On the Absence of Spurious Local Minima in Nonlinear Low-Rank Matrix Recovery Problems

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Abstract

The restricted isometry property (RIP) is a well-known condition that guarantees the absence of spurious local minima in low-rank matrix recovery problems with linear measurements. In this paper, for general low-rank matrix recovery problems with nonlinear measurements, a novel property named bound difference property (BDP) is introduced. Using RIP and BDP jointly, we propose a new criterion to certify the nonexistence of spurious local minima in the rank-1 case, and prove that it leads to a much stronger theoretical guarantee than the existing bounds on RIP.

1. Introduction

The *low-rank matrix recovery problem* plays a central role in many machine learning problems, such as recommendation systems (Koren et al., 2009) and motion detection (Zhou et al., 2013; Fattahi and Sojoudi, 2020). It also appears in engineering problems, such as power system state estimation (Zhang et al., 2018). The goal of this problem is to recover an unknown low-rank matrix $M^* \in \mathbb{R}^{n\times n}$ from certain measurements of the entries of M^*.

The basic form of the low-rank matrix recovery problem is the symmetric and noiseless one with linear measurements and the quadratic loss. The linear measurements can be represented by a linear operator $\mathcal{A} : \mathbb{R}^{n\times n} \to \mathbb{R}^m$ given by

$$\mathcal{A}(M) = (\langle A_1, M\rangle, \dots, \langle A_m, M\rangle)^T.$$

The ground-truth matrix M^* is assumed to be symmetric and positive semidefinite with $\operatorname{rank}(M^*) \le r$. The recovery problem can be formulated as follows:

$$\begin{aligned} \min \quad & \frac{1}{2}\|\mathcal{A}(M) - d\|^2 \\ \text{s.t.} \quad & \operatorname{rank}(M) \le r, \\ & M \succeq 0, \quad M \in \mathbb{R}^{n\times n}, \end{aligned}$$

where $d = \mathcal{A}(M^*)$. By factoring the decision variable M into its low-rank factors XX^T, the above problem can be rewritten as the unconstrained problem:

$$\min_{X\in\mathbb{R}^{n\times r}} \left\{ \frac{1}{2}\|\mathcal{A}(XX^T) - d\|^2 \right\}. \tag{1}$$

The optimization (1) associated with different machine learning applications is commonly solved by local search methods, such as the stochastic gradient descent (Ge et al., 2015), due to their ability in handling large-scale problems. Since (1) is generally nonconvex, local search methods may converge to a spurious (nonglobal) local minimum. To provide theoretical guarantees on the performance of local search methods for the low-rank matrix recovery, several papers have developed various conditions under which the optimization (1) is free of spurious local minima. In the following, we will briefly review the state-of-the-art results on this problem.

Given a linear operator $\mathcal{A}$, define its corresponding quadratic form $\mathcal{Q} : \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \to \mathbb{R}$ as

$$[\mathcal{Q}](K, L) = \langle \mathcal{A}(K), \mathcal{A}(L)\rangle, \tag{2}$$

for all $K, L \in \mathbb{R}^{n\times n}$.

Definition 1 (Recht et al. (2010)) *A quadratic form $\mathcal{Q} : \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \to \mathbb{R}$ satisfies the* restricted isometry property *(RIP) of rank $2r$ for a constant $\delta \in [0, 1)$, denoted as δ-RIP_{2r}, if*

$$(1 - \delta)\|K\|_F^2 \le [\mathcal{Q}](K, K) \le (1 + \delta)\|K\|_F^2$$

for all matrices $K \in \mathbb{R}^{n\times n}$ with $\mathrm{rank}(K) \le 2r$.

Ge et al. (2017) showed that the problem (1) has no spurious local minima if the quadratic form $\mathcal{Q}$ satisfies δ-RIP_{2r} with $\delta < 1/5$. Zhang et al. (2019) strengthened this result for the special case of $r = 1$ by showing that δ-RIP_{2r} with $\delta < 1/2$ is sufficient to guarantee the absence of spurious local minima for (1). It also provided an example with a spurious local minimum in case of $\delta = 1/2$ to support the tightness of the bound.

The purpose of this paper is to study the existence of spurious local minima for the general low-rank matrix recovery problem

$$\min_{X\in\mathbb{R}^{n\times r}} f(XX^T), \tag{3}$$

where $f : \mathbb{R}^{n\times n} \to \mathbb{R}$ is an arbitrary function induced by nonlinear measurements. This problem has immediate applications such as recommendation systems in which each user provides binary (like/dislike) observations (see Davenport et al. (2014); Ghadermarzy et al. (2019)). In this paper, f is always assumed to be twice continuously differentiable. The problem (1) is a special case of (3) by choosing

$$f(M) = \frac{1}{2}\|\mathcal{A}(M) - d\|^2. \tag{4}$$

In the case with linear measurements, note that $f(M^*) = 0$ and therefore M^* is a global minimizer of f. In other words, there are often infinitely many minimizers for f, but the goal is to find the ground-truth low-rank solution M^*. Similar to the linear measurement case, we assume that the problem (3) has a ground truth $M^* = ZZ^T$ with $Z \in \mathbb{R}^{n\times r}$ that is a global minimizer of f.

The Hessian $\nabla^2 f(M)$ of the function f in (3) can be also regarded as a quadratic form whose action on any two matrices $K, L \in \mathbb{R}^{n\times n}$ is given by

$$[\nabla^2 f(M)](K, L) = \sum_{i,j,k,l=1}^{n} \frac{\partial^2 f}{\partial M_{ij}\partial M_{kl}}(M)K_{ij}L_{kl}.$$

If f is considered to be equal to the special function in (4), then its corresponding Hessian $\nabla^2 f(M)$ becomes exactly the quadratic form $\mathcal{Q}$ defined in (2). Therefore, we naturally extend the definition of the δ-RIP$_{2r}$ property for quadratic forms given in Definition 1 to general functions f by restricting its Hessian.

Definition 2 *A twice continuously differentiable function* $f : \mathbb{R}^{n\times n} \to \mathbb{R}$ *satisfies the* restricted isometry property *of rank* $2r$ *for a constant* $\delta \in [0, 1)$*, denoted as* δ-RIP$_{2r}$*, if*

$$(1-\delta)\|K\|_F^2 \leq [\nabla^2 f(M)](K,K) \leq (1+\delta)\|K\|_F^2 \tag{5}$$

for all matrices $M, K \in \mathbb{R}^{n\times n}$ *with* $\operatorname{rank}(M) \leq 2r$ *and* $\operatorname{rank}(K) \leq 2r$.

It is still unknown whether the δ-RIP$_{2r}$ condition could lead to the nonexistence of spurious local minima. However, Li et al. (2019) proved that the problem (3) has no spurious local minima under a stronger condition, named δ-RIP$_{2r,4r}$ with $\delta < 1/5$, as defined below.

Definition 3 *A twice continuously differentiable function* $f : \mathbb{R}^{n\times n} \to \mathbb{R}$ *satisfies the* restricted isometry property *of rank* $(2r, 4r)$ *for a constant* $\delta \in [0, 1)$*, denoted as* δ-RIP$_{2r,4r}$*, if*

$$(1-\delta)\|K\|_F^2 \leq [\nabla^2 f(M)](K,K) \leq (1+\delta)\|K\|_F^2$$

for all matrices $M, K \in \mathbb{R}^{n\times n}$ *with* $\operatorname{rank}(M) \leq 2r$ *and* $\operatorname{rank}(K) \leq 4r$.

For the general recovery problem (3) with $r = 1$, previous results in Zhang et al. (2019) and Li et al. (2019) both have serious limitations. The bound $\delta < 1/2$ given in Zhang et al. (2019) is proved to be tight in the case when f is generated by linear measurements, but it is not applicable to nonlinear measurements. The bound $\delta < 1/5$ given in Li et al. (2019) can be applied to a general function f, but it is not tight even in the linear case. To address these issues, we develop a new criterion to guarantee the absence of spurious local minima in (3) for a general function f in the rank-1 case, which is more powerful than the previous conditions. Unlike the bound given in Li et al. (2019), our new criterion completely depends on the properties of the Hessian of the function f applied to rank-2 matrices, rather than rank-4 matrices. Note that the rank-1 case has applications in many problems, such as motion detection (Fattahi and Sojoudi, 2020) and power system state estimation (Zhang et al., 2018).

