
\[
= - (\nabla^2 f(z) + \mu \nabla^2 h(z)) \nabla h(z) \nabla h(z)^\top
\] (84)

where in the second equality, we used (80) and (76). Combining this with (82) results in
\[
J_p(z) = -\frac{1}{\alpha} \left( \nabla^2 f(z) + \mu \nabla^2 h(z) \right) \left( I - \frac{\nabla h(z) \nabla h(z)^\top}{\|\nabla h(z)\|^2} \right)
\] (85)

Note that \(z \in T\) and \(\dim(T \cap B(z, \epsilon)) = n - 1\), where \(B(z, \epsilon)\) is a ball centered at \(z\) with sufficiently small radius \(\epsilon > 0\). On the other hand, according to Theorem 4 and Lemma 8 \(\epsilon\) can be chosen such that we have \(\dim(S_p(z) \cap B(z, \epsilon)) \geq n - 1\). Invoking Lemma 7 implies that
\[
\dim(T \cap B(z, \epsilon)) = \dim(S_p(z) \cap B(z, \epsilon))
\]
\[
T \cap B(z, \epsilon) \subseteq S_p(z) \cap B(z, \epsilon)
\] (86)

Therefore, there exists \(\tilde{\epsilon} \leq \epsilon\) such that \(S_p(z) \cap B(z, \tilde{\epsilon}) = T \cap B(z, \tilde{\epsilon})\) and the proof is complete. \(\square\)

It is worthwhile to analyze the local stability for the time-varying system:
\[
\dot{x} = -\frac{1}{\alpha} \left( \nabla_x f(x, t) - \frac{\nabla h(x) \nabla_x f(x, t)}{\|\nabla h(x)\|^2} \nabla h(x) \right) + \dot{\epsilon} \frac{\nabla h(x)}{\|\nabla h(x)\|^2} := p(x, t)
\] (87)

**Lemma 9.** For \(p(x, t)\) defined as (87), we have
\[
J_p(z) = -\frac{1}{\alpha} \left( \nabla^2 f(z) + \mu \nabla^2 h(z) \right) \left( I - \frac{\nabla h(z) \nabla h(z)^\top}{\|\nabla h(z)\|^2} \right) + \frac{\nabla^2 h(z)}{\|\nabla h(z)\|^2} \left( I - \frac{2\nabla h(z) \nabla h(z)^\top}{\|\nabla h(z)\|^2} \right) \dot{\epsilon}
\] (88)

where \(\mu\) is defined as in Proposition 3.

**Proof.** Proof is similar to that of Lemma 8 and is omitted for brevity. \(\square\)

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