Notations I_n is the identity matrix of size $n \times n$, and $\operatorname{diag}(a_1, \dots, a_n)$ is the diagonal matrix whose diagonal entries are $a_1, \dots, a_n$. $\mathbf{A} = \operatorname{vec} A$ is the vector obtained from stacking the columns of a matrix A. Given a vector $\mathbf{A} \in \mathbb{R}^{n^2}$, define its symmetric matricization $\operatorname{mat_S} \mathbf{A} = (A + A^T)/2$, where $A \in \mathbb{R}^{n\times n}$ is the unique matrix satisfying $\mathbf{A} = \operatorname{vec} A$. $A \otimes B$ is the Kronecker product of A and B, which satisfies the well-known identity:

$$\operatorname{vec}(AXB^T) = (B \otimes A)\operatorname{vec} X.$$

For two matrices A, B of the same size, $\langle A, B\rangle = \operatorname{tr}(A^T B) = \langle \operatorname{vec} A, \operatorname{vec} B\rangle$. $\|v\|$ is the Euclidean norm of the vector v and $\|A\|_F = \sqrt{\langle A, A\rangle}$ is the Frobenius norm of the matrix A. In addition, $A \succeq 0$ means that A is symmetric and positive semidefinite.

2. Main Results

To obtain a tight bound for the absence of spurious local minima in problem (3), it is helpful to shed light on a distinguishing property of the function in (4) for linear measurements that does not hold in the general case: the Hessian matrices at all points are equal. If a general function f satisfies δ-RIP_{2r}, (5) intuitively states that the Hessian $\nabla^2 f(M)$ should be close to the quadratic form defined by an identity matrix, at least when applied to rank-$2r$ matrices. Hence, $\nabla^2 f(M)$ should change slowly when M alters. The above discussion inspires us to introduce a new notion below.

Definition 4 *A twice continuously differentiable function $f : \mathbb{R}^{n\times n} \to \mathbb{R}$ satisfies the* bounded difference property *of rank $2r$ for a constant $\kappa \geq 0$, denoted as κ-BDP_{2r}, if*

$$|[\nabla^2 f(M) - \nabla^2 f(M')](K, L)| \leq \kappa \|K\|_F \|L\|_F \tag{6}$$

for all matrices $M, M', K, L \in \mathbb{R}^{n\times n}$ whose ranks are at most $2r$.

It turns out that the RIP and BDP properties are not fully independent. Their relationship is summarized in the following theorems that will be proved in Section 3.

Theorem 1 *If the function f satisfies δ-RIP_{2r}, then it also satisfies 4δ-BDP_{2r}.*

Theorem 2 *If the function f satisfies δ-$\text{RIP}_{2r,4r}$, then it also satisfies 2δ-BDP_{2r}.*

The bounds in the above two theorems are tight. In Section 3, we will construct a class of functions f that satisfy the δ-RIP_{2r} property but do not satisfy the κ-BDP_{2r} property for some κ with κ/δ being arbitrarily close to 4. Similar examples can also be constructed for Theorem 2.

The main result of this paper is the following theorem, which is a brand-new criterion for the nonexistence of spurious local minima based on the RIP and BDP properties jointly. Its proof is given in Section 4.

Theorem 3 *When $r = 1$, the problem (3) has no spurious local minima if the function f satisfies the δ-RIP_2 and κ-BDP_2 properties for some constants δ and κ such that*

$$\delta < \frac{2 - 6(1+\sqrt{2})\kappa}{4 + 6(1+\sqrt{2})\kappa}.$$

In the case of linear measurements and the quadratic loss, the function f satisfies the κ-BDP_2 property with $\kappa = 0$. Hence, Theorem 3 recovers the result in Zhang et al. (2019) stating that the problem (1) has no spurious local minima if the operator $\mathcal{A}$ satisfies the δ-RIP_2 property with $\delta < 1/2$. On the other hand, by combining Theorem 1 and Theorem 3, one can immediately verify that the problem (3) has no spurious solutions if f satisfies the δ-RIP_2 property with $\delta < 0.0313$. Theorem 3 is most valuable for functions f associated with nonlinear measurements that satisfy δ-RIP_2 and κ-BDP_2 for $\delta < 1/2$ and κ being relatively small. At the end of Section 3, we will construct problems with such function f for which $\text{RIP}_{2,4}$ property does not exist, and thus the condition in Li et al. (2019) cannot be used. These are the examples for which the absence of spurious local minima can be certified by Theorem 3 but not by any of the conditions in the literature.

3. RIP and BDP Properties

In this section, the relationship between the RIP_{2r}, $\text{RIP}_{2r,4r}$ and BDP_{2r} properties of a given function f will be investigated. We will first prove Theorem 1 and Theorem 2, and then show that the bounds in these theorems are tight. The following lemma will be needed.

Lemma 4 *If a quadratic form $\mathcal{Q}$ satisfies δ-RIP_{2r}, then*

$$|[\mathcal{Q}](K,L) - \langle K,L\rangle| \leq \delta \|K\|_F \|L\|_F$$

for all matrices $K, L \in \mathbb{R}^{n\times n}$ with $\text{rank}(K) \leq r$, $\text{rank}(L) \leq r$.

Proof The reader could refer to Candès (2008); Bhojanapalli et al. (2016); Li et al. (2019), presented in different notations, or see Appendix A. ■

Proof of Theorem 2 Let M and M' be two matrices of rank at most $2r$. By the definition of δ-$\text{RIP}_{2r,4r}$ of the function f, both $\nabla^2 f(M)$ and $\nabla^2 f(M')$ satisfy δ-RIP_{4r}. After the constant r in the statement of Lemma 4 is replaced by $2r$, we obtain

$$\begin{aligned}
|[\nabla^2 f(M)](K,L) - \langle K,L\rangle| &\leq \delta \|K\|_F \|L\|_F, \\
|[\nabla^2 f(M')](K,L) - \langle K,L\rangle| &\leq \delta \|K\|_F \|L\|_F,
\end{aligned}$$

for all matrices $K, L \in \mathbb{R}^{n\times n}$ of rank at most $2r$, which leads to (6) for $\kappa = 2\delta$. ■

Proof of Theorem 1 We first prove that any quadratic form $\mathcal{Q}$ with δ-RIP_{2r} satisfies

$$|[\mathcal{Q}](K,L) - \langle K,L\rangle| \leq 2\delta \|K\|_F \|L\|_F,$$

for all matrices $K, L \in \mathbb{R}^{n\times n}$ of rank at most $2r$. Let $K = UDV^T$ be the singular value decomposition of K. Write $D = D_1 + D_2$ in which D_1 and D_2 both have at most r nonzero entries, and let $K_1 = UD_1V^T$, $K_2 = UD_2V^T$. Then, $K = K_1 + K_2$, where $\text{rank}(K_1) \leq r$, $\text{rank}(K_2) \leq r$ and $\langle K_1, K_2\rangle = 0$. Decompose $L = L_1 + L_2$ similarly. By Lemma 4, it holds that

$$\begin{aligned}
|[\mathcal{Q}](K,L) - \langle K,L\rangle| &\leq |[\mathcal{Q}](K_1,L_1) - \langle K_1,L_1\rangle| + |[\mathcal{Q}](K_1,L_2) - \langle K_1,L_2\rangle| \\
&\quad + |[\mathcal{Q}](K_2,L_1) - \langle K_2,L_1\rangle| + |[\mathcal{Q}](K_2,L_2) - \langle K_2,L_2\rangle| \\
&\leq \delta(\|K_1\|_F + \|K_2\|_F)(\|L_1\|_F + \|L_2\|_F) \\
&\leq 2\delta \sqrt{\|K_1\|_F^2 + \|K_2\|_F^2}\sqrt{\|L_1\|_F^2 + \|L_2\|_F^2} \\
&= 2\delta \|K\|_F \|L\|_F.
\end{aligned}$$

The remaining proof is exactly the same as the proof of Theorem 2. ■

In what follows, we will show that the bounds in Theorem 1 and Theorem 2 are tight. To this end, we will work on examples of function f with δ-RIP_{2r} or δ-RIP_{4r} for a small δ whose Hessian has a large variation across different points.

Consider an integer $n \geq 4$ and an integer $r \geq 1$. Let

$$A_1 = \frac{1}{\sqrt{n}} \operatorname{diag}(a_1, \dots, a_n)$$

with $a_i \in \{-1, 1\}$ whose value will be determined later. One can extend A_1 to an orthonormal basis $A_1, \dots, A_{n^2}$ of the space $\mathbb{R}^{n \times n}$. Define a linear operator $\mathcal{A} : \mathbb{R}^{n \times n} \to \mathbb{R}^{n^2-1}$ by letting

$$\mathcal{A}(M) = (\langle A_2, M \rangle, \dots, \langle A_{n^2}, M \rangle).$$

Then, for every matrix $M \in \mathbb{R}^{n \times n}$, it holds that

$$\|\mathcal{A}(M)\|^2 = \|M\|_F^2 - (\langle A_1, M \rangle)^2 \leq \|M\|_F^2.$$

Now, assume that M is a matrix with $\operatorname{rank}(M) \leq 2r$, and let $\sigma_1(M), \dots, \sigma_{2r}(M)$ denote its $2r$ largest singular values. Observe that

$$|\langle A_1, M \rangle| \leq \frac{1}{\sqrt{n}} \sum_{i=1}^{n} |M_{ii}| \leq \frac{1}{\sqrt{n}} \sum_{i=1}^{2r} \sigma_i(M) = \sqrt{\frac{2r}{n}} \sqrt{\sum_{i=1}^{2r} \sigma_i^2(M)} = \sqrt{\frac{2r}{n}} \|M\|_F,$$

which implies that

$$\|\mathcal{A}(M)\|^2 = \|M\|_F^2 - (\langle A_1, M \rangle)^2 \geq \left(1 - \frac{2r}{n}\right) \|M\|_F^2.$$

Define a scaled linear operator $\bar{\mathcal{A}}$ as

$$\bar{\mathcal{A}}(M) = \sqrt{\frac{n}{n-r}} \mathcal{A}(M), \quad \forall M \in \mathbb{R}^{n \times n}.$$

Thus, the relation

$$\left(1 - \frac{r}{n-r}\right) \|M\|_F^2 \leq \|\bar{\mathcal{A}}(M)\|^2 \leq \left(1 + \frac{r}{n-r}\right) \|M\|_F^2 \tag{7}$$

holds for all $M \in \mathbb{R}^{n \times n}$ with $\operatorname{rank}(M) \leq 2r$.

After choosing $A_1 = (1/\sqrt{n}) I_n$ in the above argument, let $\mathcal{A}$ be the resulting linear operator and $\mathcal{Q}$ be the quadratic form in (2) that corresponds to the scaled linear operator $\bar{\mathcal{A}}$. By the same argument, another linear operator $\mathcal{A}'$ and the corresponding quadratic form $\mathcal{Q}'$ can also be obtained after choosing

$$A_1' = \frac{1}{\sqrt{n}} \operatorname{diag}(1, 1, -1, -1, 1, \dots, 1). \tag{8}$$

Now, we select $K = \operatorname{diag}(1, 1, 0, 0, 0, \dots, 0)$ and $L = \operatorname{diag}(0, 0, 1, 1, 0, \dots, 0)$. Then,

$$\begin{aligned} |[\mathcal{Q} - \mathcal{Q}'](K, L)| &= \frac{n}{n-r} |\langle \mathcal{A}(K), \mathcal{A}(L) \rangle - \langle \mathcal{A}'(K), \mathcal{A}'(L) \rangle| \\ &= \frac{n}{n-r} |-\langle A_1, K \rangle \langle A_1, L \rangle + \langle A_1', K \rangle \langle A_1', L \rangle| \\ &= \frac{4}{n-r} \|K\|_F \|L\|_F. \end{aligned} \tag{9}$$

In the case $r = 1$, it follows from (7) that both of the constructed quadratic forms $\mathcal{Q}$ and $\mathcal{Q}'$ satisfy δ-RIP$_2$ with $\delta = 1/(n-1)$. If one can find a twice continuously differentiable function f satisfying δ-RIP$_2$ such that

$$\nabla^2 f(M) = \mathcal{Q}, \quad \nabla^2 f(M') = \mathcal{Q}'$$

hold at two particular points $M, M' \in \mathbb{R}^{n\times n}$ with $\operatorname{rank}(M) \le 2$ and $\operatorname{rank}(M') \le 2$, then by (9) the function f cannot satisfy κ-BDP$_2$ for $\kappa < 4\delta$. Since the design of such function is cumbersome, we will use a weaker result that serves the same purpose. This result, to be formalized in Lemma 5, states that for any $\epsilon > 0$, one can find a twice continuously differentiable function f with $(\delta+\epsilon)$-RIP$_2$ and two matrices $M, M' \in \mathbb{R}^{n\times n}$ of rank at most 1 satisfying the following inequalities:

$$\begin{aligned} |[\nabla^2 f(M) - \mathcal{Q}](K,L)| &\le \epsilon \|K\|_F \|L\|_F, \\ |[\nabla^2 f(M') - \mathcal{Q}'](K,L)| &\le \epsilon \|K\|_F \|L\|_F. \end{aligned} \tag{10}$$

Combining (9) and (10) yields that

$$|[\nabla^2 f(M) - \nabla^2 f(M')](K,L)| \le (4\delta + 2\epsilon)\|K\|_F \|L\|_F.$$

Therefore, the function f cannot satisfy the κ-BDP$_2$ property for any $\kappa < 4\delta + 2\epsilon$. Since ϵ can be made arbitrarily small, this shows that the constant 4δ in Theorem 1 cannot be improved. Similarly, by choosing $r = 2$ instead of $r = 1$ and repeating the above argument, one can show that the constant 2δ in Theorem 2 cannot be improved either.

Lemma 5 *Consider two quadratic forms $\mathcal{Q}, \mathcal{Q}'$ satisfying the δ-RIP$_{2r}$ property. For every $\epsilon > 0$, there exists a twice continuously differentiable function $f : \mathbb{R}^{n\times n} \to \mathbb{R}$ and two matrices $M, M' \in \mathbb{R}^{n\times n}$ with $\operatorname{rank}(M) \le 1$ and $\operatorname{rank}(M') \le 1$ such that f satisfies the $(\delta+\epsilon)$-RIP$_{2r}$ property and that (10) holds for all $K, L \in \mathbb{R}^{n\times n}$.*

Proof Given $\epsilon > 0$, let f be given as

$$f(V) = \frac{1}{2}[\mathcal{Q}'](V,V) + \frac{1}{2}H(\|V\|_F^2)[\Delta](V,V),$$

where $\Delta = \mathcal{Q} - \mathcal{Q}'$ and $H : \mathbb{R} \to \mathbb{R}$ is defined as

$$H(t) = \begin{cases} 0, & \text{if } t \le 0, \\ \exp(-1/t^\gamma), & \text{if } t > 0. \end{cases}$$

Here, $\gamma \in (0,1)$ is a constant that will be determined later. It is straightforward to verify that H is twice continuously differentiable and

$$H'(0) = H''(0) = 0, \tag{11a}$$

$$|tH'(t)| \le \frac{\gamma}{\mathrm{e}}, \quad |t^2 H''(t)| \le \frac{4\gamma}{\mathrm{e}}, \quad \forall t \in \mathbb{R}. \tag{11b}$$

The basic idea behind the above construction of f is that when γ is chosen to be small, the growth of the function H becomes so slow that it can be regarded as a constant when computing the Hessian of the above function f. As a result, the Hessian is approximately a linear combination of two quadratic forms $\mathcal{Q}$ and $\mathcal{Q}'$ with the δ-RIP$_{2r}$ property. Formally, the Hessian $\nabla^2 f(V)$ of f at a particular matrix $V \in \mathbb{R}^{n\times n}$, when applied to arbitrary $K, L \in \mathbb{R}^{n\times n}$, is given by

$$\begin{aligned}[\nabla^2 f(V)](K,L) = {} & 2H''(\|V\|_F^2)[\Delta](V,V)\langle V,K\rangle\langle V,L\rangle + H'(\|V\|_F^2)[\Delta](V,V)\langle K,L\rangle \\ & + 2H'(\|V\|_F^2)([\Delta](L,V)\langle V,K\rangle + [\Delta](K,V)\langle V,L\rangle) \\ & + [\mathcal{Q}' + H(\|V\|_F^2)\Delta](K,L).\end{aligned} \tag{12}$$

By compactness, there exists a constant $C > 0$ such that

$$|[\Delta](A,B)| \le C\|A\|_F\|B\|_F \tag{13}$$

holds for all $A, B \in \mathbb{R}^{n\times n}$. We choose a sufficiently small γ such that $26\gamma C/\mathrm{e} \le \epsilon$. By (11b), (12), (13) and the Cauchy-Schwartz inequality, we have

$$|[\nabla^2 f(V) - \mathcal{Q}' - H(\|V\|_F^2)\Delta](K,L)| \le \frac{13\gamma C}{\mathrm{e}}\|K\|_F\|L\|_F \le \frac{\epsilon}{2}\|K\|_F\|L\|_F. \tag{14}$$

To prove that the function f satisfies $(\delta+\epsilon)$-RIP$_{2r}$, assume for now that $K = L$ and $\operatorname{rank}(K) \le 2r$. The inequality $0 \le H(\|V\|_F^2) \le 1$ and the δ-RIP$_{2r}$ property of $\mathcal{Q}$ and $\mathcal{Q}'$ imply that

$$(1-\delta)\|K\|_F^2 \le [\mathcal{Q}' + H(\|V\|_F^2)\Delta](K,K) \le (1+\delta)\|K\|_F^2.$$

By (14) and the above inequality, the function f satisfies the $(\delta+\epsilon)$-RIP$_{2r}$ property. To prove the existence of M and M' satisfying (10), we select $M' = 0$ and

$$M = \operatorname{diag}(s, 0, \dots, 0).$$

For any $K, L \in \mathbb{R}^{n\times n}$, it follows from (11a) and (12) that

$$[\nabla^2 f(M') - \mathcal{Q}'](K,L) = 0. \tag{15}$$

Moreover, (13) and (14) yield that

$$\begin{aligned}|[\nabla^2 f(M) - \mathcal{Q}](K,L)| &\le \frac{\epsilon}{2}\|K\|_F\|L\|_F + |[\mathcal{Q}' + H(\|M\|_F^2)\Delta - \mathcal{Q}](K,L)| \\ &\le \left(\frac{\epsilon}{2} + (1 - H(\|M\|_F^2))C\right)\|K\|_F\|L\|_F.\end{aligned}$$

Since $H(\|M\|_F^2) \to 1$ as $s \to +\infty$, (10) is satisfied as long as s is sufficiently large. ■

The above argument also provides examples of the function f whose corresponding recovery problem (3) can be certified to have no spurious local minima via Theorem 3, while the existing results in the literature fail to do so. Following the above construction, choose $n = 4$, $r = 1$, and let

$$\tilde{f}(V) = \frac{1-\lambda}{2}[\mathcal{Q}'](V,V) + \lambda f(V),$$

for some $\lambda \in [0,1]$. The Hessian can be written as

$$\nabla^2 \tilde{f}(V) = (1-\lambda)\mathcal{Q}' + \lambda \nabla^2 f(V). \tag{16}$$

If $\lambda > 0$, the Hessian of $\tilde{f}$ is not a constant, and therefore the condition in Zhang et al. (2019) cannot be applied. On the other hand, it follows from (15) that

$$[\nabla^2 \tilde{f}(0)](A_1', A_1') = [\mathcal{Q}'](A_1', A_1') = 0,$$

for the matrix A_1' of rank 4 defined in (8). Thus, the function $\tilde{f}$ cannot satisfy the δ-RIP$_{2,4}$ property for any $0 \le \delta < 1$. This implies that the condition in Li et al. (2019) cannot be applied either. In contrast, note that the quadratic form $\mathcal{Q}'$ satisfies the 1/3-RIP$_2$ property and the function f satisfies the $(1/3+\epsilon)$-RIP$_2$ property. Therefore, it can be concluded from (16) that the function $\tilde{f}$ also satisfies the $(1/3+\epsilon)$-RIP$_2$ property. In light of Theorem 1, f satisfies $4(1/3+\epsilon)$-BDP$_2$ and thus f' satisfies $4\lambda(1/3+\epsilon)$-BDP$_2$. Hence, Theorem 3 certifies the absence of spurious local minima as long as λ and ϵ jointly satisfy

$$\frac{1}{3} + \epsilon < \frac{2 - 6(1+\sqrt{2})4\lambda(1/3+\epsilon)}{4 + 6(1+\sqrt{2})4\lambda(1/3+\epsilon)}.$$

4. Proof of Theorem 3

The proof of Theorem 3 consists of two major steps. The first step is to find a necessary condition that the function f must satisfy if the corresponding problem (3) has a local minimizer X such that $XX^T \neq M^*$, where M^* is the ground truth. The second step is to develop certain conditions on δ and κ that rule out the satisfaction of the above necessary condition.

Before proceeding with the proof, we need to introduce some notations. Given two matrices $X, Z \in \mathbb{R}^{n\times r}$, define

$$\mathbf{e} = \operatorname{vec}(XX^T - ZZ^T) \in \mathbb{R}^{n^2},$$

and let $\mathbf{X} \in \mathbb{R}^{n^2 \times nr}$ be the matrix satisfying

$$\mathbf{X} \operatorname{vec} U = \operatorname{vec}(XU^T + UX^T), \quad \forall U \in \mathbb{R}^{n\times r}.$$

Similarly, let $\mathbf{H} \in \mathbb{R}^{n^2\times n^2}$ be the matrix satisfying

$$(\operatorname{vec} K)^T \mathbf{H} \operatorname{vec} L = [\nabla^2 f(XX^T)](K, L),$$

for all $K, L \in \mathbb{R}^{n\times n}$. The desired necessary condition for the existence of spurious local minima in (3) is stated in the following lemma.

Lemma 6 *Assume that the function f in the problem* (3) *satisfies the δ-*RIP$_{2r}$ *and κ-*BDP$_{2r}$ *properties. If X is a local minimizer of* (3) *and Z is a global minimizer of* (3) *with $M^* = ZZ^T$, then*

1. $\|\mathbf{X}^T \mathbf{H}\mathbf{e}\| \le 2\kappa \|X\|_F \|\mathbf{e}\|$;

2. $2I_r \otimes \mathrm{mat_S}(\mathbf{He}) + \mathbf{X}^T\mathbf{H}\mathbf{X} \succeq -2\kappa\|\mathbf{e}\| I_{nr}$;

3. $\mathbf{H}$ *satisfies the* δ-RIP_{2r} *property, i.e, for each matrix* $U \in \mathbb{R}^{n\times n}$ *with* $\mathrm{rank}(U) \le 2r$, *it holds that*
$$(1-\delta)\|\mathbf{U}\|^2 \le \mathbf{U}^T\mathbf{H}\mathbf{U} \le (1+\delta)\|\mathbf{U}\|^2,$$
where $\mathbf{U} = \mathrm{vec}\, U$.

Proof Condition 3 follows immediately from the δ-RIP_{2r} property of the function f. To prove the remaining two conditions, define $g(Y) = f(YY^T)$ and $M = XX^T$. Since X is a local minimizer of the function g, for every $U \in \mathbb{R}^{n\times r}$ with $\mathbf{U} = \mathrm{vec}\, U$, the first-order optimality condition implies that
$$0 = \langle \nabla g(X), U\rangle = \langle \nabla f(M), XU^T + UX^T\rangle. \tag{17}$$
Define an auxiliary function $h : \mathbb{R}^{n\times n} \to \mathbb{R}$ by letting
$$h(V) = \langle \nabla f(V), XU^T + UX^T\rangle.$$
By the mean value theorem, there exists a matrix ξ on the segment between M and M^* such that
$$[\nabla^2 f(\xi)](M - M^*, XU^T + UX^T) = \langle \nabla h(\xi), M - M^*\rangle = h(M) - h(M^*) = 0, \tag{18}$$
in which the last equality follows from (17) and $\nabla f(M^*) = 0$. Since $\mathrm{rank}(M) \le r$ and $\mathrm{rank}(M^*) \le r$, we have $\mathrm{rank}(\xi) \le 2r$ and $\mathrm{rank}(M - M^*) \le 2r$. Applying the κ-BDP_{2r} property to the Hessian of f at matrices M and ξ, together with (18), one can obtain
$$\begin{aligned}|\mathbf{e}^T\mathbf{H}\mathbf{X}\mathbf{U}| &= |[\nabla^2 f(M)](M - M^*, XU^T + UX^T)| \\ &\le \kappa\|M - M^*\|_F\|XU^T + UX^T\|_F \\ &\le 2\kappa\|\mathbf{e}\|\|X\|_F\|\mathbf{U}\|.\end{aligned}$$
Condition 1 can be proved by setting $\mathbf{U} = \mathbf{X}^T\mathbf{He}$.

For every $U \in \mathbb{R}^{n\times r}$ with $\mathbf{U} = \mathrm{vec}\, U$, the second-order optimality condition implies that
$$0 \le [\nabla^2 g(X)](U, U) = [\nabla^2 f(M)](XU^T + UX^T, XU^T + UX^T) + 2\langle \nabla f(M), UU^T\rangle. \tag{19}$$
The first term on the right-hand side can be equivalently written as $(\mathbf{XU})^T\mathbf{H}(\mathbf{XU})$. A similar argument can be made to conclude that there exists another matrix ξ' on the segment between M and M^* such that
$$\begin{aligned}\langle \nabla f(M), UU^T\rangle &= \langle \nabla f(M) - \nabla f(M^*), UU^T\rangle \\ &= [\nabla^2 f(\xi')](M - M^*, UU^T) \\ &\le [\nabla^2 f(M)](M - M^*, UU^T) + \kappa\|M - M^*\|_F\|UU^T\|_F \\ &= \mathrm{vec}(UU^T)\mathbf{He} + \kappa\|\mathbf{e}\|\|\mathbf{U}\|^2 \\ &= \frac{1}{2}(\mathrm{vec}\, U)^T\,\mathrm{vec}((W + W^T)U) + \kappa\|\mathbf{e}\|\|\mathbf{U}\|^2 \\ &= \mathbf{U}^T(I_r \otimes \mathrm{mat_S}(\mathbf{He}))\mathbf{U} + \kappa\|\mathbf{e}\|\|\mathbf{U}\|^2,\end{aligned} \tag{20}$$

in which $W \in \mathbb{R}^{n\times n}$ is the unique matrix satisfying $\operatorname{vec} W = \mathbf{He}$. Condition 2 can be obtained by combining (19) and (20). ■

For given $X, Z \in \mathbb{R}^{n\times r}$ and $\kappa \geq 0$, one can construct an optimization problem based on the conditions in Lemma 6 as follows:

$$\begin{aligned} \min \quad & \delta \\ \text{s.t.} \quad & \|\mathbf{X}^T\mathbf{He}\| \leq a, \\ & 2I_r \otimes \operatorname{mat}_S(\mathbf{He}) + \mathbf{X}^T\mathbf{HX} \succeq -bI_{nr}, \\ & \mathbf{H} \text{ is symmetric and satisfies } \delta\text{-RIP}_{2r}, \end{aligned} \tag{21}$$

where

$$a = 2\kappa\|X\|_F\|\mathbf{e}\|, \quad b = 2\kappa\|\mathbf{e}\|. \tag{22}$$

Let $\delta(X, Z; \kappa)$ be the optimal value of (21). Assume that f in the original problem (3) satisfies δ-RIP$_{2r}$ and κ-BDP$_{2r}$. By Lemma 6, if X is a local minimizer of (3) and Z is a global minimizer of (3) with $M^* = ZZ^T$, then $\delta \geq \delta(X, Z; \kappa)$. As a result, by defining

$$\begin{aligned} \delta^*(\kappa) = \min \quad & \delta(X, Z; \kappa) \\ \text{s.t.} \quad & XX^T \neq ZZ^T, \end{aligned}$$

the problem (3) is guaranteed to have no spurious local minima as long as $\delta < \delta^*(\kappa)$.

The remaining task is to compute $\delta(X, Z; \kappa)$ and $\delta^*(\kappa)$. First, by the property of the Schur complement, the first constraint in (21) can be equivalently written as

$$\begin{bmatrix} I_{nr} & \mathbf{X}^T\mathbf{He} \\ (\mathbf{X}^T\mathbf{He})^T & a^2 \end{bmatrix} \succeq 0.$$

The major difficulty of solving (21) comes from the last constraint, since it is NP-hard to verify whether a given quadratic form satisfies δ-RIP$_{2r}$ (Tillmann and Pfetsch, 2014). Instead, we tighten the last constraint of (21) by requiring $\mathbf{H}$ to have a norm-preserving property for all matrices instead of just for matrices with rank at most $2r$, i.e.,

$$(1-\delta)\|\mathbf{U}\|^2 \leq \mathbf{U}^T\mathbf{HU} \leq (1+\delta)\|\mathbf{U}\|^2, \quad \forall \mathbf{U} \in \mathbb{R}^{n^2},$$

which leads to following semidefinite program:

$$\begin{aligned} \min \quad & \delta \\ \text{s.t.} \quad & \begin{bmatrix} I_{nr} & \mathbf{X}^T\mathbf{He} \\ (\mathbf{X}^T\mathbf{He})^T & a^2 \end{bmatrix} \succeq 0, \\ & 2I_r \otimes \operatorname{mat}_S(\mathbf{He}) + \mathbf{X}^T\mathbf{HX} \succeq -bI_{nr}, \\ & (1-\delta)I_{n^2} \preceq \mathbf{H} \preceq (1+\delta)I_{n^2}. \end{aligned} \tag{23}$$

Similar to the case of linear measurements studied in Zhang et al. (2019), due to the symmetry under orthogonal projections, the problems (21) and (23) actually have the same optimal value.

Lemma 7 *For given $X, Z \in \mathbb{R}^{n\times r}$ and $\kappa \geq 0$, the optimization problems* (21) *and* (23) *have the same optimal value.*

Proof See Appendix B. ■

Even if the value of $\delta(X, Z; \kappa)$ for given X, Z and κ can now be efficiently calculated by solving the semidefinite program (23), to further compute $\delta^*(\kappa)$, an analytical expression is still needed for $\delta(X, Z; \kappa)$. For our purpose, it is sufficient to find a lower bound on $\delta(X, Z; \kappa)$. In the remainder of this section, we will focus on the problem of lower bounding $\delta(X, Z; \kappa)$ and $\delta^*(\kappa)$ in the case $r = 1$.

When $r = 1$, X and Z are vectors and henceforth will be denoted as x and z with

$$\mathbf{e} = x \otimes x - z \otimes z, \quad \mathbf{X}u = x \otimes u + u \otimes x.$$

Moreover,

$$\|\mathbf{X}u\|^2 = 2\|x\|^2\|u\|^2 + 2(x^T u)^2, \quad \forall u \in \mathbb{R}^n. \tag{24}$$

Given two vectors $x, z \in \mathbb{R}^n$ with $x \neq 0$ and $xx^T \neq zz^T$, one can find a unit vector $w \in \mathbb{R}^n$ such that w is orthogonal to x and $z = c_1 x + c_2 w$. Then,

$$\mathbf{e} = \mathbf{X}\tilde{y} - c_2^2(w \otimes w),$$

in which

$$\tilde{y} = \frac{1 - c_1^2}{2} x - c_1 c_2 w.$$

Note that $\mathbf{X}\tilde{y}$ is orthogonal to $w \otimes w$. Furthermore, since $\tilde{y} \neq 0$ by $xx^T \neq zz^T$ and thus $\mathbf{X}\tilde{y} \neq 0$ by (24), one can rescale $\tilde{y}$ into $\hat{y}$ such that $\|\mathbf{X}\hat{y}\| = 1$ and

$$\mathbf{e} = \|\mathbf{e}\|(\sqrt{1 - \alpha^2}\mathbf{X}\hat{y} - \alpha(w \otimes w)), \tag{25}$$

with

$$\alpha := \frac{c_2^2}{\|\mathbf{e}\|} = \frac{\|z\|^2 - (x^T z/\|x\|)^2}{\|\mathbf{e}\|}. \tag{26}$$

In addition, (24) also implies

$$\|\hat{y}\| \leq \frac{\|\mathbf{X}\hat{y}\|}{\sqrt{2}\|x\|} = \frac{1}{\sqrt{2}\|x\|}. \tag{27}$$

Lemma 8 *Let $x, z \in \mathbb{R}^n$ with $xx^T \neq zz^T$. The optimal value $\delta(x, z; \kappa)$ of* (23) *satisfies*

$$\delta(x, z; \kappa) \geq \frac{1 - \eta_0(x, z) - 2(1 + \sqrt{2})\kappa}{1 + \eta_0(x, z) + 2(1 + \sqrt{2})\kappa},$$

in which

$$\eta_0(x, z) = \begin{cases} \dfrac{1 - \sqrt{1 - \alpha^2}}{1 + \sqrt{1 - \alpha^2}}, & \text{if } \beta \geq \dfrac{\alpha}{1 + \sqrt{1 - \alpha^2}}, \\ \dfrac{\beta(\beta - \alpha)}{\beta\alpha - 1}, & \text{if } \beta \leq \dfrac{\alpha}{1 + \sqrt{1 - \alpha^2}}, \end{cases}$$

with α defined in (26)[1] *and $\beta = \|x\|^2/\|\mathbf{e}\|$.*

1. When $x = 0$, α is defined to be $\|z\|^2/\|\mathbf{e}\|$.

Proof Define $\eta(x, z; \kappa)$ to be the optimal value of the following optimization problem:

$$\begin{aligned} \max \quad & \eta \\ \text{s.t.} \quad & \begin{bmatrix} I_{nr} & \mathbf{X}^T\mathbf{He} \\ (\mathbf{X}^T\mathbf{He})^T & a^2 \end{bmatrix} \succeq 0, \\ & 2\,\mathrm{mat}_\mathrm{S}(\mathbf{He}) + \mathbf{X}^T\mathbf{X} \succeq -bI_{nr}, \\ & \eta I_{n^2} \preceq \mathbf{H} \preceq I_{n^2}. \end{aligned} \tag{28}$$

It can be verified that

$$\eta(x, z; \kappa) \geq \frac{1-\delta(x, z; \kappa)}{1+\delta(x, z; \kappa)}, \tag{29}$$

because given any feasible solution $(\delta, \mathbf{H})$ to (23), the point

$$\left(\frac{1-\delta}{1+\delta}, \frac{1}{1+\delta}\mathbf{H}\right)$$

is also a feasible solution to (28): The first and the last constraint in (28) are trivial to verify, and the second constraint is satisfied since

$$2\,\mathrm{mat}_\mathrm{S}\left(\frac{1}{1+\delta}\mathbf{He}\right) + \mathbf{X}^T\mathbf{X} \succeq \frac{1}{1+\delta}(2\,\mathrm{mat}_\mathrm{S}(\mathbf{He}) + \mathbf{X}^T\mathbf{HX}) \succeq -\frac{b}{1+\delta}I_{nr} \succeq -bI_{nr}.$$

Therefore, to find a lower bound on $\delta(x, z; \kappa)$, we only need to find an upper bound on $\eta(x, z; \kappa)$.

The dual problem of (28) can be written as

$$\begin{aligned} \min \quad & \mathrm{tr}(U_2) + \langle \mathbf{X}^T\mathbf{X} + bI_n, V\rangle + a^2\lambda + \mathrm{tr}(G), \\ \text{s.t.} \quad & \mathrm{tr}(U_1) = 1, \\ & (\mathbf{X}y - v)\mathbf{e}^T + \mathbf{e}(\mathbf{X}y - v)^T = U_1 - U_2, \\ & \begin{bmatrix} G & -y \\ -y^T & \lambda \end{bmatrix} \succeq 0, \\ & U_1 \succeq 0, \quad U_2 \succeq 0, \quad V \succeq 0, \quad v = \mathrm{vec}\,V. \end{aligned} \tag{30}$$

By weak duality, the dual objective value associated with any feasible solution to the dual problem (30) is an upper bound on $\eta(x, z; \kappa)$.

In the case when $x \neq 0$, fix a constant $\gamma \in [0, \alpha]$ and choose

$$y = \frac{\sqrt{1-\gamma^2}}{\|\mathbf{e}\|}\hat{y}, \quad v = \frac{\gamma}{\|\mathbf{e}\|}(w \otimes w),$$

where $\hat{y}$ and w are the vectors defined before (25). Since $\|\mathbf{X}\hat{y}\| = 1$, $\|w \otimes w\| = 1$ and $\mathbf{X}\hat{y}$ is orthogonal to $w \otimes w$, it holds that

$$\|\mathbf{X}y - v\| = \frac{1}{\|\mathbf{e}\|}.$$

Combined with (25), one can obtain

$$\mathbf{e}^T(\mathbf{X}y - v) = \psi(\gamma),$$

with $\psi(\gamma)$ is given by

$$\psi(\gamma) = \gamma\alpha + \sqrt{1-\gamma^2}\sqrt{1-\alpha^2}.$$

Now, define

$$M = (\mathbf{X}y - v)\mathbf{e}^T + \mathbf{e}(\mathbf{X}y - v)^T$$

and decompose

$$M = [M]_+ - [M]_-,$$

in which both $[M]_+ \succeq 0$ and $[M]_- \succeq 0$. Let θ be the angle between $\mathbf{e}$ and $\mathbf{X}y - v$. Subsequently,

$$\begin{aligned}\mathrm{tr}([M]_+) &= \|\mathbf{e}\|\|\mathbf{X}y - v\|(1+\cos\theta) = 1 + \psi(\gamma),\\ \mathrm{tr}([M]_-) &= \|\mathbf{e}\|\|\mathbf{X}y - v\|(1-\cos\theta) = 1 - \psi(\gamma)\end{aligned}$$

(see (Zhang et al., 2019, Lemma 15)). Then, it is routine to verify that

$$\begin{aligned} U_1^* = \frac{[M]_+}{\mathrm{tr}([M]_+)}, &\quad U_2^* = \frac{[M]_-}{\mathrm{tr}([M]_+)},\\ y^* = \frac{y}{\mathrm{tr}([M]_+)}, &\quad v^* = \frac{v}{\mathrm{tr}([M]_+)}\\ \lambda^* = \frac{\|y^*\|}{a}, &\quad G^* = \frac{1}{\lambda^*}y^* y^{*T}\end{aligned}$$

forms a feasible solution to the dual problem (30) whose objective value is equal to

$$\frac{\mathrm{tr}([M]_-) + \langle \mathbf{X}^T\mathbf{X} + bI_n, V\rangle + 2a\|y\|}{\mathrm{tr}([M]_+)}. \tag{31}$$

By (24) and (27), one can write

$$\langle \mathbf{X}^T\mathbf{X} + bI_n, V\rangle = \frac{\gamma}{\|\mathbf{e}\|}(\|\mathbf{X}w\|^2 + b) = \frac{\gamma}{\|\mathbf{e}\|}(2\|x\|^2 + b) = 2(\beta+\kappa)\gamma, \tag{32}$$

$$2a\|y\| \leq \frac{2a\|\hat{y}\|}{\|\mathbf{e}\|} \leq 2\sqrt{2}\kappa, \tag{33}$$

where a and b are defined in (22). Substituting (32) and (33) into (31) yields that

$$\eta(x,z;\kappa) \leq \Psi(\gamma) + 2(1+\sqrt{2})\kappa,$$

where

$$\Psi(\gamma) = \frac{2\beta\gamma + 1 - \psi(\gamma)}{1 + \psi(\gamma)}.$$

A simple calculation shows that the function $\Psi(\gamma)$ has at most one stationary point over the interval $(0, \alpha)$ and

$$\min_{0\leq\gamma\leq\alpha} \Psi(\gamma) = \eta_0(x,z).$$

In the case when $x = 0$, we have $\eta_0(x, z) = 0$, and

$$U_1 = \frac{\mathbf{e}\mathbf{e}^T}{\|\mathbf{e}\|^2}, \quad U_2 = 0, \quad V = \frac{zz^T}{2\|\mathbf{e}\|^2},$$
$$y = 0, \quad \lambda = 0, \quad G = 0$$

forms a feasible solution to the dual problem (30), which implies

$$\eta(x, z, \kappa) \leq \langle bI_n, V\rangle = \kappa.$$

In either case, it holds that

$$\eta(x, z; \kappa) \leq \eta_0(x, z) + 2(1 + \sqrt{2})\kappa,$$

which gives the desired result after combining it with (29). ■

Proof of Theorem 3 By Lemma 7 and the discussion after Lemma 6, we only need to show that

$$\delta(x, z; \kappa) \geq \frac{2 - 6(1 + \sqrt{2})\kappa}{4 + 6(1 + \sqrt{2})\kappa}, \tag{34}$$

for all $x, z \in \mathbb{R}^n$ with $xx^T \neq zz^T$. By a similar approach used in proof of (Zhang et al., 2019, Theorem 3), we can show that the $\eta_0(x, z)$ function defined in the statement of Lemma 8 has maximum value 1/3 attained by any x and z that are orthogonal to each other with $\|x\|/\|z\| = 1/2$, and consequently (34) holds by Lemma 8. ■

5. Conclusions

In this paper, we first propose the bounded difference property (BDP) in order to study the symmetric low-rank matrix recovery problem with nonlinear measurements. The relationship between the BDP and RIP is carefully investigated. Then, a novel criterion for the nonexistence of spurious local minima is proposed based on RIP and BDP jointly. It is shown that the developed criterion is superior to the existing conditions relying only on RIP.

Appendix A. Proof of Lemma 4

Without loss of generality, assume that $\|K\|_F = \|L\|_F = 1$. By the δ-RIP$_{2r}$ property of $\mathcal{Q}$, we have

$$(1 - \delta)\|K - L\|_F^2 \leq [\mathcal{Q}](K - L, K - L) \leq (1 + \delta)\|K - L\|_F^2,$$
$$(1 - \delta)\|K + L\|_F^2 \leq [\mathcal{Q}](K + L, K + L) \leq (1 + \delta)\|K + L\|_F^2.$$

Taking the difference between above two inequalities, one can obtain

$$4[\mathcal{Q}](K, L) \leq (1 + \delta)\|K + L\|_F^2 - (1 - \delta)\|K - L\|_F^2 = 4\delta + 4\langle K, L\rangle,$$
$$-4[\mathcal{Q}](K, L) \leq (1 + \delta)\|K - L\|_F^2 - (1 - \delta)\|K + L\|_F^2 = 4\delta - 4\langle K, L\rangle,$$

which proves the desired inequality.

Appendix B. Proof of Lemma 7

Let

$$\begin{aligned} \mathrm{OPT}(X,Z) = \min \quad & \delta \\ \text{s. t.} \quad & \|\mathbf{X}^T\mathbf{He}\| \le a, \\ & 2I_r \otimes \mathrm{mat_S}(\mathbf{He}) + \mathbf{X}^T\mathbf{HX} \succeq -bI_{nr}, \\ & \mathbf{H} \text{ is symmetric and satisfies } \delta\text{-RIP}_{2r}, \end{aligned} \tag{35}$$

and

$$\begin{aligned} \mathrm{LMI}(X,Z) = \min \quad & \delta \\ \text{s. t.} \quad & \begin{bmatrix} I_{nr} & \mathbf{X}^T\mathbf{He} \\ (\mathbf{X}^T\mathbf{He})^T & a^2 \end{bmatrix} \succeq 0, \\ & 2I_r \otimes \mathrm{mat_S}(\mathbf{He}) + \mathbf{X}^T\mathbf{HX} \succeq -bI_{nr}, \\ & (1-\delta)I_{n^2} \preceq \mathbf{H} \preceq (1+\delta)I_{n^2}. \end{aligned} \tag{36}$$

As mentioned in the paper, the first constraint in (35) and the first constraint in (36) are interchangeable. Our goal is to prove $\mathrm{OPT}(X,Z) = \mathrm{LMI}(X,Z)$ for given $X, Z \in \mathbb{R}^{n\times r}$. Let $(v_1,\dots,v_n)$ be an orthogonal basis of $\mathbb{R}^n$ such that $(v_1,\dots,v_d)$ spans the column spaces of both X and Z. Note that $d \le 2r$. Let $P \in \mathbb{R}^{n\times d}$ be the matrix with columns $(v_1,\dots,v_d)$ and $P_\perp \in \mathbb{R}^{n\times(n-d)}$ be the matrix with columns $(v_{d+1},\dots,v_n)$. Then,

$$\begin{gathered} P^TP = I_d, \quad P_\perp^TP_\perp = I_{n-d}, \quad P_\perp^TP = 0, \quad P^TP_\perp = 0, \\ PP^T + P_\perp P_\perp^T = I_n, \quad PP^TX = X, \quad PP^TZ = Z. \end{gathered}$$

Define $\mathbf{P} = P \otimes P$. Next, consider the auxiliary optimization problem:

$$\begin{aligned} \overline{\mathrm{LMI}}(X,Z) = \min \quad & \delta \\ \text{s. t.} \quad & \begin{bmatrix} I_{nr} & \mathbf{X}^T\mathbf{He} \\ (\mathbf{X}^T\mathbf{He})^T & a^2 \end{bmatrix} \succeq 0, \\ & 2I_r \otimes \mathrm{mat_S}(\mathbf{He}) + \mathbf{X}^T\mathbf{HX} \succeq -bI_{nr}, \\ & (1-\delta)I_{d^2} \preceq \mathbf{P}^T\mathbf{HP} \preceq (1+\delta)I_{d^2}. \end{aligned} \tag{37}$$

Given an arbitrary symmetric matrix $\mathbf{H} \in \mathbb{R}^{n^2\times n^2}$, if $\mathbf{H}$ satisfies the last constraint in (36), then it obviously satisfies δ-RIP_{2r} and subsequently the last constraint in (35). On the other hand, if H satisfies the last constraint in (35), for every matrix $Y \in \mathbb{R}^{d\times d}$ with $\mathbf{Y} = \operatorname{vec} Y$, since $\operatorname{rank}(PYP^T) \le d \le 2r$ and $\operatorname{vec}(PYP^T) = \mathbf{PY}$, by δ-RIP_{2r} property, one arrives at

$$(1-\delta)\|\mathbf{Y}\|^2 = (1-\delta)\|\mathbf{PY}\|^2 \le (\mathbf{PY})^T\mathbf{HPY} \le (1+\delta)\|\mathbf{PY}\|^2 = (1+\delta)\|\mathbf{Y}\|^2,$$

which implies that $\mathbf{H}$ satisfies the last constraint in (37). The above discussion implies that

$$\mathrm{LMI}(X,Z) \ge \mathrm{OPT}(X,Z) \ge \overline{\mathrm{LMI}}(X,Z).$$

Let

$$\hat{X} = P^TX, \quad \hat{Z} = P^TZ.$$

Lemma 10 and Lemma 11 to be stated later will show that

$$\mathrm{LMI}(X,Z) \le \mathrm{LMI}(\hat{X},\hat{Z}) \le \overline{\mathrm{LMI}}(X,Z),$$

which completes the proof.

The following lemma will be used in the proofs of Lemma 10 and Lemma 11:

Lemma 9 *Define* $\hat{\mathbf{e}}$ *and* $\hat{\mathbf{X}}$ *in the same way as* $\mathbf{e}$ *and* $\mathbf{X}$*, except that* X *and* Z *are replaced by* $\hat{X}$ *and* $\hat{Z}$*, respectively. Then, it holds that*

$$\begin{aligned} \mathbf{e} &= \mathbf{P}\hat{\mathbf{e}}, \\ \mathbf{X}(I_r \otimes P) &= \mathbf{P}\hat{\mathbf{X}}, \\ \mathbf{P}^T\mathbf{X} &= \hat{\mathbf{X}}(I_r \otimes P)^T. \end{aligned}$$

Proof Observe that

$$\begin{aligned} \mathbf{e} &= \operatorname{vec}(XX^T - ZZ^T) = \operatorname{vec}(P(\hat{X}\hat{X}^T - \hat{Z}\hat{Z}^T)P^T) = \mathbf{P}\hat{\mathbf{e}}, \\ \mathbf{X}(I_r \otimes P)\operatorname{vec}\hat{U} &= \mathbf{X}\operatorname{vec}(P\hat{U}) = \operatorname{vec}(X\hat{U}^TP^T + P\hat{U}X^T) \\ &= \operatorname{vec}(P(\hat{X}\hat{U}^T + \hat{U}\hat{X}^T)P^T) = \mathbf{P}\hat{\mathbf{X}}\operatorname{vec}\hat{U}, \\ \hat{\mathbf{X}}(I_r \otimes P)^T\operatorname{vec}U &= \hat{\mathbf{X}}\operatorname{vec}(P^TU) = \operatorname{vec}(\hat{X}U^TP + P^TU\hat{X}^T) \\ &= \operatorname{vec}(P^T(XU^T + UX^T)P) = \mathbf{P}^T\mathbf{X}\operatorname{vec}U, \end{aligned}$$

where $U \in \mathbb{R}^{n\times r}$ and $\hat{U} \in \mathbb{R}^{d\times r}$ are arbitrary matrices. ∎

Lemma 10 *The inequality* $\mathrm{LMI}(\hat{X},\hat{Z}) \ge \mathrm{LMI}(X,Z)$ *holds.*

Proof Let $(\delta,\hat{\mathbf{H}})$ be an arbitrary feasible solution to the optimization problem defining $\mathrm{LMI}(\hat{X},\hat{Z})$ with $\delta \le 1$. It is desirable to show that $(\delta,\mathbf{H})$ with

$$\mathbf{H} = \mathbf{P}\hat{\mathbf{H}}\mathbf{P}^T + (I_{n^2} - \mathbf{P}\mathbf{P}^T)$$

is a feasible solution to the optimization problem defining $\mathrm{LMI}(X,Z)$, which directly proves the lemma.

First,

$$\mathbf{H} - (1-\delta)I_{n^2} = \mathbf{P}(\hat{\mathbf{H}} - (1-\delta)I_{d^2})\mathbf{P}^T + \delta(I_{n^2} - \mathbf{P}\mathbf{P}^T),$$

which is positive semidefinite because

$$\begin{aligned} I_{n^2} - \mathbf{P}\mathbf{P}^T &= (PP^T + P_\perp P_\perp^T) \otimes (PP^T + P_\perp P_\perp^T) - (PP^T) \otimes (PP^T) \\ &= (PP^T) \otimes (P_\perp P_\perp^T) + (P_\perp P_\perp^T) \otimes (PP^T) + (P_\perp P_\perp^T) \otimes (P_\perp P_\perp^T) \succeq 0. \end{aligned}$$

Similarly,

$$\mathbf{H} - (1+\delta)I_{n^2} \preceq 0,$$

and therefore the last constraint in (36) is satisfied and $\mathbf{H}$ is always positive semidefinite. Next, since

$$\mathbf{X}^T\mathbf{H}\mathbf{e} = \mathbf{X}^T\mathbf{H}\mathbf{P}\hat{\mathbf{e}} = \mathbf{X}^T\mathbf{P}\hat{\mathbf{H}}\hat{\mathbf{e}} = (I_r \otimes P)\hat{\mathbf{X}}^T\hat{\mathbf{H}}\hat{\mathbf{e}},$$

we have

$$\|\mathbf{X}^T\mathbf{H}\mathbf{e}\|^2 = (\hat{\mathbf{X}}^T\hat{\mathbf{H}}\hat{\mathbf{e}})^T(I_r \otimes P^T)(I_r \otimes P)(\hat{\mathbf{X}}^T\hat{\mathbf{H}}\hat{\mathbf{e}}) = \|\hat{\mathbf{X}}^T\hat{\mathbf{H}}\hat{\mathbf{e}}\|^2,$$

and thus the first constraint in (36) is satisfied. Finally, by letting $W \in \mathbb{R}^{d\times d}$ be the vector satisfying $\operatorname{vec} W = \hat{\mathbf{H}}\hat{\mathbf{e}}$, one can write

$$\operatorname{vec}(PWP^T) = \mathbf{P}\operatorname{vec} W = \mathbf{P}\hat{\mathbf{H}}\hat{\mathbf{e}}.$$

Hence,

$$\begin{aligned}2I_r \otimes \operatorname{mat}_{\mathrm{S}}(\mathbf{H}\mathbf{e}) &= 2I_r \otimes \operatorname{mat}_{\mathrm{S}}(\mathbf{H}\mathbf{P}\hat{\mathbf{e}}) = 2I_r \otimes \operatorname{mat}_{\mathrm{S}}(\mathbf{P}\hat{\mathbf{H}}\hat{\mathbf{e}}) = I_r \otimes (P(W + W^T)P^T)\\ &= 2I_r \otimes (P\operatorname{mat}_{\mathrm{S}}(\hat{\mathbf{H}}\hat{\mathbf{e}})P^T) = 2(I_r \otimes P)(I_r \otimes \operatorname{mat}_{\mathrm{S}}(\hat{\mathbf{H}}\hat{\mathbf{e}}))(I_r \otimes P)^T.\end{aligned}$$

In addition,

$$\mathbf{X}^T\mathbf{H}\mathbf{X}(I_r \otimes P) = \mathbf{X}^T\mathbf{H}\mathbf{P}\hat{\mathbf{X}} = \mathbf{X}^T\mathbf{P}\hat{\mathbf{H}}\hat{\mathbf{X}} = (I_r \otimes P)\hat{\mathbf{X}}^T\hat{\mathbf{H}}\hat{\mathbf{X}}.$$

Therefore, by defining

$$\mathbf{S} := 2I_r \otimes \operatorname{mat}_{\mathrm{S}}(\mathbf{H}\mathbf{e}) + \mathbf{X}^T\mathbf{H}\mathbf{X} + bI_{nr},$$

we have

$$\begin{aligned}(I_r \otimes P)^T\mathbf{S}(I_r \otimes P) &= 2I_r \otimes \operatorname{mat}_{\mathrm{S}}(\hat{\mathbf{H}}\hat{\mathbf{e}}) + \hat{\mathbf{X}}^T\hat{\mathbf{H}}\hat{\mathbf{X}} + bI_{dr} \succeq 0,\\ (I_r \otimes P_\perp)^T\mathbf{S}(I_r \otimes P_\perp) &= (I_r \otimes P_\perp)^T\mathbf{X}^T\mathbf{H}\mathbf{X}(I_r \otimes P_\perp) + bI_{(n-d)r} \succeq 0,\\ (I_r \otimes P_\perp)^T\mathbf{S}(I_r \otimes P) &= 0.\end{aligned}$$

Since the columns of $I_r \otimes P$ and $I_r \otimes P_\perp$ form a basis for $\mathbb{R}^{nr}$, the above inequalities imply that $\mathbf{S}$ is positive semidefinite, and thus the second constraint in (36) is satisfied. ■

Lemma 11 *The inequality* $\overline{\mathrm{LMI}}(X, Z) \geq \mathrm{LMI}(\hat{X}, \hat{Z})$ *holds.*

Proof The dual problem of the optimization problem defining $\mathrm{LMI}(\hat{X}, \hat{Z})$ can be expressed as

$$\begin{aligned}\max \quad & \operatorname{tr}(\hat{U}_1 - \hat{U}_2) - \operatorname{tr}(\hat{G}) - a^2\hat{\lambda} - b\operatorname{tr}(\hat{V})\\ \text{s.t.} \quad & \operatorname{tr}(\hat{U}_1 + \hat{U}_2) = 1,\\ & \sum_{j=1}^r (\hat{\mathbf{X}}\hat{y} - \operatorname{vec}\hat{V}_{j,j})\hat{\mathbf{e}}^T + \sum_{j=1}^r \hat{\mathbf{e}}(\hat{\mathbf{X}}\hat{y} - \operatorname{vec}\hat{V}_{j,j})^T - \hat{\mathbf{X}}\hat{V}\hat{\mathbf{X}}^T = \hat{U}_1 - \hat{U}_2,\\ & \begin{bmatrix}\hat{G} & -\hat{y}\\ -\hat{y}^T & \hat{\lambda}\end{bmatrix} \succeq 0,\\ & \hat{U}_1 \succeq 0, \quad \hat{U}_2 \succeq 0, \quad \hat{V} = \begin{bmatrix}\hat{V}_{1,1} & \cdots & \hat{V}_{r,1}\\ \vdots & \ddots & \vdots\\ \hat{V}_{r,1}^T & \cdots & \hat{V}_{r,r}\end{bmatrix} \succeq 0.\end{aligned} \tag{38}$$

Since

$$\hat{U}_1 = \frac{1}{2d^2} I_{d^2} - \frac{\epsilon}{2} M, \quad \hat{U}_2 = \frac{1}{2d^2} I_{d^2} + \frac{\epsilon}{2} M, \quad \hat{V} = \epsilon I_{dr}, \quad \hat{G} = I_{dr}, \quad \hat{\lambda} = 1, \quad \hat{y} = 0,$$

where

$$M = r((\operatorname{vec} I_d)\hat{\mathbf{e}}^T + \hat{\mathbf{e}}(\operatorname{vec} I_d)^T) + \hat{\mathbf{X}}\hat{\mathbf{X}}^T,$$

is a strict feasible solution to the above dual problem (38) as long as $\epsilon > 0$ is sufficiently small, Slater's condition implies that strong duality holds for the optimization problem defining $\operatorname{LMI}(\hat{X}, \hat{Z})$. Therefore, we only need to prove that the optimal value of (38) is smaller than or equal to the optimal value of the dual of the optimization problem defining $\overline{\operatorname{LMI}}(X, Z)$ given by:

$$\begin{aligned}
\max \quad & \operatorname{tr}(U_1 - U_2) - \operatorname{tr}(G) - a^2\lambda - b\operatorname{tr}(V) \\
\text{s.t.} \quad & \operatorname{tr}(U_1 + U_2) = 1, \\
& \sum_{j=1}^{r} (\mathbf{X}y - \operatorname{vec} V_{j,j})\mathbf{e}^T + \sum_{j=1}^{r} \mathbf{e}(\mathbf{X}y - \operatorname{vec} V_{j,j})^T - \mathbf{X}V\mathbf{X}^T = \mathbf{P}(U_1 - U_2)\mathbf{P}^T, \\
& \begin{bmatrix} G & -y \\ -y^T & \lambda \end{bmatrix} \succeq 0, \\
& U_1 \succeq 0, \quad U_2 \succeq 0, \quad V = \begin{bmatrix} V_{1,1} & \cdots & V_{r,1} \\ \vdots & \ddots & \vdots \\ V_{r,1}^T & \cdots & V_{r,r} \end{bmatrix} \succeq 0.
\end{aligned} \tag{39}$$

The above claim can be verified by noting that given any feasible solution

$$(\hat{U}_1, \hat{U}_2, \hat{V}, \hat{G}, \hat{\lambda}, \hat{y})$$

to (38), the matrices

$$U_1 = \hat{U}_1, \quad U_2 = \hat{U}_2, \quad V = (I_r \otimes P)\hat{V}(I_r \otimes P)^T,$$
$$\begin{bmatrix} G & -y \\ -y^T & \lambda \end{bmatrix} = \begin{bmatrix} I_r \otimes P & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{G} & -\hat{y} \\ -\hat{y}^T & \hat{\lambda} \end{bmatrix} \begin{bmatrix} (I_r \otimes P)^T & 0 \\ 0 & 1 \end{bmatrix}$$

form a feasible solution to (39), and both solutions have the same optimal value. ■

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